A Nonlinear Heat Equation with Temperature-Dependent Parameters

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Abstract

A nonlinear partial differential equation of the following form is considered:

$$u' - \operatorname{div}\Big(a(u)\nabla u\Big) + b(u)\,|\nabla u|^2 = 0,$$

which arises from the heat conduction problems with strong temperature-dependent material parameters, such as mass density, specific heat and heat conductivity. Existence, uniqueness and asymptotic behavior of initial boundary value problems under appropriate assumptions on the material parameters are established. Both one-dimensional and two-dimensional cases are considered.

1 Introduction

Metallic materials present a complex behavior during heat treatment processes involving phase changes. In a certain temperature range, change of temperature induces a phase transformation of metallic structure, which alters physical properties of the material. Indeed, measurements of specific heat and conductivity show a strong temperature dependence during processes such as quenching of steel.

Several mathematical models, as solid mixtures and thermal-mechanical coupling, for problems of heat conduction in metallic materials have been proposed, among them, [8], [6] and [13]. In this paper, we take a simpler approach without thermal-mechanical coupling of deformations, by considering the nonlinear temperature dependence of thermal parameters as the sole effect due to those complex behaviors.

The above discussion of phase transformation of metallic materials serves only as a motivation for the strong temperature-dependence of material properties. In general, thermal properties of materials do depend on the temperature, and the present formulation of heat conduction problem may be served as a mathematical model when the temperature-dependence of material parameters becomes important.

More specifically, we are interested in a nonlinear heat equation with temperature-dependent material parameters, in contrast to the usual linear heat equation with constant coefficients.

2 A nonlinear heat equation

Let $\theta(x,t)$ be the temperature field, then we can write the conservation of energy in the following form:

$$\rho\,\varepsilon' + \operatorname{div} q = 0, \tag{1}$$

where prime denotes the time derivative.

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The mass density $\rho = \rho(\theta) > 0$ may depend on temperature due to possible change of material structure, while the heat flux q is assumed to be given by the Fourier law with temperature-dependent heat conductivity,

$$q = -\kappa \nabla \theta, \qquad \kappa = \kappa(\theta) > 0. \tag{2}$$

The internal energy density $\varepsilon = \varepsilon(\theta)$ generally depends on the temperature, and the specific heat c is assume to be positive defined by

$$c(\theta) = \frac{\partial \varepsilon}{\partial \theta} > 0, \tag{3}$$

which is not necessarily a constant.

By the assumption (3), we can reformulate the equation (1) in terms of the energy ε instead of the temperature θ. Rewriting Fourier law as

$$q = -\kappa(\theta)\nabla\theta = -\alpha(\varepsilon)\nabla\varepsilon, \tag{4}$$

and observing that

$$\nabla\varepsilon = \frac{\partial\varepsilon}{\partial\theta}\nabla\theta = c(\theta)\nabla\theta,$$

we have $c(\theta)\alpha(\varepsilon) = \kappa(\theta)$, and hence

$$\alpha = \alpha(\varepsilon) > 0.$$

Now let u be defined as $u = \varepsilon(\theta)$, then the equation (1) becomes

$$\rho(u)\, u' - \operatorname{div}(\alpha(u)\nabla u) = 0.$$

Since $\rho(u) > 0$, dividing the equation by ρ, and using the relation,

$$\frac{1}{\rho(u)} \operatorname{div}\big(\alpha(u)\nabla u\big) = \operatorname{div}\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) - \Big(\nabla\frac{1}{\rho(u)}\Big)\cdot\Big(\alpha(u)\nabla u\Big),$$

we obtain

$$u' - \operatorname{div}\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) + \Big(\nabla\frac{1}{\rho(u)}\Big)\cdot\Big(\alpha(u)\nabla u\Big) = 0. \tag{5}$$

Since $c = \dfrac{\partial\varepsilon}{\partial\theta} = \dfrac{du}{d\theta}$,

$$\Big(\nabla\frac{1}{\rho(u)}\Big) = -\frac{1}{\rho(u)^2}\frac{d\rho(u)}{du}\nabla u = -\frac{1}{\rho(u)^2}\frac{d\rho}{d\theta}\frac{d\theta}{du}\nabla u = -\frac{1}{\rho(u)^2}\frac{d\rho}{d\theta}\frac{1}{c}\nabla u.$$

Substituting into equation (5) we obtain

$$u' - \operatorname{div}\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) - \Big(\frac{1}{\rho(u)^2}\frac{d\rho}{d\theta}\frac{1}{c}\nabla u\Big)\cdot\Big(\alpha(u)\nabla u\Big) = 0,$$

which is equivalent to

$$u' - \operatorname{div}\big(a(u)\nabla u\big) + b(u)\,|\nabla u|^2 = 0, \tag{6}$$

where

$$a(u) = \frac{\alpha(u)}{\rho(u)} = \frac{\alpha(u)c(u)}{\rho(u)c(u)} = \frac{k(u)}{\rho(u)c(u)} > 0 \tag{7}$$

and

$$b(u) = -\frac{\alpha(u)}{c\,\rho(u)^2}\frac{d\rho}{d\theta} > 0. \tag{8}$$

The positiveness of $a(u)$ and $b(u)$ is the consequence of thermodynamic considerations, see [10], and reasonable physical experiences: the specific heat $c > 0$, the thermal conductivity $\kappa > 0$, the mass density $\rho > 0$, and the thermal expansion $d\rho/d\theta < 0$. In this paper we shall formulate the problem based on the nonlinear heat equation (6), using $a(x, u)$ and $b(x, u)$ functions more general in stead of the $a(u)$ and $b(u)$.

Formulation of the Problem

Let Ω be a bounded open set of $I\!R^n$, $n = 1, 2$, with regular boundary. We represent by $Q = \Omega \times (0,T)$ for $T > 0$, a cylindrical domain, whose lateral boundary we represent by $\Sigma = \Gamma \times (0,T)$. We shall consider the following non-linear problem:

$$\begin{cases} u' - \operatorname{div}\Big(a(x,u)\nabla u\Big) + b(x,u)\,|\nabla u|^2 = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{9}$$

Mathematical models of semi-linear and nonlinear parabolic equations under Dirichlet or Neumann boundary conditions has been considered in several papers, among them, let us mention ([1], [2], [3]) and ([4], [5], [11], [14]), respectively.

Feireisl, Petzeltová and Simondon in [7] prove that with non-negative initial data, the function $a(x,u) \equiv 1$ and $g(u,\nabla u) \leq h(u)(1 + |\nabla u|^2)$, instead of the non-linear term $b(x,u)\,|\nabla u|^2$ in $(9)_1$, there exists an admissible solution positive in some maximal interval $[0, T_{\max})$ and if $T_{\max} < +\infty$ then

$$\lim_{t \to T_{\max}} \|u(t,.)\|_\infty = \infty.$$

For Problem (9) we will prove global existence, uniqueness and asymptotic behavior for the one-dimensional case and existence for the two-dimensional case, for small enough initial data.

3 Existence and Uniqueness: One-dimensional Case

In this section we investigate the existence and uniqueness of solutions for the one-dimensional case of Problem (9).

Let $(\!(\cdot,\cdot)\!)$, $\|\cdot\|$ and $(\cdot,\cdot)$, $|\cdot|$ be respectively the scalar product and the norms in $H_0^1(\Omega)$ and $L^2(\Omega)$. In this section we investigate the existence and uniqueness of solutions for the one-dimensional case. For this we need the following hypotheses:

H1: $a(x,u)$ and $b(x,u)$ belongs the $C^1(\overline{\Omega} \times [0,T])$ and there are positive constants a_0, a_1 such that,

$$a_0 \leq a(x,u) \leq a_1 \quad \text{and} \quad b(x,u)u \geq 0.$$

H2: There is positive constant $M > 0$ such that

$$\max_{x \in \overline{\Omega},\ s \in I\!R} \left\{ \Big|\frac{\partial a}{\partial x}(x,s)\Big|;\ \Big|\frac{\partial a}{\partial u}(x,s)\Big|;\ \Big|\frac{\partial b}{\partial u}(x,s)\Big| \right\} \leq M.$$

H3: $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $|\Delta u_0| < \varepsilon$

Theorem 1 *Under the hypotheses* (H1), (H2) *and* (H3), *there exist a positive constant* ε_0 *such that, if* $0 < \varepsilon \leq \varepsilon_0$ *then the problem* (9) *admits a unique solution* $u : Q \to I\!R$, *satisfying the following conditions:*

i. $u \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega))$,

ii. $u' \in L^2(0,T; H_0^1(\Omega))$,

iii. $u' - \operatorname{div}(a(x,u)\nabla u) + b(x,u)\,|\nabla u|^2 = 0$, *in* $L^2(Q)$,

iv. $u(0) = u_0$.

Remark. The positive constant ε_0 will be determined by (22), (23), (29) and (54) .

Proof. To prove the theorem, we employ Galerkin method with the Hilbertian basis from $H_0^1(\Omega)$, given by the eigenvectors (w_j) of the spectral problem: $((w_j,v)) = \lambda_j(w_j,v)$ for all $v \in V$ and $j = 1,2,\cdots$. We represent by V_m the subspace of V generated by vectors $\{w_1,w_2,...,w_m\}$. We propose the following approximate problem: Determine $u_m \in V_m$, so that

$$\begin{cases} (u_m', v) + \Big(a(u_m)\nabla u_m, \nabla v\Big) + \Big(b(u_m)\,|\nabla u_m|^2, v\Big) = 0, \quad \forall v \in V_m \\ u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega)\cap H^2(\Omega). \end{cases} \tag{10}$$

Existence

The system (10) has local solution in the interval $(0,T_m)$. To extend the local solution to the interval $(0,T)$ independent of m the following a priori estimates are needed.

Estimate I: Taking $v = u_m(t)$ in the equation $(10)_1$ and integrating over $(0,T)$, we obtain

$$\frac{1}{2}|u_m|^2 + a_0\int_0^T \|u_m\|^2 + \int_0^T\int_\Omega b(x,u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2 \tag{11}$$

where we have used hypothesis (H1). Therefore, we have the following estimate:

$$\begin{aligned} (u_m) \quad &\text{is bounded in} \quad L^\infty\left(0,T;L^2(\Omega)\right), \\ (u_m) \quad &\text{is bounded in} \quad L^2\left(0,T;H_0^1(\Omega)\right). \end{aligned} \tag{12}$$

Estimate II: Taking $v = u_m'(t)$ in the equation $(10)_1$ and integrating over $(0,T)$, we obtain

$$\begin{aligned} |u_m'|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a(x,u_m)|\nabla u_m|^2 = \frac{1}{2}\int_\Omega \frac{\partial a}{\partial u}(x,u_m)u_m'|\nabla u_m|^2 \\ + \int_\Omega b(x,u_m)u_m'|\nabla u_m|^2. \end{aligned} \tag{13}$$

But, from hypothesis (H2), we have the following inequality,

$$\frac{1}{2}\int_\Omega \frac{\partial a}{\partial u}(x,u_m)u_m'|\nabla u_m|^2 \le \frac{1}{2}MC_0\|u_m'\|\,\|u_m\|^2, \tag{14}$$

and since $|\cdot|_{L^\infty(\Omega)} \le \|\cdot\|$ and $|b(x,u_m)| \le M|u_m|$, we obtain

$$\int_\Omega b(x,u_m)u_m'|\nabla u_m|^2 \le MC_0^2\|u_m'\|\|u_m\|^3, \tag{15}$$

where $C_0 = C_0(\Omega)$ is constant depending on Ω.

Substituting (14) and (15) into the left hand side of (13) we get

$$\begin{aligned} |u_m'|^2 \quad &+\frac{1}{2}\frac{d}{dt}\int_\Omega a(x,u_m)|\nabla u_m|^2 \le \frac{1}{2}MC_0\|u_m'\|\,\|u_m\|^2 + MC_0^2\|u_m'\|\|u_m\|^3 \\ &\le \frac{1}{4}(MC_0)^2\|u_m'\|^2\,\|u_m\|^2 + \frac{1}{2}\|u_m\|^2 + (MC_0^2)^2\|u_m'\|^2\|u_m\|^4. \end{aligned} \tag{16}$$

Now taking the derivative of the equation $(10)_1$ with respect to t and making $v = u_m'(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m'|^2 + \int_\Omega a(x,u_m)|\nabla u_m'|^2 = -\int_\Omega \frac{\partial b}{\partial u}|u_m'|^2|\nabla u_m|^2 - \int_\Omega \frac{\partial a}{\partial u}u_m'|\nabla u_m|^2 \\ &-2\int_\Omega b(x,u_m)\nabla u_m\nabla u_m' u_m' \le M(3C_0^2 + C_0)\|u_m'\|^2\|u_m\|^2. \end{aligned} \tag{17}$$

From the inequality (16) and (17), we have

$$\begin{aligned}\frac{d}{dt}\left\{\frac{1}{2}|u_m'|^2+\frac{1}{2}\int_\Omega a(x,u_m)|\nabla u_m|^2\right\}+|u_m'|+\frac{a_0}{2}\|u_m'\|^2+\\ \|u_m'\|^2\left\{\frac{a_0}{2}-\alpha_0\|u_m\|^4-\alpha_1\|u_m\|^2\right\}\leq\frac{1}{2}\|u_m\|^2,\end{aligned}\tag{18}$$

where we have defined $\alpha_0=(MC_0^2)^2$ and $\alpha_1=MC_0\left(\frac{1}{4}MC_0+3C_0+1\right)$.

Now, suppose that the following inequality,

$$\alpha_0\|u_m\|^4+\alpha_1\|u_m\|^2<\frac{a_0}{4}\qquad\forall\, t\geq 0,\tag{19}$$

is true. Under this condition, the coefficients of the term $\|u_m'\|$ in the relation (18) is positive and we can integrate it with respect to t,

$$|u_m'|^2+\int_\Omega a(x,u_m)|\nabla u_m|^2+\int_0^t|u_m'|^2+a_0\int_0^t\|u_m'\|^2\leq C.\tag{20}$$

Therefore, we obtain the following estimate:

$$\begin{aligned}&(u_m')\quad\text{is bounded in}\quad L^\infty\left(0,T;L^2(\Omega)\right),\\ &(u_m')\quad\text{is bounded in}\quad L^2\left(0,T;H_0^1(\Omega)\right).\end{aligned}\tag{21}$$

Now we want to prove that the inequality (19) is true if the initial data are sufficiently small. Suppose that

$$\frac{\alpha_0}{a_0}\left\{S_0+a_1\|u_0\|^2+\frac{1}{2a_0}|u_0|^2\right\}^2+\alpha_1\left\{\frac{1}{a_0}S_0+\frac{a_1}{a_0}\|u_0\|^2+\frac{1}{2a_0}\|u_0\|^2\right\}<\frac{a_0}{4}\tag{22}$$

and

$$\alpha_0\|u_0\|^4+\alpha_1\|u_0\|^2<\frac{a_0}{4},\tag{23}$$

where, we have denoted $S_0=\Big(M(\|u_0\|+|\Delta u_0|^2+\|u_0\||\Delta u_0|^2\ )+a_1|\Delta u_0|\Big)^2$.

We shall prove by contradiction. Suppose that (19) is false, then there is a t^* such that

$$\alpha_0\|u_m(t)\|^4+\alpha_1\|u_m(t)\|^2<\frac{a_0}{4}\quad\text{if}\quad 0<t<t^*\tag{24}$$

and

$$\alpha_0\|u_m(t^*)\|^4+\alpha_1\|u_m(t^*)\|^2=\frac{a_0}{4}.\tag{25}$$

Integrating (18) from 0 to t^*, we obtain

$$\begin{aligned}\frac{1}{2}|u_m'(t^*)|^2\quad&+\int_\Omega a(x,u_m)|\nabla u_m(t^*)|^2\leq\frac{1}{2}|u_m'(0)|^2+\int_\Omega a(x,0)|\nabla u_m(0)|^2\\ &+\int_0^{t^*}\|u_m\|^2\leq\Big(M(\ \|u_0\|+|\Delta u_0|^2+\|u_0\||\Delta u_0|^2\ )+a_1|\Delta u_0|^2\Big)^2\\ &+a_1\|u_0\|^2+\frac{1}{2a_0}|u_0|^2,\end{aligned}\tag{26}$$

and consequently,

$$\|u_m(t^*)\|^2\leq\frac{1}{a_0}S_0+\frac{a_1}{a_0}\|u_0\|^2+\frac{1}{2a_0^2}|u_0|^2.\tag{27}$$

Using (22) and (23) we obtain

$$\begin{aligned}\alpha_0\|u_m(t^*)\|^2 \quad &+\alpha_1\|u_m(t^*)\| \leq \frac{\alpha_0}{a_0^2}\left\{S_0 + a_1\|u_0\|^2 + \frac{1}{2a_0}|u_0|^2\right\}^2 \\ &+\alpha_1\left\{\frac{1}{a_0}S_0 + \frac{a_1}{a_0}\|u_0\|^2 + \frac{1}{2a_0}\|u_0\|^2\right\} < \frac{a_0}{4},\end{aligned} \tag{28}$$

hence, comparing with (25), we have a contradiction.

Estimate III: Taking $v = -\Delta u_m$ in the equation $(10)_1$, and using hypothesis H1, we obtain

$$\frac{d}{dt}\|u_m\|^2 + \int_\Omega a(x, u_m)|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m| + b_1|\Delta u|^2\ \|u_m\|$$

where we have used the following inequality,

$$\begin{aligned}\int_\Omega b(x, u_m)|\nabla u_m|^2\ |\Delta u_m| \quad &\leq \bar{b}_1|\nabla u_m|_{L^\infty}\int_\Omega |\nabla u_m|\ |\Delta u_m| \leq \bar{b}_1|\nabla u_m|_{H^1(\Omega)}\ \|u_m\|\ |\Delta u_m| \\ &\leq \bar{b}_1|\Delta u_m|\ \|u_m\|\ |\Delta u_m| = \bar{b}_1\|u_m\|\ |\Delta u_m|^2.\end{aligned}$$

In this expression we have denoted $\bar{b}_1 = \sup|b(x, s)|$, for $x \in \overline{\Omega}$ and $s \in [-\frac{a_0}{4\alpha_1}, \frac{a_0}{4\alpha_1}]$.

Note that, from (19), we have $|u_m|_{I\!R} \leq \|u_m\| < a_0/4\alpha_1$ and $b_1 = \bar{b}_1 + M$.

Using hypothesis (H1), we obtain

$$\frac{d}{dt}\|u_m\|^2 + \frac{a_0}{2}|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m| + b_1|\Delta u_m|^2\ \|u_m\|,$$

or equivalently,

$$\frac{d}{dt}\|u_m\|^2 + \Big(\frac{a_0}{2} - b_1\|u_m\|\Big)\,|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m|.$$

Note that the constant C from (20) is given by $C = S_0 + a_1\|u_0\|^2 + \dfrac{1}{2a_0}|u_0|^2$. Considering

$$b_1\Big(\frac{1}{a_0}(S_0 + a_1\|u_0\|^2 + \frac{1}{2a_0}|u_0|^2)^{1/2}\Big) < \frac{a_0}{4} \tag{29}$$

and from (20) we obtain $\dfrac{a_0}{4} \leq \Big(\dfrac{a_0}{2} - b_1\|u_m\|\Big)$, it implies that

$$\frac{d}{dt}\|u_m\|^2 + \frac{a_0}{4}|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m| \leq \frac{a_0}{8}|\Delta u_m|^2 + C\|u_m\|^2. \tag{30}$$

Now, integrating from 0 to t, we obtain the estimate

$$\|u_m\|^2 + \int_0^T |\Delta u_m|^2 \leq \widehat{C}.$$

Hence, we have

$$\begin{aligned}(u_m) \quad &\text{is bounded in} \quad L^\infty\left(0, T; H_0^1(\Omega)\right), \\ (u_m) \quad &\text{is bounded in} \quad L^2\left(0, T; H_0^1(\Omega) \cap H^2(\Omega)\right).\end{aligned} \tag{31}$$

Limit of the approximate solutions

From the estimates (12), (21) and (31), we can take the limit of the nonlinear system (10). In fact, there exists a subsequence of $(u_m)_{m\in N}$, which we denote as the original sequence, such that

$$\begin{aligned}
&u'_m \longrightarrow u' \quad \text{weak star} && \text{in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&u'_m \longrightarrow u' \quad \text{weak} && \text{in } L^2\left(0,T;H_0^1(\Omega)\right),\\
&u_m \longrightarrow u \quad \text{weak star} && \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right),\\
&u_m \longrightarrow u \quad \text{weak} && \text{in } L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right).
\end{aligned} \tag{32}$$

Thus, by compact injection of $H_0^1(\Omega\times]0,T[)$ into $L^2(\Omega\times]0,T[)$ it follows by compactness arguments of Aubin-Lions [9], we can extract a subsequence of $(u_m)_{m\in N}$, still represented by $(u_m)_{m\in N}$ such that

$$\begin{aligned}
&u_m \longrightarrow u && \text{strong} && \text{in } L^2\left(0,T;H_0^1(\Omega)\right),\\
&\nabla u_m \longrightarrow \nabla u && \text{strong} && \text{in } L^2(Q),\\
&u_m \longrightarrow u && \text{a.e.} && \text{in } Q,\\
&\nabla u_m \longrightarrow \nabla u && \text{a.e.} && \text{in } Q.
\end{aligned} \tag{33}$$

Let us analysis the nonlinear terms from the approximate system (10). From the first term, we know that

$$\int_\Omega |a(x,u_m)\nabla u_m|^2 \le a_1 \int_\Omega |\nabla u_m|^2 \le a_1 C, \tag{34}$$

and since that $u_m \to u$ a.e. in Q and that $a(x,.)$ is continuous, we get

$$a(x,u_m) \longrightarrow a(x,u) \quad \text{and} \quad \nabla u_m \longrightarrow \nabla u \quad \text{a.e. in } Q. \tag{35}$$

Hence, we also have

$$|a(x,u_m)\nabla u_m|^2 \longrightarrow |a(x,u)\nabla u|^2 \quad \text{a.e. in } Q. \tag{36}$$

From (34) and (36), and Lions Lemma, we obtain

$$a(x,u_m)\nabla u_m \longrightarrow a(x,u)\nabla u \quad \text{weak in } L^2(Q). \tag{37}$$

From the second term, we know that

$$\begin{aligned}
&\int_0^T \int_\Omega |b(x,u_m)|\nabla u_m|^2|^2 \le (b_1)^2 \int_0^T |\nabla u_m|^2_{L^\infty} \int_\Omega |\nabla u_m|^2\\
&\le (b_1)^2 C_0 \int_0^T \|u_m\|^2 |\nabla u_m|^2_{H^1} \le (b_1)^2 C_0 \|u_m\|^2 \int_0^T |\Delta u_m|^2 \le C.
\end{aligned} \tag{38}$$

By the same argument as (36) we get

$$|b(x,u_m)|\nabla u_m|^2|^2 \longrightarrow |b(x,u)|\nabla u|^2|^2 \quad \text{a.e. in} \quad \text{Q} \tag{39}$$

Hence, from (38) and (39) we obtain the convergence,

$$b(x,u_m)|\nabla u_m|^2 \longrightarrow b(x,u)|\nabla u|^2 \quad \text{weak in } L^2(Q). \tag{40}$$

Taking into account (32), (37) and (40) into $(10)_1$, there exists a function $u = u(x,t)$ defined over $\Omega\times[0,T[$ with value in $I\!R$ satisfying (9). Moreover, from the convergence results obtained, we have that $u_m(0) = u_{0m} \to u_0$ in $H_0^1(\Omega)\cap H^2(\Omega)$ and initial condition is well defined.

Hence, we conclude that equation (9) is given in the sense of $L^2(0,T;L^2(\Omega))$.

Uniqueness

Let $w = u - v$, where u and v are solution of Problem (9). Then we have

$$\begin{cases} w' - \text{div}\,\big(a(x,u)\nabla w\big) - \text{div}(a(x,u) - a(x,v))\nabla v \\ \quad + b(x,u)\,\big(|\nabla u|^2 - |\nabla v|^2\big) + (b(x,u) - b(x,v))|\nabla v|^2 = 0 \quad \text{in } Q, \\ w = 0 \quad \text{on } \Sigma, \\ w(x,0) = 0 \quad \text{in } \Omega. \end{cases} \tag{41}$$

Multiplying by w, integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d|w|^2}{dt} + \int_\Omega a(x,u)|\nabla w|^2 \le \int_\Omega |\frac{\partial a}{\partial u}(x,\widehat{u})|\,|w|\,|\nabla v|\,|\nabla w| \\ &+ \int_\Omega |b(x,u)|\Big(|\nabla u| + |\nabla v|\Big)\,|\nabla w|\,|w| + \int_\Omega |\nabla v|^2|\,|\frac{\partial b}{\partial u}(x,\overline{u})|\,|w|^2 \\ &\le M\int_\Omega |\nabla w|\,|w|\,|\nabla v| + C_0\int_\Omega \Big(|\nabla u| + |\nabla v|\Big)\,|\nabla w|\,|w| + M\int_\Omega |\nabla v|^2\,|w|^2 \\ &\le M|\nabla v|_{L^\infty(\Omega)}\,\|w\|\,|w| + C_0\Big(|\nabla u|_{L^\infty(\Omega)} + |\nabla v|_{L^\infty(\Omega)}\Big)\,\|w\|\,|w| \\ &+ M|\nabla v|^2_{L^\infty(\Omega)}\,|w|^2 \le \frac{a_0}{2}\|w\|^2 + \overline{C}(\Delta u|^2 + |\Delta v|^2)|w|^2, \end{aligned} \tag{42}$$

where we have used

(a) The generalized mean-value theorem, i.e.,
$$|a(x,u) - a(x,v)| = \Big|\frac{\partial a}{\partial s}(x,\widehat{u})(u-v)\Big| \le |\frac{\partial a}{\partial s}(x,\widehat{u})|\,|w|, \qquad u \le \widehat{u} \le v,$$

(b) $\Big|b(x,u)\,(|\nabla u|^2 - |\nabla v|^2)\Big| \le |b(x,u)|\,|\nabla w|\Big(|\nabla u| + |\nabla v|\Big)$,

(c) $|\nabla v|_{L^\infty(\Omega)} \le \|\nabla v\|_{H^1(\Omega)} \le \|v\|_{H^2(\Omega)} \le |\Delta v|_{L^2(\Omega)}$.

The last inequality is valid only for one-dimensional case.

Integrating (42) from o to t, we obtain

$$\frac{1}{2}|w(t)|^2 + \frac{a_0}{2}\int_0^t \|w\|^2 \le \frac{1}{2}|w(0)|^2 + \overline{C}\int_0^t (|\Delta u|^2 + |\Delta v|^2)|w|^2,$$

where $\overline{C}$ denotes a different positive constant. Since $w(0) = u_0 - v_0 = 0$, then using the Gronwall's inequality, we obtain

$$|w(t)|^2 + \int_0^t \|w\|^2 = 0$$

Hence $w = u - v = 0$ and the uniqueness is proved. □

Asymptotic behavior

In the following we shall prove that the solution $u(x,t)$ of the Problem (9) decays exponentially when time $t \to \infty$, using the same procedure developed in Lions [9] and Prodi [12].

Theorem 2 *Let $u(x,t)$ the solution of the Problem (9). Then there are positive constants $C = C(\|u_0\|, |\Delta u_0|)$ and $\widehat{s}_0$ such that*

$$\|u\|^2 + |u'|^2 \le C\exp^{-\widehat{s}_0\,t} \tag{43}$$

Proof. To prove the theorem, complementary estimates are needed.

Estimate I': Consider the approximate system (10). Using the same argument as Estimate I, i.e, taking $v = u_m(t)$, we have

$$\frac{d}{dt}|u_m|^2 + \int_\Omega a(x,u_m)|\nabla u_m|^2 + \int_\Omega b(x,u_m)u_m|\nabla u_m|^2 = 0 \tag{44}$$

Integrating (45) from $(0,T)$, we obtain

$$\frac{1}{2}|u_m|^2 + a_0\int_0^T \|u_m\|^2 + \int_0^T\int_\Omega b(x,u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2 \tag{45}$$

From H1 hypothesis, (44) and (45) we conclude

$$a_0\|u_m\|^2 \leq 2|u_m'|\;|u_m| \leq 2|u_m'|\;|u_0| \tag{46}$$

Estimate II': Taking derivative of the system (10) with respect to t we have

$$\begin{aligned}&(u_m'', v) + (a(x,u_m)\nabla u_m', \nabla v) + (a_u(x,u_m)u_m'\nabla u_m, \nabla v) + \\ &\Big(b_u(x,u_m)u_m'\;|\nabla u_m|^2, v\Big) + 2\Big(b(x,u_m)\;\nabla u_m\;\nabla u_m', v\Big) = 0\end{aligned} \tag{47}$$

Taking $v = u_m'$ in (47), we obtain

$$\begin{aligned}\frac{d}{dt}|u_m'|^2 + a_0\|u_m'\|^2 &\leq C\int_\Omega |u_m'|^2|\nabla u_m| + \int_\Omega |u_m'||\nabla u_m|^2 + C\int_\Omega |\nabla u_m||\nabla u_m'||u_m'| \\ &\leq C_1\Big(\|u_m'\|^2\|u_m\| + \|u_m'\|^2\|u_m\|^2\end{aligned} \tag{48}$$

So,

$$\frac{d}{dt}|u_m'|^2 + \frac{a_0}{2}\|u_m'\|^2 + \|u_m'\|^2\left(\frac{a_0}{2} - C_1\|u_m\| - C_1\|u_m\|^2\right) \leq 0 \tag{49}$$

Using (46) then we can write the inequality (49) in the form

$$\frac{d}{dt}|u_m'|^2 + \frac{a_0}{2}\|u_m'\|^2 + \|u_m'\|^2\left(\frac{a_0}{2} - C_1\left(\frac{2|u_0|}{a_0}\right)^{1/2}|u_m'|^{1/2} - C_1\frac{2|u_0|}{a_0}|u_m'|\right) \leq 0 \tag{50}$$

Let $v = u_m'(0)$ in (10). Then

$$|u_m'(0)|^2 \leq \left(C\|u_0\| + C_2|\Delta u_0| + |\Delta u_0)|^2\right)|u_m'(0)|$$

and then

$$|u_m'(0)|^2 \leq \Big(C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0)|^2 + |\Delta u_0|^3\right)\Big)^2 \tag{51}$$

We define the operator

$$\begin{aligned}J(u_0) = \;&\left(\frac{2|u_0|}{a_0}\right)^{1/2}\Big(C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right)\Big)^{1/2} \\ &\frac{2|u_0|}{a_0}\;C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right)\end{aligned} \tag{52}$$

Then we can show that

$$\left(\frac{2|u_0|}{a_0}\right)^{1/2}|u_m'(0)|^{1/2} + \frac{2|u_0|}{a_0}\;|u_m'(0)| \leq J(u_0) \tag{53}$$

Consider c_1 a positive constant and u_0 enough small, that is,

$$c_1 J(u_0) < \frac{a_0}{4}. \tag{54}$$

In this conditions we have are true the following inequality,

$$c_1 \left(\frac{2|u_0|}{a_0} \right)^{1/2} |u_m'(t)|^{1/2} + c_1 \frac{2|u_0|}{a_0} |u_m'(t)| < \frac{a_0}{4}, \qquad \forall t \geq 0 \tag{55}$$

In fact, we will proof for contradiction. Suppose that there is a t^* such that

$$c_1 \Big(\frac{2|u_0|}{a_0} \Big)^{1/2} |u_m'(t^*)|^{1/2} + c_1 \frac{2|u_0|}{a_0} |u_m'(t^*)| = \frac{a_0}{4} \tag{56}$$

Integrating (50) from 0 to t^* we obtain

$$|u'(t^*)|^2 \leq |u'(0)|^2,$$

From (54) and (55), we conclude that

$$\begin{aligned} & c_1 \Big(\frac{2|u_0|}{a_0} \Big)^{1/2} |u_m'(t^*)|^{1/2} + c_1 \frac{2|u_0|}{a_0} |u_m'(t^*)| \\ & \leq c_1 \Big(\frac{2|u_0|}{a_0} \Big)^{1/2} |u_m'(0)|^{1/2} + c_1 \frac{2|u_0|}{a_0} |u_m'(0)| \leq c_1 J(u_0) < \frac{a_0}{4}. \end{aligned} \tag{57}$$

So, we have a contradiction by (56).

From (50), (55) and using the Poincaré inequality, we obtain

$$\frac{d}{dt} |u_m'|^2 + s_0 |u_m'|^2 \leq 0 \tag{58}$$

where $s_0 = (a_0 c_0)/2$ and c_0 is a positive constant such that $\| \, . \, \|_{H_0^1(\Omega)} \geq c_0 | \cdot |_{L^2(\Omega)}$.

Then, we have

$$\frac{d}{dt} \Big\{ \exp(s_0 t) \; |u_m'|^2 \Big\} \leq 0 \tag{59}$$

and hence

$$\begin{aligned} |u_m'(t)|^2 \leq |u_m'(0)|^2 \, \exp(-s_0 t) \quad & \leq C \Big(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 \Big)^2 \, \exp(-s_0 t) \\ & \leq \widetilde{C} \Big(\|u_0\|, |\Delta u_0| \Big) \, \exp(-s_0 t). \end{aligned} \tag{60}$$

We also have from (46) that

$$\|u_m\|^2 \leq \frac{2}{a_0} |u_0| \; |u_m'| \leq \frac{2}{a_0} \; C \Big(\|u_0\|, |\Delta u_0| \Big) |u_0| \; \exp(-\frac{s_0}{2} t). \tag{61}$$

Defining $\widehat{s}_0 = s_0/2$ then the result follows from (60), (61) inequality and of the Banach-Steinhauss theorem. □

4 Existence: Two-dimensional Case

In order to prove the existence of the solution of Problem (9) for the two-dimensional case and for this we need the following hypotheses:

H1: $a(x,u)$ and $b(x,u)$ belongs the $C^1(\overline{\Omega}\times[0,T])$ and there are positive constants a_0, a_1 such that,

$$a_0 \leq a(x,u) \leq a_1 \quad \text{and} \quad b(x,u)u \geq 0.$$

H2: There is positive constant $M > 0$ such that

$$\max_{x\in\overline{\Omega},\ s\in I\!R}\left\{\left|\nabla_x a(x,s)\right|;\ \left|\frac{\partial a}{\partial u}(x,s)\right|\right\} \leq M.$$

H3: $\dfrac{\partial a}{\partial u}(x,0) = 0, \qquad \forall x \in \partial\Omega$

H4: $u_0 \in H_0^1(\Omega)$ such that $\|u_0\| < \varepsilon$

Theorem 3 *Under the hypotheses* (H1) - (H4)*, there exist a positive constant* ε_0 *such that, if* $0 < \varepsilon \leq \varepsilon_0$ *then the problem* (9) *admits a solution* $u : Q \to I\!R$*, satisfying the following conditions:*

i. $u \in L^\infty(0,T; H_0^1(\Omega)\cap H^2(\Omega))$,

ii. $u' \in L^\infty(0,T; L^2(\Omega))$,

iii. $u' - div(a(x,u)\nabla u + b(x,u)\,|\nabla u|^2 = 0 \quad in\ L^2(Q)$,

iv. $u(0) = u_0$.

Remark. The positive constant ε_0 will be determined by (72).

Proof. To prove the theorem, we employ Galerkin method with the Hilbertian basis from $H_0^1(\Omega)$, given by the eigenvectors (w_j) of the spectral problem: $((w_j,v)) = \lambda_j(w_j,v)$, for all $v \in V$ and $j = 1,2,\cdots$. We represent by V_m the subspace of V generated by vectors $\{w_1, w_2, ..., w_m\}$. Considerer the local solution $u_m(x,t)$ of the approximate problem (10). In a similar manner for the one-dimensional case, in order to extend the local solution to the interval $(0,T)$ independent of m, the following a priori estimates are needed.

Estimate I: From (11), we have that

$$\frac{1}{2}|u_m|^2 + a_0\int_0^T \|u_m\|^2 + \int_0^T\int_\Omega b(x,u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2 \tag{62}$$

and hence we have

$$\begin{array}{lll} (u_m) & \text{is bounded in} & L^\infty\left(0,T;L^2(\Omega)\right), \\ (u_m) & \text{is bounded in} & L^2\left(0,T;H_0^1(\Omega)\right). \end{array} \tag{63}$$

Estimativa II: Taking $v = -\Delta u_m$ in (10), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_m\|^2 \quad &+ \int_\Omega a(x,u_m)|\Delta u_m|^2 + \int_\Omega b(x,u_m)|\nabla u_m|^2\Delta u_m \\ &= \int_\Omega \nabla a\nabla u_m\Delta u_m + \int_\Omega \frac{\partial a}{\partial u}|\nabla u_m|^2\Delta u_m. \end{aligned} \tag{64}$$

When the space dimension is 2 then the following inequality is true (see p.70 in [9]),

$$\|v\|_{L^4(\Omega)} \leq C(\Omega)\,\|v\|^{1/2}_{H_0^1(\Omega)}\,\|v\|^{1/2}_{L^2(\Omega)}, \qquad \forall v \in H_0^1(\Omega). \tag{65}$$

Let $\widetilde{b}(x,u_m) = -b(x,u_m) + \dfrac{\partial a}{\partial u}(x,u_m)$. Then

$$\Big| \int_\Omega \widetilde{b}(x,u_m)|\nabla u_m|^2 \, \Delta u_m \Big| \le \Big(\int_\Omega \widetilde{b}(x,u_m)^2 \, |\nabla u_m|^4 \Big)^{1/2} \, |\Delta u_m|. \tag{66}$$

We also have that

$$|\nabla u_m|^4 = \Big(|\nabla u_m|^2 \Big)^2 = \quad \Big\{ \Big(\frac{\partial u_m}{\partial x} \Big)^2 + \Big(\frac{\partial u_m}{\partial y} \Big)^2 \Big\}^2 \le 2 \Big\{ \Big(\frac{\partial u_m}{\partial x} \Big\}^4 + \Big(\frac{\partial u_m}{\partial y} \Big)^4 \Big\} \tag{67}$$

and using (67), (65) and Hypothesis H3, we get

$$\begin{aligned} \int_\Omega \widetilde{b}(x,u_m)^2 |\nabla u_m|^4 \quad & \le 2 \int_\Omega \widetilde{b}(x,u_m)^2 \Big\{ \Big(\frac{\partial u_m}{\partial x} \Big)^4 + \Big(\frac{\partial u_m}{\partial y} \Big)^4 \Big\} \\ & \le 2 \int_\Omega \Big| \, \nabla (\widetilde{b}(x,u_m))^{1/2} \, \frac{\partial u_m}{\partial x}) \Big|^2 \, \int_\Omega \Big| \frac{\partial u_m}{\partial x} \Big|^2 \\ & + \int_\Omega \Big| \, \nabla \Big(\widetilde{b}(x,u_m)^{1/2} \, \frac{\partial u_m}{\partial y} \Big) \Big|^2 \, \int_\Omega \Big| \frac{\partial u_m}{\partial y} \Big|^2 \\ & \le c \, |\Delta u_m|^2 \, \|u_m\|^2. \end{aligned} \tag{68}$$

Note that the term $\Big(\widetilde{b}(x,u_m)^{1/2} \, \dfrac{\partial u_m}{\partial x} \Big) \in H_0^1(\Omega)$, since $\widetilde{b}(x,0) = 0$ for all $x \in \partial\Omega$ by hypothesis H1 and H3.

From (66) and (68) we conclude that

$$\Big| \int_\Omega \widetilde{b}(x,u_m)|\nabla u_m|^2 |\Delta u_m| \, \Big| \le c \, \|u_m\| \, |\Delta u_m|^2. \tag{69}$$

We also have that

$$| \int_\Omega \nabla a \nabla u_m \Delta u_m | \le \frac{a_0}{2} |\Delta u_m|^2 + c_1 \|u_m\|^2. \tag{70}$$

Substituting (69) and (70) in the inequality (64), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \frac{a_0}{4} |\Delta u_m|^2 + \Big(\frac{a_0}{4} - c_0 \|u_m\| \Big) |\Delta u_m|^2 \le c_1 \|u_m\|^2. \tag{71}$$

Suppose that

$$\begin{cases} \|u_0\| < \dfrac{a_0}{8c_0} \\ \|u_0\| < \dfrac{a_0^{3/2}}{8c_0(a_0 + c_1c_2)^{1/2}} \end{cases} \tag{72}$$

then we can confirm that

$$\|u_m(t)\| < \frac{a_0}{8c_0}, \qquad \forall t \ge 0. \tag{73}$$

Indeed, by presuming absurdity, there is a t^* such that

$$\|u_m(t)\| < \frac{a_0}{8c_0}, \quad \text{if} \quad 0 < t < t^* \quad \text{and} \quad \|u_m(t^*)\| = \frac{a_0}{8c_0}.$$

Then, by integrating (71) from 0 to t^*, we have

$$\begin{aligned} \|u_m(t^*)\|^2 \le \|u_{0m}\|^2 + \frac{c_1}{a_0} \|u_{0m}\|^2 \; & \le \|u_{0m}\|^2 + \frac{c_1 c_2}{a_0} \|u_{0m}\|^2 \\ & \le \Big(1 + \frac{c_1 c_2}{a_0} \Big) \|u_0\|^2 \end{aligned} \tag{74}$$

and hence

$$\|u_m(t^*)\| \leq \left(\frac{a_0 + c_1c_2}{a_0}\right)^{1/2} \|u_0\| < \frac{a_0}{8c_0}. \tag{75}$$

This leads to a contradiction.

Since that (73) is true, then the term $\left(\frac{a_0}{4} - c_0\|u_m\|\right)$ in the left hand-side of (71) is also positive. Then, integrating from 0 to T we obtain;

$$\|u_m(t)\|^2 + \int_0^T |\ \Delta u_m|^2 \leq c \tag{76}$$

Then,

$$\begin{array}{lll} (u_m) & \text{is bounded in} & L^\infty\left(0,T;H_0^1(\Omega)\right), \\ (u_m) & \text{is bounded in} & L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right) \end{array} \tag{77}$$

Estimate III: Taking $v = u_m'$ from (10) and integrating in Ω we obtain

$$|u_m'|^2 = \int_\Omega |\nabla a|\ |\nabla u_m||u_m'| + \int_\Omega |a(x,u_m)|\ |\Delta u_m|\ |u_m'| + \int_\Omega |\widetilde{b}(x,u_m)|\ |\nabla u_m|^2\ |u_m'|. \tag{78}$$

By Hypothesis (H1) and (H2), we have that $|\nabla a| \leq M$ and $|a(x,u_m)| \leq a_1$. Then using (68), (76) and (78), we obtain

$$\frac{1}{2}\int_0^T |u_m'|_{L^2(\Omega)} \leq c\int_0^T |\Delta u_m|^2 + \frac{1}{2}\int_0^T |\Delta u_m|^2\ \|u_m\| \leq c_1\int_0^T |\Delta u_m|^2 \tag{79}$$

So we conclude that

$$(u_m') \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). \tag{80}$$

The limit of the approximate solutions can be obtained following the same arguments similar to (32), (33), (37) and (40), i.e, we obtain the solution in $L^2(Q)$. $\square$

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