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# EXISTENCE FOR CALCULUS OF VARIATIONS AND OPTIMAL CONTROL PROBLEMS ON TIME SCALES

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Abstract. *In this paper we prove the existence for optimal control problems with terminal constraints on time scales. A definition of the solution of semi-linear control systems involving Sobolev space $W_{\mathbb{T}}^{1,2}$ is proposed and new existence and uniqueness results of this kind of dynamic systems on time scales are presented under a weaker assumption. According to $L_{\mathbb{T}}^2$ strong-weak lower semi-continuity of integral functionals, we establish the existence of optimal controls. In particular, the existence for calculus of variations on time scales is derived.*

**Keywords:** Time scales, Existence and uniqueness of solutions of control systems, Calculus of variations, Optimal control

1. **Introduction.** Continuous-time modelling and discrete-time modelling are two main approaches which dominate the methodology of mathematical modelling. System dynamics can be analyzed by using either approach. For example, both continuous-time recurrent neural networks and their discrete-time analogue have been studied in the literature (see, for example, [1] and the references therein). However, in practice, some processes consist of both continuous and discrete elements. A simple example of this kind of hybrid continuous-discrete time system is seasonally breeding population whose generations do not overlap. Temperate zone insects (including many economically important crop and orchid pests) are of this kind. These insects lay their eggs just before the generation dies out at the end of the season, with the eggs laying dormant, hatching at the start of the next season, and giving rise to a new, nonoverlapping generation. During each generation the population varies continuously (due to mortality, resource consumption, predation, interaction, etc.), while the population varies in a discrete fashion between the end of one generation and the beginning of the next [2,3].

On the other hand, sometimes certain real world phenomena cannot be described by continuous or discrete dynamics only. For example, the received income, the adjustment of asset holdings, and the consumption, etc. are with discrete features. Moreover, the consumption and the saving decisions can be modelled to occur with arbitrary, time-varying frequency. Therefore, there is a need to find a more flexible mathematical framework to accurately model the aforementioned hybrid dynamical systems [4].

Calculus on the so-called time scales initiated by S. Hilger [5] unifies the theory of differential equations and difference equations and it even allows dynamic systems to be both partially continuous and partially discrete. Hence, it provides a possible theory to investigate optimal control problems on arbitrary time scales in a unified way. To our best knowledge, results in the literature for time scales relevant to optimal control are restricted to calculus of variations problems. The pioneering work of calculus of variations on time scales was done by Bohner [6]. The theory has been developed in several different directions; see [7] for non-fixed boundary conditions, [8] for two independent variables, [9] for higher-order delta derivatives. These works seem to be of special interest to practical applications, in particular, in economics [4,10]. Problems studied in the above mentioned works typically are in the general form of minimizing Lagrange type cost functionals involving the so-called delta or nabla derivatives in $C_{rd}^k$ (see [6] for the definitions).

However, even though the existence of the variational problem on time scales is one of the most important issues for optimal controls, it seems that there are few works on it. The existence of the LQ problem on time scales was studied in [11]. In [12], the existence of unconstrained Lagrange optimal control governed by a class of the first-order linear dynamic systems on time scales has been shown. In this paper, we will consider the following problem,

$$\mathcal{L}[y(\cdot)] = \int_{[a,b)_{\mathbb{T}}} l(t, y(t), y^{\Delta}(t))\Delta t \to \inf, \quad y(a) = \xi, y(b) = \eta, \tag{1}$$

where $\xi$, $\eta \in \mathbb{R}$. A natural function space for this problem is the space of absolutely continuous functions $AC([a,b]_{\mathbb{T}}, \mathbb{R})$, or the space of functions of the bounded variation $BV([a,b]_{\mathbb{T}}, \mathbb{R})$. This is because of the fact that such functions are differentiable $\Delta$-almost everywhere and the derivatives are locally integrable. However, we may have difficulties to prove that a minimizing sequence has a weakly convergent subsequence. A more convenient function space is the Sobolev space $W_{\mathbb{T}}^{1,2}([a,b]_{\mathbb{T}}, \mathbb{R})$. Our problem is to find a solution $y(\cdot) \in W_{\mathbb{T}}^{1,2}([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfying the boundary conditions $y(a) = \xi$, $y(b) = \eta$.

First, we consider an optimal problem with a terminal state constraint on time scales.

Let $\mathbb{T}$ be a time scale with $\min \mathbb{T} = a > -\infty$ and $\max \mathbb{T} = b < \infty$. Consider a control system on $\mathbb{T}$

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = f(t, y(t)) + m(t)u(t), & \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}, \\ y(a) = y_0, \end{cases} \tag{2}$$

where $m(\cdot) \in L_{\mathbb{T}}^2([a,b)_{\mathbb{T}}, \mathbb{R})$, $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ is a known regressive rd-continuous function and $u(\cdot)$ is taken from

$$\mathcal{U}_{ad} := \{u(t), t \in [a,b)_{\mathbb{T}} : u \quad \text{is } \Delta\text{-measurable and} \quad u(t) \in \mathcal{U} \quad \Delta\text{-a.e.}\}, \tag{3}$$

where $\mathcal{U}$ is a convex and compact subset of $\mathbb{R}$. Any element in $\mathcal{U}_{ad}$ is called an admissible control.

Let $\mathcal{S} \subseteq \mathbb{R}$ be a final target set, and

$$y(b) \in \mathcal{S} \tag{4}$$

be the terminal constraint. Next, we introduce the following cost functional

$$J(u(\cdot)) = \int_{[a,b)_{\mathbb{T}}} l(t, y(t|u), u(t))\Delta t. \tag{5}$$

Under proper conditions (which will be assumed later), for any $y_0 \in \mathbb{R}, u(\cdot) \in \mathcal{U}_{ad}$, the control system admits a unique solution $y(\cdot;|u)$.

**Problem (P).** Find $u^*(\cdot) \in \mathcal{U}_{ad}$ and a corresponding state trajectory $y(\cdot|u^*) \in W_{\mathbb{T}}^{1,2}([a, b]_{\mathbb{T}}, \mathbb{R})$ satisfying the constraint (4), such that

$$J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u) = m.$$

2. **Preliminaries.** A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If there is the finite $\max \mathbb{T}$, then $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ and if there exists the finite $\min \mathbb{T}$, then $\rho(\min \mathbb{T}) = \min \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense (left-scattered, right-dense, right-scattered) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$) holds. If $\mathbb{T}$ has a left-scattered maximum value $M$, then we denote $\mathbb{T}^k := \mathbb{T} - \{M\}$. Otherwise, $\mathbb{T}^k := \mathbb{T}$. Throughout the paper, we denote $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. On $[a, b]_{\mathbb{T}}$, let $I_{[a,b]_{\mathbb{T}}} := \{i \in I : t_i \in [a, b]_{\mathbb{T}}\}$ be the index of all right-scattered points of the set $[a, b)_{\mathbb{T}}$. Let $\mu_\Delta$ be the Lebesgue $\Delta$-measure on time scales defined in terms of the Caratheodory extension (see [13, 14] for details). Here, we enumerate some crucial definitions which will be used in this paper. For each $t_0 \in \mathbb{T}\backslash\{\max \mathbb{T}\}$, the single-point set $\{t_0\}$ is $\Delta$-measurable, and its $\Delta$-measure is given by

$$\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0).$$

Let $P$ denote a proposition w.r.t. $t \in \mathbb{T}$ and $A$ a subset of $\mathbb{T}$. If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that $P$ holds on $A\backslash E_1$, then $P$ is said to hold $\Delta - a.e$ on $A$.

Let $f : [a, b)_{\mathbb{T}} \to \mathbb{R}$. Define $\widetilde{f} : [a, b] \to \mathbb{R}$ to be the extension of $f(\cdot)$ to real interval $[a, b]$ by

$$\widetilde{f}(t) = \begin{cases} f(t), & \text{if } t \in [a, b)_{\mathbb{T}}, \\ f(t_i), & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I_{[a,b)_{\mathbb{T}}}. \end{cases} \tag{6}$$

For any $f : [a, b)_{\mathbb{T}} \to \mathbb{R}$, if $\widetilde{f}(\cdot) \in L^1([a, b], \mathbb{R})$ then $f(\cdot)$ is said to be Lebesgue $\Delta-$integrable on $[a, b)_{\mathbb{T}}$, and define

$$\int_{[a,b)_{\mathbb{T}}} f(t)\Delta t := \int_{[a,b]} \widetilde{f}(t)dt. \tag{7}$$

The set of all Lebesgue $\Delta-$integrable function on $[a, b)_{\mathbb{T}}$ is denoted by $L^1_{\mathbb{T}}([a, b], \mathbb{R})$.

Furthermore, we say that $f$ belongs to $L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$ provided that

$$\int_{[a,b)_{\mathbb{T}}} |f(t)|^2 \Delta t < \infty.$$

The set $L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$ is a Banach space together with the norm defined as

$$\|f\|_{L^2_{\mathbb{T}}([a,b)_{\mathbb{T}},\mathbb{R})} = \left(\int_{[a,b)_{\mathbb{T}}} |f\,(t)\,|^2 \Delta t\right)^{1/2}.$$

A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is said to be right-dense continuous ($rd-$continuous, for short) on $\mathbb{T}$ if it is continuous at right-dense point sets and has finite left-sided limits at all left-dense points. We have a similar definition for the left-dense continuous function.

Let $C([a, b]_{\mathbb{T}}, \mathbb{R}))$ be the set of continuous functions $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ (namely, both $rd-$continuous and left-dense continuous). Define

$$\| f \|_0 = \sup_{t\in[a,b]_{\mathbb{T}}} | f(t) | .$$

Clearly, the above is a norm under which $C([a, b]_{\mathbb{T}}, \mathbb{R}))$ is Banach spaces.

A function $f : \mathbb{T} \to \mathbb{R}$ is $\Delta$ differentiable at $t \in \mathbb{T}^k$ if there exists a number $\alpha$ with the following property: for any $\varepsilon > 0$ there exists a neighborhood $U \subseteq \mathbb{T}$ of $t$ such that

$$|f(\sigma(t)) - f(s) - \alpha[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We denote such $\alpha$ by $f^{\Delta}(t)$.

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. For regressive functions $p : \mathbb{T} \to \mathbb{R}$, $\ominus p$ is defined as

$$\ominus p := -\frac{p}{1 + \mu p}.$$

Now, for a regressive and right-dense continuous function $p(\cdot)$ on the time scale $\mathbb{T}$, consider the following initial value problem

$$\begin{cases} y^{\Delta} = p(t)y, & t \in \mathbb{T}, \\ y(a) = 1. \end{cases}$$

One can show that the above admits a unique solution, denoted by $e_p(\cdot, a)$ which is given by

$$e_p(t, s) = \exp\left\{\int_{[s,t)_{\mathbb{T}}} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\}, \quad \xi_h(z) = \begin{cases} \frac{\ln(1+hz)}{h}, & \text{if} \quad h \neq 0, \\ z, & \text{if} \quad h = 0. \end{cases}$$

We call $e_p(\cdot, a)$ a generalized exponential function.

**Lemma 2.1.** [15] *(**Gronwall inequality**) Let $y(\cdot)$ be rd-continuous and $\beta, \gamma \in \mathbb{R}$ with $\gamma > 0$. Then*

$$y(t) \le \beta + \gamma \int_{[a,t)_{\mathbb{T}}} y(\tau)\Delta\tau, \quad \forall t \in \mathbb{T},$$

*implies*

$$y(t) \le \beta e_{\gamma}(t, a), \quad \forall t \in \mathbb{T}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $\{[a_k, b_k)_{\mathbb{T}}\}_{k=1}^n$, with $a_k, b_k \in \mathbb{T}$, is a finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

then

$$\sum_{k=1}^{n} | f(b_k) - f(a_k) | < \varepsilon.$$

We denote all absolutely continuous functions on $[a, b]_{\mathbb{T}}$ by $AC([a, b]_{\mathbb{T}}, \mathbb{R})$.

For an absolutely continuous function, the following integration by parts formula on time scales is true.

**Lemma 2.2.** [14] *If $f, g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ are absolutely continuous functions on $[a, b]_{\mathbb{T}}$, then $f \cdot g$ is absolutely continuous on $[a, b]_{\mathbb{T}}$ and the following equality is valid.*

$$\int_{[a,b)_{\mathbb{T}}} (f^{\Delta} g + f^{\sigma} g^{\Delta})(s)\Delta s = f(b)g(b) - f(a)g(a) = \int_{[a,b)_{\mathbb{T}}} (f g^{\Delta} + f^{\Delta} g^{\sigma})(s)\Delta s. \tag{8}$$

3. **Existence and Uniqueness of the Solution for a Control System.** In this section, we study the existence and uniqueness of solutions for the control system (2). Firstly, we consider the dynamic Cauchy problem

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = f(t, y(t)), & \Delta\text{-a.e. } t \in [a, b)_{\mathbb{T}}, \\ y(a) = y_0. \end{cases} \tag{9}$$

We assume that

**[HF:** ]

$(i)$ for every $r \in \mathbb{R}$, $f(\cdot, r)$ is $\Delta$-measurable in $[a, b)_{\mathbb{T}}$ and $f(t, \cdot) \in C(\mathbb{R}; \mathbb{R})$ for $t \in [a, b)_{\mathbb{T}}$,

$(ii)$ there is a constant $L_1 > 0$ such that

$$|f(t, y)| \le L_1(1 + |y|), \quad \forall \ t \in [a, b)_{\mathbb{T}}, y \in \mathbb{R},$$

$(iii)$ and there is a constant $L_2 > 0$ such that

$$|f(t, x) - f(t, y)| \le L_2|x - y|, \quad \forall \ t \in [a, b)_{\mathbb{T}}, x, y \in \mathbb{R}.$$

We begin with the following first-order Sobolev's space on $[a, b]_{\mathbb{T}}$ equipped with the Lebesgue $\Delta$-measure.

**Definition 3.1.** [14] *Let $u : [a, b]_{\mathbb{T}} \to \mathbb{R}$, then we say that $u(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ if and only if $u(\cdot) \in L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$ and there exists $g : [a, b)_{\mathbb{T}} \to \mathbb{R}$ such that $g(\cdot) \in L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$ and*

$$\int_{[a,b)_{\mathbb{T}}} (u \cdot \varphi^{\Delta})(s)\Delta s = -\int_{[a,b)_{\mathbb{T}}} (g \cdot \varphi^{\sigma})(s)\Delta s \quad \forall \varphi \in C^1_{0,rd}([a, b]_{\mathbb{T}}, \mathbb{R}) \tag{10}$$

*with*

$$C^1_{0,rd}([a, b]_{\mathbb{T}}, \mathbb{R}) := \{f : [a, b]_{\mathbb{T}} \to \mathbb{R} : f \in C^1_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}), f(a) = 0 = f(b)\}$$

*and $C^1_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ is the set of all continuous functions on $[a, b]_{\mathbb{T}}$ such that they are $\Delta$-differentiable on $[a, b)_{\mathbb{T}}$ and their $\Delta$-derivatives are $rd-$continuous on $[a, b)_{\mathbb{T}}$.*

**Lemma 3.1.** [14] *The set $W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ is a Banach space together with the norm defined for every $u(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ as*

$$\|u\|_{W^{1,2}_{\mathbb{T}}} := \|u\|_{L^2_{\mathbb{T}}} + \|u^{\Delta}\|_{L^2_{\mathbb{T}}}.$$

The following lemma asserts that $W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ is continuously immersed into $C([a, b]_{\mathbb{T}}, \mathbb{R})$ equipped with the supremum norm $\| \cdot \|_0$.

**Lemma 3.2.** [14] *There exists a constant $K > 0$, only depending on $b - a$, such that the inequality*

$$\|u\|_0 \le K \cdot \|u\|_{W^{1,2}_{\mathbb{T}}}$$

*holds for all $u(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ and therefore, the immersion $W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R}) \hookrightarrow C([a, b]_{\mathbb{T}}, \mathbb{R})$ is continuous.*

From now on, we consider the dynamic Cauchy problem (9).

**Definition 3.2.** *A solution of Cauchy problem* (9) *will be defined as a function $y(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ satisfying* (9) *for $\Delta$-a.e. $t \in [a, b)_{\mathbb{T}}$.*

We firstly show that the Problem (9) is equivalent to delta integral equation.

**Lemma 3.3.** *Let assumptions* $(i)$ *and* $(ii)$ *of* $[HF]$ *hold, then*
$(a)$ *if* $y(\cdot)$ *is a solution of Cauchy problem* (9),

$$y(t) = e_{\ominus p}(t,a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)f(\tau, y(\tau))\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}, \tag{11}$$

$(b)$ *and if* $y(\cdot) \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ *satisfies* (11), $y(\cdot)$ *is a solution of* (9).

**Proof:** $(a)$ If $y(\cdot)$ is a solution of Cauchy problem (9), then Lemma 3.2 implies $y(\cdot) \in C([a,b]_{\mathbb{T}}, \mathbb{R})$. Define the function $h : [a,b)_{\mathbb{T}} \to \mathbb{R}$ as

$$h(s) = f(s, y(s)), \qquad \text{for any } s \in [a,b)_{\mathbb{T}}.$$

Then, by conditions $[HF](i)$ and $(ii)$, we derive that $h(\cdot) \in L^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$. Hence, (11) is well defined. Now, we multiply both sides of Equation (9) by the integrating factor $e_p(\cdot, a)$ and obtain

$$\begin{aligned} [e_p(\cdot,a)y]^{\Delta} &= e_p(\cdot,a)y^{\Delta} + pe_p(\cdot,a)y^{\sigma} \\ &= e_p(\cdot,a)f(\cdot, y(\cdot)). \end{aligned}$$

Integrating both sides from $a$ to $t$ and using $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$, we obtain

$$y(t) = e_{\ominus p}(t,a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)f(\tau, y(\tau))\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}.$$

$(b)$ On the other hand, let $y(\cdot) \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ and (11) hold. By applying Lemma 2.10 in [12] and Lemma 2.6 in [16] to (11), we have $y(\cdot) \in AC([a,b]_{\mathbb{T}}, \mathbb{R})$ and

$$\begin{cases} y^{\Delta}(t) + p(t)y^{\sigma}(t) = f(t, y(t)), & \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}, \\ y(a) = y_0. \end{cases}$$

Denote $g(\cdot)$ in (10) by $g(t) = -p(t)y^{\sigma}(t) + f(t, y(t))$ on $[a,b)_{\mathbb{T}}$. Using assumptions (i) and (ii), we infer that $g(\cdot)$ is bounded and $g(\cdot) \in L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$. Furthermore, it follows from the integration by parts formula (Lemma 2.2) that

$$\int_{[a,b)_{\mathbb{T}}} (y \cdot \varphi^{\Delta})(s)\Delta s = -\int_{[a,b)_{\mathbb{T}}} (g \cdot \varphi^{\sigma})(s)\Delta s \quad \forall \varphi \in C^1_{0,rd}([a,b]_{\mathbb{T}}, \mathbb{R}) \tag{12}$$

Therefore, Definition 3.1 and Definition 3.2 imply that $y(\cdot) \in W^{1,2}_{\mathbb{T}}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $y(\cdot)$ is a solution of Cauchy problem (9).

Define

$$M_1 := \sup_{t \in [a,b]_{\mathbb{T}}} |e_{\ominus p}(t,a)| \text{ and } M_2 := \sup_{t,\tau \in [a,b]_{\mathbb{T}}} |e_{\ominus p}(t,\tau)|.$$

**Theorem 3.1.** ***(Local existence of solution)*** *Suppose that assumptions* $[HF](i)$ *and* $(ii)$ *hold. Then the dynamic Cauchy problem* (9) *has at least one solution in some interval* $a \le t \le a+h$, *where* $a < h \le \frac{K - M_1|y_0|}{M_2 L_1(1+K)}$ *for every* $K \ge (M_1+1)|y_0|$.

**Proof:** If $a$ is right scatted, it follows from Equation (9) that we obtain

$$y(\sigma(a)) = \frac{(y_0 + f(a,y_0))(\sigma(a)-a)}{(1+p(a)(\sigma(a)-a))}.$$

If $a$ is a right dense, we define the operator $F : C([a,a+h]_{\mathbb{T}}, \mathbb{R}) \to C([a,a+h]_{\mathbb{T}}, \mathbb{R})$ as

$$(Fy)(t) := e_{\ominus p}(t,a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)f(\tau, y(\tau))\Delta\tau, \quad t \in [a,a+h]_{\mathbb{T}}. \tag{13}$$

It follows from Lemma 3.3 that $y(\cdot)$ is a fixed point of the operator $F$ if and only if $y(\cdot)$ is a solution of Cauchy problem (9) on $[a, a+h]_{\mathbb{T}}$. Introduce a set

$$X := \overline{B}(0, K) := \{x \in C([a, a+h]_{\mathbb{T}}, \mathbb{R}), \| x \|_0 \leq K\}.$$

Then, it is easy to show that $F$ maps from $X$ into $F(X) \subseteq X$, $F(X)$ is uniformly bounded with $\| Fy \|_0 \leq K$ for all $y \in X$ and $F(X)$ is equicontinuous. Hence, Arzela-Ascoli theorem (Lemma 2.1 in [12]) implies that $F(X)$ is a precompact subset of $X$.

Moreover, $F$ is continuous. Otherwise, there exists a sequence $\{y_n(\cdot)\}$ convergent to $y(\cdot)$ in $X$, and $\{Fy_n(\cdot)\}$ is not uniformly convergent to $(Fy)(\cdot)$ in $[a, a+h]_{\mathbb{T}}$. Then we can find $\epsilon > 0$ and a sequence $\{t_n\} \subset [a, a+h]_{\mathbb{T}}$ such that

$$|(Fy_n)(t_n) - (Fy)(t_n)| \geq \epsilon.$$

On the other hand,

$$\begin{aligned} & |(Fy_n)(t_n) - (Fy)(t_n)| \\ = \ & |\int_{[a,a+h)_{\mathbb{T}}} \chi_{[a,t_n]_{\mathbb{T}}}(\tau) e_{\ominus p}(t_n, \tau)[f(\tau, y_n(\tau)) - f(\tau, y(\tau))]\Delta\tau| \\ \leq \ & M_2 \int_{[a,a+h)_{\mathbb{T}}} |f(\tau, y_n(\tau)) - f(\tau, y(\tau))|\Delta\tau, \end{aligned}$$

where $\chi_{[a,t_n]_{\mathbb{T}}}$ is the characteristic function of $[a, t_n]_{\mathbb{T}}$. By the continuity of $f$ w.r.t. to $y$, $[HF]$(ii), and Lebesgue dominated convergence theorem (Lemma 2.6 in [12]), we have

$$|(Fy_n)(t_n) - (Fy)(t_n)| \to 0 \text{ as } n \to \infty,$$

which contradicts the above inequality. Therefore, $F$ is continuous on $X$.

Now, we can use Schauder's fixed-point theorem (Theorem 2.0.14 in [17]) and derive that $F$ has at least one fixed point in $X$. Hence the Cauchy problem (9) also has at least one solution in $[a, a+h]_T$.

For the global existence of the solutions for (9), we need the following lemma.

For some fixed positive $\tilde{R} > 0$, we denote

$$\overline{B}(0, \tilde{R}) := \{y \in C([a, b]_{\mathbb{T}}, \mathbb{R}) : \| y \|_0 \leq \tilde{R}\}.$$

**Lemma 3.4.** ***(A priori estimate)*** *Assume that $[HF]$(i) and (ii) hold. If $y(\cdot)$ is a solution of the Cauchy problem* (9) *in $[a, b]_{\mathbb{T}}$, then $y(\cdot) \in \overline{B}(0, \tilde{R})$, with $\tilde{R} = [M_1|y_0| + M_2L_1(b-a)]e_{M_2L_1}(b, a)$.*

**Proof:** If $y(\cdot)$ is a solution of (9) in $[a, b]_{\mathbb{T}}$, then it follows from Lemma 3.3 that

$$y(t) = e_{\ominus p}(t, a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t, \tau) f(\tau, y(\tau))\Delta\tau, \quad t \in [a, b]_{\mathbb{T}}.$$

By condition $(ii)$, we have

$$\begin{aligned} |y(t)| \ & \leq \ |e_{\ominus p}(t, a)||y_0| + \int_{[a,t)_{\mathbb{T}}} |e_{\ominus p}(t, \tau)||f(\tau, y(\tau))|\Delta\tau \\ & \leq \ M_1|y_0| + \int_{[a,t)_{\mathbb{T}}} M_2L_1(1 + |y(\tau)|\Delta\tau. \end{aligned}$$

Now the Gronwall inequality (Lemma 2.1) implies

$$\| y \|_0 = \sup_{t \in [a,b]_{\mathbb{T}}} |y(t)| \leq [M_1|y_0| + M_2L_1(b-a)]e_{M_2L_1}(b, a) := \tilde{R}.$$

**Theorem 3.2.** ***(Global existence of solution)*** *Let the assumptions (i) and (ii) of $[HF]$ hold. Then the Cauchy problem* (9) *has at least one solution in $C([a, b]_{\mathbb{T}}, \mathbb{R})$.*

**Proof:** If $h$ is left-scattered, then $y(\rho(h))$ exists. By Equation (9), we have

$$y(h) = \frac{y(\rho(h) + (h - \rho(h))f(\rho(h), y(\rho(h)))}{1 + (h - \rho(h))p(\rho(h))}.$$

If $h$ is left-dense, then $y(h) = \lim_{t \to h^-} y(t)$ exists and $|y(h)| \leq \tilde{R}$. Using Theorem 3.1, we may solve the system Equation (9) with the initial condition $y(h) = y(h)$ and obtain the solution in $[h, \bar{h}]_{\mathbb{T}}$. Repeating the process, until $y(\bar{h}) = \infty$ or $\bar{h} = b$. But Lemma 3.4 ensures that $y(\bar{h}) \neq \infty$. Consequently, we obtain the global existence of solution for the Cauchy problem (9) in $C([a, b]_{\mathbb{T}}, \mathbb{R})$.

**Theorem 3.3.** ***(Uniqueness of the solutions)*** *Suppose that assumptions* $[HF]$ *hold. Then the Cauchy problem* (9) *has a unique solution* $y(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$*, which is equivalent to the following*

$$y(t) = e_{\ominus p}(t, a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t, \tau)f(\tau, y(\tau))\Delta\tau, \quad t \in [a, b]_{\mathbb{T}}.$$

**Proof:** By virtue of Theorem 3.2 and Lemma 3.3, let $y_1(\cdot)$ and $y_2(\cdot)$ be solutions of (9) in $[a, b]_{\mathbb{T}}$. Then,

$$\begin{aligned} |y_1(t) - y_2(t)| &= \left| \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t, \tau)[f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))]\Delta\tau \right| \\ &\leq M_2 L_2 \int_{[a,t)_{\mathbb{T}}} |y_1(\tau) - y_2(\tau)|\Delta\tau, \quad \forall t \in [a, b]_{\mathbb{T}}. \end{aligned}$$

It follows from the Gronwall inequality that

$$|y_1(t) - y_2(t)| = 0, \quad \forall t \in [a, b]_{\mathbb{T}}.$$

This completes the proof.

Next, we establish the existence and uniqueness of the solutions of the control system (2). For any fixed $u(\cdot) \in \mathcal{U}_{ad}$, we define

$$F_u(t) = f(t, y(t)) + m(t)u(t), \quad t \in [a, b)_{\mathbb{T}}.$$

Obviously, $\mathcal{U}_{ad}$ is a bounded subset of $L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$. If $m(\cdot) \in L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$, then the Hölder's inequality on time scales (Lemma 2.8 in [12]) implies $F_u(\cdot) \in L^1_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$. Therefore, by virtue of Theorem 3.3, we have the following corollary.

**Corollary 3.1.** ***(Existence and uniqueness of solutions for control system (2))*** *Suppose assumptions* $[HF]$ *hold. Furthermore,* $m(\cdot) \in L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$*. Then, for every fixed* $u(\cdot) \in \mathcal{U}_{ad}$*, the control system* (2) *has a unique solution* $y(\cdot|u) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$ *which satisfies*

$$y(t|u) = e_{\ominus p}(t, a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t, \tau)[f(\tau, y(\tau|u)) + m(\tau)u(\tau)]\Delta\tau, \quad t \in [a, b]_{\mathbb{T}}.$$

## 4. Main Results.

In this section, we study the existence of optimal solutions of Problem (P) and Lagrange Variational Problem (1).

To this end, we first introduce some notations and assumptions for Problem (P).

For the solution $y(\cdot|u)$ of the control system (2) corresponding to the control $u(\cdot) \in \mathcal{U}_{ad}$, define

$$\mathcal{Y}_{ad} := \{y(\cdot) \in W^{1,2}_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R}) : y(\cdot) = y(\cdot|u) \quad \text{for some} \ \ u(\cdot) \in \mathcal{U}_{ad} \quad \text{and} \ \ y(b) \in \mathcal{S}\},$$

and

$$\mathcal{A} := \{(y, u) \in \mathcal{Y}_{ad} \times \mathcal{U}_{ad} : y(\cdot) = y(\cdot|u)\}.$$

Assume that:

**[HL:** ]

($i$) the set $\mathcal{S}$ is a closed subset in $\mathbb{R}$, and

($ii$) $l = l(t, y, w)$ is lower-semicontinuity in $(y, w)$ for $\Delta$-a.e. $t$ in $[a, b)_{\mathbb{T}}$ and $l$ is $\Delta$-measurable on $[a, b)_{\mathbb{T}}$ for each $(y, w) \in \mathbb{R} \times \mathbb{R}$. Furthermore, $l$ is convex in $w$ for each fixed $t, y$ and there exists $h(\cdot) \in L^1_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$, $\lambda, \mu > 0$, such that

$$l(t, y, w) \geq h(t) + \lambda|y|^2 + \mu|w|^2, \quad \forall \ \ y, w \in \mathbb{R}. \tag{14}$$

Now, we introduce an operator $T : L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R}) \to L^2([a, b], \mathbb{R})$ given for every $f(\cdot) \in L^2_{\mathbb{T}}([a, b]_{\mathbb{T}}, \mathbb{R})$

$$Tf := \widetilde{f} \tag{15}$$

defined in (6). We also need the following lemmas.

**Lemma 4.1.** [12] $L^2_{\mathbb{T}}([a, b)_{\mathbb{T}}, \mathbb{R})$ *is a reflexive Banach space and* $T(\mathcal{U}_{ad})$ *is bounded and weakly closed in* $L^2([a, b], \mathbb{R})$.

**Lemma 4.2.** *(**Relative compact of the set of trajectories in** $C([a, b]_{\mathbb{T}}, \mathbb{R})$) Under the conditions of Corollary 3.1, the set of trajectories for the control system (2) is relatively compact in* $C([a, b]_{\mathbb{T}}, \mathbb{R})$.

**Proof:** For any $\{u_n(\cdot)\} \subset \mathcal{U}_{ad}$, it follows from Corollary 3.1 that

$$y(t|u_n) = e_{\ominus p}(t, a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t, \tau)[f(\tau, y(\tau|u_n)) + m(\tau)u_n(\tau)]\Delta\tau, \quad t \in [a, b]_{\mathbb{T}}.$$

By the boundedness of $\mathcal{U}_{ad}$, [HF](ii) and Hölder's inequality, we obtain

$$\begin{aligned} |y(t|u_n)| &\leq |e_{\ominus p}(t, a)||y_0| + \int_{[a,t)_{\mathbb{T}}} |e_{\ominus p}(t, \tau)||f(\tau, y(\tau|u_n)) + m(\tau)u_n(\tau)|\Delta\tau \\ &\leq M_1|y_0| + M_2 \int_{[a,t)_{\mathbb{T}}} L_1(1 + |y(\tau|u_n)|)\Delta\tau + M_2 \int_{[a,b)_{\mathbb{T}}} |m(\tau)u_n(\tau)|\Delta\tau \\ &\leq [M_1|y_0| + M_2\|m\|_{L^2_{\mathbb{T}}}\|u_n\|_{L^2_{\mathbb{T}}} + M_2L_1(b - a)] + M_2L_1 \int_{[a,t)_{\mathbb{T}}} |y(\tau|u_n)|\Delta\tau. \end{aligned}$$

Let $M$ be a bound for $\|u_n\|_{L^2_{\mathbb{T}}}$, then it follows from the Gronwall inequality that

$$\begin{aligned} \| \ y(\cdot|u_n) \ \|_0 &= \sup_{t\in[a,b]_{\mathbb{T}}} |y(t|u_n)| \\ &\leq [M_1|y_0| + MM_2\|m\|_{L^2_{\mathbb{T}}} + M_2L_1(b - a)]e_{M_2L_1}(b, a). \end{aligned}$$

That is, $\{y(\cdot|u_n)\}$ is uniformly bounded on $[a, b]_{\mathbb{T}}$. Taking arbitrary points $t_1$ and $t_2$ of the segment $[a, b]_{\mathbb{T}}$, we have

$$\begin{aligned} &|y(t_1|u_n) - y(t_2|u_n)| \\ \leq\ & M_1|y_0||e_{\ominus p}(t_2, t_1) - 1| + M_2 \int_{[t_1,t_2)_{\mathbb{T}}} |[f(\tau, y(\tau|u_n) + m(\tau)u_n(\tau)]|\Delta\tau \\ &+ \int_{[a,t_1)_{\mathbb{T}}} |e_{\ominus p}(t_2, \tau) - e_{\ominus p}(t_1, \tau)||[f(\tau, y(\tau|u_n) + m(\tau)u_n(\tau)]|\Delta\tau \end{aligned}$$

By using the absolute continuity of integration, the boundedness of $\mathcal{U}_{ad}$ and the Hölder's inequality, we obtain

$$\sup_n |y(t_1; u_n) - y(t_2; u_n)| \to 0 \text{ as } |t_1 - t_2| \to 0.$$

thus, $\{y(\cdot|u_n)\}$ is equicontinuous in $[a,b]_{\mathbb{T}}$. Therefore, by the Arzela-Ascoli theorem, there exists a continuous function $\bar{y}(\cdot) \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ such that

$$\|y(\cdot|u_n) - \bar{y}(\cdot)\|_0 \to 0.$$

**Lemma 4.3.** *If, under the conditions of Corollary 3.1, let $\{u_n(\cdot)\} \subset \mathcal{U}_{ad}$ converges weakly to $u^*(\cdot) \in \mathcal{U}_{ad}$ in $L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$ and $y(\cdot|u^*)$ is a solution of the control system corresponding to $u^*(\cdot)$, then $y(\cdot|u_n)$ converges strongly to $y(\cdot|u^*)$ in $L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$.*

**Proof:** By Corollary 3.1, it follows that

$$y(t|u_n) = e_{\ominus p}(t,a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)[f(\tau, y(\tau|u_n)) + m(\tau)u_n(\tau)]\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}.$$

For every fixed $t \in [a,b]_{\mathbb{T}}$, define

$$\varphi(\tau) = \begin{cases} e_{\ominus p}(t,\tau)m(\tau), & \tau \in [a,t)_{\mathbb{T}}, \\ 0, \quad \tau \in [t,b)_{\mathbb{T}}, \end{cases}$$

then we have $\varphi(\cdot) \in L^2_{\mathbb{T}}([a,t)_{\mathbb{T}}, \mathbb{R})$. Hence the weak convergence of $\{u_n(\cdot)\}$ implies that

$$\lim_{n\to\infty} e_{\ominus p}(t,a) \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)m(\tau)u_n(\tau)\Delta\tau = e_{\ominus p}(t,a) \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)m(\tau)u^*(\tau)\Delta\tau.$$

By Lemma 4.2, there exists a $\bar{y}(\cdot) \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ such that

$$\begin{aligned} \bar{y}(t) \quad &= \quad \lim_{n\to\infty} y(t|u_n) = e_{\ominus p}(t,a)y_0 + \lim_{n\to\infty} e_{\ominus p}(t,a) \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(a,\tau) f(\tau, y(\tau|u_n))\Delta\tau \\ &\quad + \lim_{n\to\infty} e_{\ominus p}(t,a) \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(a,\tau)m(\tau)u_n(\tau)\Delta\tau, \quad t \in [a,\sigma(b)]_T. \end{aligned}$$

Therefore, it follows from the Lebesgue dominated convergence theorem that

$$\bar{y}(t) = e_{\ominus p}(t,a)y_0 + \int_{[a,t)_{\mathbb{T}}} e_{\ominus p}(t,\tau)[f(\tau, \bar{y}(\tau)) + m(\tau)u^*(\tau)]\Delta\tau, \quad t \in [a,b]_{\mathbb{T}}.$$

That is, $\bar{y}(\cdot) = y(\cdot|u^*)$. Again, using the Lebesgue dominated convergence theorem and the boundedness of solutions (Lemma 4.2), we obtain

$$\|y(\cdot|u_n) - y(\cdot|u^*)\|^2_{L^2_{\mathbb{T}}} = \int_{[a,b)_{\mathbb{T}}} |y(t|u_n) - y(t|u^*)|^2 \Delta t \to 0, \text{ as } n \to \infty.$$

Next, consider a functional

$$J(y,u) = \int_{[a,b)_{\mathbb{T}}} l(t, y(t), u(t))\Delta t. \tag{16}$$

Let $\mathcal{D}$ denote the set of elements $(y,u)$ in $L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}) \times L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$ for which the integral in (16) exists and is finite. We present the following $L^2_{\mathbb{T}}$ strong-weak lower semicontinuity of the integral functional.

**Lemma 4.4.** *Let the assumption $[HL](ii)$ holds and $\{y_n, u_n\} \subset \mathcal{D}$. If $u_n \to u$ weakly in $L^2_{\mathbb{T}}$, $y_n \to y$ strongly in $L^2_{\mathbb{T}}$ and $\liminf_n J(y_n, u_n) > -\infty$, then*

$$J(y,u) \le \liminf_n J(y_n, u_n).$$

**Proof:** Throughout the proof of this lemma we will choose the subsequences of various sequences. Unless stated otherwise, we shall relabel the subsequence with the labeling of the original sequence.

Since $\{y_n(\cdot)\}$ and $\{u_n(\cdot)\}$ are strongly and weakly convergent respectively in $L^2_{\mathbb{T}}$, the sequences $\{y_n(\cdot)\}$ and $\{u_n(\cdot)\}$ are bounded in norm in the space $L^2_{\mathbb{T}}([a,b)_{\mathbb{T}},\mathbb{R})$. It follows from (14) that

$$J(y_n,u_n) \geq \int_{[a,b)_{\mathbb{T}}} h(t)\Delta t + \lambda \int_{[a,b)_{\mathbb{T}}} |y_n(t)|^2 \Delta t + \mu \int_{[a,b)_{\mathbb{T}}} |u_n(t)|^2 \Delta t > -\infty.$$

Let

$$\gamma = \liminf J(y_n,u_n) > -\infty.$$

Then, there exists a subsequence $\{(y_n,u_n)\}$, such that

$$\gamma = \lim J(y_n,u_n).$$

Since

$$u_n \to u \quad weakly \quad in \quad L^2_{\mathbb{T}}([a,b)_{\mathbb{T}},\mathbb{R}) \qquad as \quad n \to \infty.$$

By virtue of the Mazur's Theorem (Corollary A.7.5 in [18]), for every positive integer $i$ and integer $n(i)$, $m(i)$ increasing with $i$, and $n(i)$, $m(i) \to +\infty$ as $i \to +\infty$, one can construct a suitable convex combination of $\{u_n(\cdot)\}$, such that

$$\widehat{u}_i(\cdot) := \sum_{j=1}^{m(i)} a_{i,j} u_{n(i)+j}(\cdot) \to u(\cdot) \quad in \quad L^2_{\mathbb{T}}([a,b)_{\mathbb{T}},\mathbb{R}) \quad as \quad i \to \infty.$$

where,

$$a_{i,j} \geq 0, \sum_{j=1}^{m(i)} a_{i,j} = 1 \quad \text{for all positive integer} \quad i.$$

Furthermore, it follows from Lemma 4.2 in [12] that there exists a subsequence $\{\widehat{u}_i(\cdot)\}$, such that

$$\widehat{u}_i(t) \to u(t) \quad \Delta\text{-a.e.} \quad t \in [a,b)_{\mathbb{T}} \text{ as } i \to \infty. \tag{17}$$

Similarly, since

$$y_{n(i)+j} \to y \text{ strongly in } L^2_{\mathbb{T}} \text{ as } i \to \infty,$$

again by Lemma 4.2 in [12], we infer that there exists a subsequence $\{y_{n(i)+j}(\cdot)\}$, such that

$$y_{n(i)+j}(t) \to y(t) \quad \Delta\text{-a.e.} \quad t \in [a,b)_{\mathbb{T}} \text{ as } \quad i \to \infty. \tag{18}$$

Now, define

$$\widehat{l}_i(t) := \sum_{j=1}^{m(i)} a_{i,j} l(t, y_{n(i)+j}(t), u_{n(i)+j}(t)), \quad \Delta\text{-a.e.} \quad t \in [a,b)_{\mathbb{T}}.$$

and

$$l^*(t) := \liminf \widehat{l}_i(t), \quad \Delta\text{-a.e.} \quad t \in [a,b)_{\mathbb{T}}. \tag{19}$$

It follows from (14) that

$$\begin{aligned} \widehat{l}_i(t) &= \sum_{j=1}^{m(i)} a_{i,j} l(t, y_{n(i)+j}(t), u_{n(i)+j}(t)) \\ &\geq \sum_{j=1}^{m(i)} a_{i,j} h(t) = h(t), \quad \Delta\text{-a.e.} \quad t \in [a,b)_{\mathbb{T}}. \end{aligned}$$

Thus the function $l^*$ is well defined and $l^*(t) \geq h(t)$ for $\Delta$-a.e. $t \in [a,b)_{\mathbb{T}}$. And Fatou's lemma and (19) imply that

$$\begin{aligned}\int_{[a,b)_{\mathbb{T}}} l^*(t)\Delta t &\leq \liminf \int_{[a,b)_{\mathbb{T}}} \widehat{l}_i(t)\Delta t \\ &\leq \liminf \sum_{j=1}^{m(i)} a_{i,j} \int_{[a,b)_{\mathbb{T}}} l(t, y_{n(i)+j}(t), u_{n(i)+j}(t))\Delta t \\ &\leq \liminf \sum_{j=1}^{m(i)} a_{i,j} J(y_{n(i)+j}, u_{n(i)+j}) = \gamma.\end{aligned}$$

Therefore, $l^*(\cdot) \in L^1_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$.

Then, by virtue of (17), (18), (19) and l.s.c. of $l(t,y,w)$ w.r.t. $(y,w)$ for $\Delta$-a.e. $t$ and convexity w.r.t. $w$ for each fixed $t, y$, we obtain

$$\begin{aligned}l(t,y(t),u(t)) &= \lim_{i\to+\infty} l(t, y_{n(i)+j}(t), \widehat{u}_i(t)) \\ &= \lim_{i\to+\infty} l(t, y_{n(i)+j}(t), \sum_{j=1}^{m(i)} a_{i,j} u_{n(i)+j}(t)) \\ &\leq \lim_{i\to+\infty} \sum_{j=1}^{m(i)} a_{i,j} l(t, y_{n(i)+j}(t), u_{n(i)+j}(t)) \\ &= \lim_{i\to+\infty} \widehat{l}_i(t) = l^*(t), \quad \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}.\end{aligned}$$

Therefore, combining the two last inequalities and (14) we have

$$\int_{[a,b)_{\mathbb{T}}} h(t)\Delta t \leq \int_{[a,b)_{\mathbb{T}}} l(t,y(t),u(t))\Delta t \leq \int_{[a,b)_{\mathbb{T}}} l^*(t)\Delta t \leq \gamma.$$

Hence, $(y,u) \in \mathcal{D}$ and

$$J(y,u) \leq \liminf J(y_n, u_n).$$

Now we are in a position of prove the following principal theorems.

**Theorem 4.1.** *(**Existence for Optimal Control Problem (P)**). Assume assumptions $[HF]$ and $[HL]$ hold. If $\mathcal{Y}_{ad} \neq \emptyset$, then the optimal control Problem $(P)$ has an optimal solution.*

**Proof:** Let

$$m = \inf\{J(u(\cdot)) : (y(\cdot|u), u(\cdot)) \in \mathcal{A}\}.$$

If $m = +\infty$, we do not need to prove. Hence, let $m < \infty$. Since $\mathcal{U}$ is compact, it follows from (14) and Lemma 4.2 that

$$J(u(\cdot)) \geq \int_{[a,b)_{\mathbb{T}}} h(t)\Delta t + \lambda \int_{[a,b)_{\mathbb{T}}} |y(t|u)|^2 \Delta t + \mu \int_{[a,b)_{\mathbb{T}}} |u(t)|^2 \Delta t > -\infty$$

for any $(y(\cdot|u), u(\cdot)) \in \mathcal{A}$.

Since $\mathcal{Y}_{ad} \neq \emptyset$, there exists a minimizing sequence $\{(y(\cdot|u_n), u_n(\cdot))\} \subset \mathcal{A}$ such that

$$y(b|u_n) \in \mathcal{S} \text{ and } \lim_{n\to\infty} J(u_n(\cdot)) = m. \tag{20}$$

It is clear that $\{u_n(\cdot)\}$ is bounded in $L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$, and so $\{(Tu_n)(\cdot)\} \subset T(\mathcal{U}_{ad})$ is also bounded. Hence, by Lemma 4.1, there exists a subsequence, relabeled as $\{(Tu_n)(\cdot)\}$, such that $(Tu_n)(\cdot) \to z(\cdot)$ weakly in $L^2([a,b], \mathbb{R})$ and $z(\cdot) \in T(\mathcal{U}_{ad})$. Hence, there exists

$u^*(\cdot) \in \mathcal{U}_{ad}$ such that $z(\cdot) = (Tu^*)(\cdot)$. Furthermore, $\forall v(\cdot) \in L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R})$, we have $(Tv)(\cdot) \in L^2([a,b], \mathbb{R})$. It follows from

$$\int_{[a,b)_{\mathbb{T}}} v(t)(u_n(t) - u^*(t))\Delta t = \int_{[a,b]} (Tv)(t)(Tu_n - Tu^*)(t)dt \to 0 \quad as \quad n \to \infty$$

that

$$u_n \to u^* \text{ weakly in } L^2_{\mathbb{T}}([a,b)_{\mathbb{T}}, \mathbb{R}) \text{ as } n \to \infty. \tag{21}$$

By virtue of Corollary 3.1 and Lemma 4.3, we select the subsequence relabeled as $\{y(\cdot|u_n)\}$ such that

$$y(\cdot|u_n) \to y(\cdot|u^*) \text{ in } L^2_{\mathbb{T}}([a,b]_{\mathbb{T}}, \mathbb{R}), \text{ as } n \to +\infty. \tag{22}$$

Furthermore, since $\mathcal{S}$ is closed, (20) and Lemma 4.2 imply that

$$y(b|u_n) \to y(b|u^*) \in S, \text{ as } n \to +\infty. \tag{23}$$

Hence,

$$(y(\cdot|u^*), u^*(\cdot)) \in \mathcal{A}. \tag{24}$$

On the other hand, it follows from (20), (21), (22), (24) and Lemma 4.4 that

$$m \leq J(u^*) = J(y(\cdot|u^*), u^*(\cdot)) \leq \liminf J(y(\cdot|u_n), u_n) = \lim J(u_n) = m.$$

Therefore, the control $u^*(\cdot)$ is optimal. This completes the proof.

As an application of Theorem 4.1, we return to the problem of calculus of variations:

$$\mathcal{L}[y(\cdot)] = \int_{[a,b)_{\mathbb{T}}} l(t, y(t), y^{\Delta}(t))\Delta t \to \inf, \quad y(a) = \xi, y(b) = \eta, \tag{25}$$

where $\xi, \eta \in \mathbb{R}$.

Denote

$$\mathcal{N} := \{y \in W^{1,2}_{\mathbb{T}}([a,b]_{\mathbb{T}}, \mathbb{R}) : y(a) = \xi, y(b) = \eta\}.$$

Our problem is to find a solution $y(\cdot) \in \mathcal{N}$ of Problem (25). This problem can be reduced to a particular optimal control problem:

$$J(u(\cdot)) = \int_{[a,b)_{\mathbb{T}}} l(t, y(t|u), u(t))\Delta t \to \inf, \tag{26}$$

subject to

$$y^{\Delta}(t) = u(t), \quad \Delta\text{-a.e. } t \in [a,b)_{\mathbb{T}}, \quad y(a) = \xi, \tag{27}$$

$$y(b) = \eta. \tag{28}$$

**Theorem 4.2.** ***(Solutions for Calculus of Variations Problem (25))*** *Suppose assumptions $[HL]$ hold. Then Calculus of Variations Problem (25) admits a solution $y^*(\cdot) \in \mathcal{N}$.*

**Proof:** In this case, Problem (26), (27) and (28) satisfies all the assumptions in Theorem 4.1, hence, it admits an optimal pair $(y(\cdot|u^*), u^*(\cdot)) \in \mathcal{A}$ and $y^*(\cdot) := y(\cdot|u^*) \in \mathcal{N}$ is a solution of variations Problem (25).

**Example 4.1.** ***(Optimal inventory problem in discrete time)****. In an inventory problem, there is a product that can be acquired (either produced or purchased) at some specified cost per unit, and that is consumed based upon demands at specified times. There is also an inventory "holding" cost for storing products that are not consumed. We may formulate such an inventory problem as an $N$-stage sequential decision process, where at each stage $k$ a decision must be made to acquire $u(k)$ units at an acquisition cost $C(k, x)$, that may depend on the stage $k$ and on the number of units acquired $u(k)$. The state is $y(k)$, where $y(k)$ is the size of the inventory, i.e., how many units of the product are available at the start of the stage. The demand $D(k)$ generally depends on the stage.*

*If the decision in state $k$ is to acquire $u(k)$ units, the next-state $k+1$ is $y(k+1) = y(k) + u(k) - D(k)$. The inventory holding cost will be $l(k, y(k))$ for $y(k) > 0$.*

*Here, we consider the case that the restrictions on capacity (how many units may be acquired or produced in each stage) are fixed constant $M$, an inventory limit on terminate stage is $y_1$ and there is no penalty cost for $y(k) < 0$ in each stage. Then, the optimal inventory problem can be described as the following discrete optimal control problem.*

*To find the optimal control policy $\{u^*(k)\}$, $k = 0, 1, 2, \ldots, N$ such that*

$$\min J = \sum_{k=0}^{N} [C(k, u(k)) + l(k, y(k))], \tag{29}$$

*subject to*

$$y(k+1) = y(k) + u(k) - D(k), \quad y(0) = y_0, \tag{30}$$

*with control and terminal state constraints*

$$u(k) \in [0, M], \quad y(N+1) = y_1. \tag{31}$$

*Some reasonable assumptions for cost functionals are as follows: $C$, $l$ are continuous w.r.t state $y$ and control $u$, and $C$ is convex w.r.t $u$. Therefore, in this case, the optimal problem satisfies all the assumptions in Theorem 4.1, hence, it admits an optimal control policy $\{u^*(k)\}$, $k = 0, 1, 2, \ldots, N$.*

*If we let $c(k, u(k)) = 0.005u^2(k)$, $h(k, y(k+1)) = y(k)$ and the initial and terminal inventory be $y(0) = 0$, $y(N+1) = 0$. Together with $N = 4$, $M = 1000$, $s(0) = 600$, $s(1) = 700$, $s(2) = 500$, $s(3) = 1200$. Then, the problem can be solved using the discrete maximum principle:*

*The Hamiltonian*

$$H(k, y(k), u(k), \lambda(k+1)) = (0.005u^2(k) + y(k)) + \lambda(k+1)(y(k) + u(k) - s(k)).$$

*Optimal inventory model satisfies the following necessary conditions:*

$$H(k, y^*(k), u^*(k), \lambda(k+1)) = \min_{u \in [0,M]} H(k, y^*(k), u, \lambda(k+1)),$$

$$\lambda(k) = -\frac{\partial H(k, y^*(k), u^*(k), \lambda(k+1))}{\partial y(k)},$$

$$y(k+1) = \frac{\partial H(k, y^*(k), u^*(k), \lambda(k+1))}{\partial \lambda(k+1)},$$

*with initial and boundary conditions*

$$\lambda(N+1) = \mu, \quad y(0) = 0, \quad y(N+1) = 0.$$

*Therefore, the optimal inventory is*

$$u^* = [600, 700, 800, 900], \quad y^* = [0, 0, 0, 300, 0].$$

**Example 4.2.** *Consider a system in the form of*

$$\begin{cases} y^{\Delta}(t) = f(y(t)) + c(t)u(t), & \Delta\text{-}a.e. \ \ t \in [a, b)_{\mathbb{T}}, \\ y(a) = y_0, \quad y(b) = y_1, & \\ u(t) \in [-1, 1], & \end{cases} \tag{32}$$

*where $f(\cdot)$ is bounded, continuously differentiable on $\mathbb{R}$, and $c(\cdot)$ is bounded measurable on $[a, b)_{\mathbb{T}}$. Let $y_d \in \mathbb{R}$ be the desired value, the cost functional considered here is to minimize*

$$J(u) = \int_{[a,b)_{\mathbb{T}}} |y(t|u) - y_d|^2 \Delta t + \int_{[a,b)_{\mathbb{T}}} |u(t)|^2 \Delta t, \tag{33}$$

*subject to the dynamic system Equation* (29). *Then, given* $b > a$, *it is clear that* $l = |y - y_d|^2 + |u|^2$ *satisfies the assumption* $[HL](ii)$. *Hence, Corollary 3.1 and Theorem 4.1 imply that the problem has an optimal solution.*

5. **Conclusion.** This paper has studied the problem of the existence for optimal control problems with terminal state constraints on time scales. New conditions for the existence and uniqueness of the control system on time scales are presented based on a weaker assumption. Furthermore, the existence of optimal controls for the optimal control problems and Lagrange variational problems are analyzed as well. Finally, two examples are presented to illustrate our results obtained.

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