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Chaos synchronization between linearly coupled chaotic systems

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Abstract

This paper investigates the chaos synchronization between two linearly coupled chaotic systems. Some sufficient conditions of global asymptotic synchronization are attained from rigorously mathematical theory. Also, a new method for analyzing the stability of synchronization solution is presented. Using this method, some sufficient conditions of linear stability of the synchronization chaotic solution are gained. The influence of coupling coefficients on chaos synchronization is further studied for three typical chaotic systems: Lorenz system, Chen system, and newly found Lü system. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied in the last three decades [10,13]. It is found to be useful or has great potential in many disciplines such as in collapse prevention of power systems, biomedical engineering applications to the human brain and heart, thorough liquid mixing with low power consumption, secret communication technology, to name just a few [10,13,24].

Over the last decade, many new types of synchronization have appeared: chaotic synchronization [3,4], lag synchronization [9], adaptive synchronization [2], phase synchronization [6], and generalized synchronization [9], to mention only a few. Since the discovery of chaos synchronization [3], there has been tremendous interest in studying the synchronization of chaotic systems [10]. Recently, synchronization of coupled chaotic systems has received considerable attention [1,2,5,7]. Especially, a typical study of synchronization is the coupled identical chaotic systems [1,6].

In 1963, Lorenz found the first classical chaotic attractor [12]. In 1999, Chen found another similar but topologically not equivalent chaotic attractor [11,21,22], as the *dual* of the Lorenz system, in a sense defined by Vaněček and Čelikovský [23]: The Lorenz system satisfies the condition $a_{12}a_{21} > 0$ while Chen system satisfies $a_{12}a_{21} < 0$. Very recently, Lü et al. produced a new chaotic system [14,15], which satisfies the condition $a_{12}a_{21} = 0$, thereby bridging the gap between the Lorenz and Chen attractors [15–17].

Some numerical results were reported about the coupled identical Lorenz systems [6]. And two kinds of methods were applied for analyzing the stability of synchronization solution [6]. However, both methods concern the computation of Lyapunov exponents. In this paper, we further study the chaos synchronization between two linearly coupled chaotic systems from rigorously mathematical theory. Several sufficient conditions of global asymptotic synchronization are given. Furthermore, a new method is introduced for analyzing the stability of synchronization solution. That is, the in-phase solution decomposition method, which can transform the stability of synchronization solution into lower-

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dimensional problem. Using this method, a sufficient condition of linear stability of the synchronization solution is attained. The impact of coupling coefficients on chaos synchronization is investigated for the classical Lorenz system, Chen system, and newly found Lü system. Computer simulations are also given for illustration and verification.

2. Three kinds of representative chaotic systems

The Lorenz system is known to be a simplified model of several physical systems. At the origin, it was derived from a model of the earth's atmospheric convection flow heated from below and cooled from above [12]. Furthermore, it has been reported that Lorenz equations may describe such different systems as laser devices, disk dynamos and several problems related to convection [8]. Recently, the Lorenz attractor has just been mathematically confirmed to exist [20]. The Lorenz system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz, \end{cases} \quad (1)$$

which has a chaotic attractor as shown in Fig. 1(a) when $a = 10$, $b = 8/3$, $c = 28$.

Chen system is a typical chaos anti-control model, which has a more complicated topological structure than Lorenz attractor [11]. The Chen system can be obtained by merging together two simple attractors after performing a mirror operation [17]. Furthermore, it has been implemented by circuitry [25] and has widely applicable prospect in secret communication. The nonlinear differential equations that describe the Chen system are:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz, \end{cases} \quad (2)$$

which has a chaotic attractor as shown in Fig. 1(b) when $a = 35$, $b = 3$, $c = 28$.

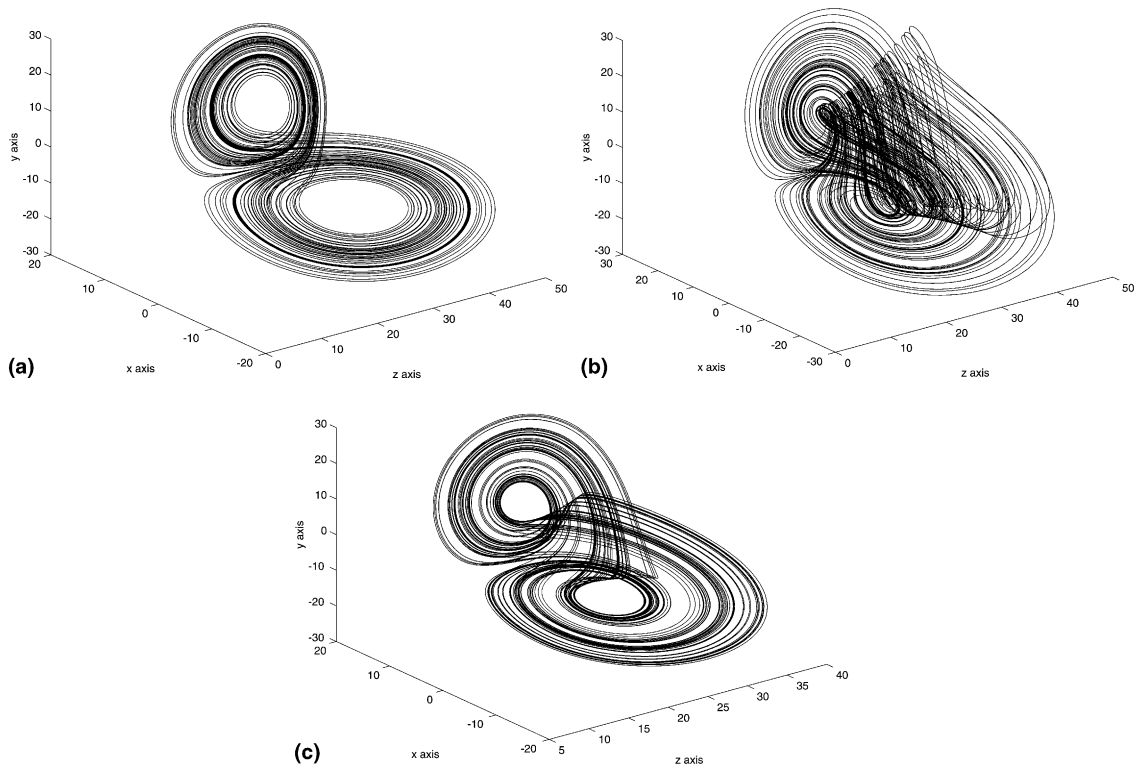


Fig. 1. (a) Lorenz chaotic attractor; (b) Chen chaotic attractor; (c) Lü chaotic attractor.

Lü system is a typical transition system, which connects the Lorenz and Chen attractors and represents the transition from one to the other [15,16,18,19]. The Lü system is described by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{3}$$

which has a chaotic attractor as shown in Fig. 1(c) when $a = 36, b = 3, c = 20$.

In the following, we will further study the chaos synchronization of two identical coupled chaotic systems based on three typical chaotic systems. Also, we will compare the influence of coupling coefficients on chaos synchronization for three systems.

3. Chaos synchronization between linear coupled systems

Synchronization of two identical Lorenz systems has been widely investigated under different conditions over the last decade [1]. However, most studies have only provided numerical or local dynamical analysis about the coupled Lorenz system. In the following, we will consider a general linear coupled Lorenz system and give some global asymptotic synchronization conditions for two linearly coupled Lorenz systems from theory.

Consider the following linear coupling of two identical Lorenz systems:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + d_1(y_1 - x_1), \\ \dot{x}_2 = cx_1 - x_1x_3 - x_2 + d_2(y_2 - x_2), \\ \dot{x}_3 = x_1x_2 - bx_3 + d_3(y_3 - x_3), \\ \dot{y}_1 = a(y_2 - y_1) + d_1(x_1 - y_1), \\ \dot{y}_2 = cy_1 - y_1y_3 - y_2 + d_2(x_2 - y_2), \\ \dot{y}_3 = y_1y_2 - by_3 + d_3(x_3 - y_3), \end{cases} \tag{4}$$

where x_j, y_j ($j = 1, 2, 3$) are status variables, provided that the orbit is close enough to the basin of attraction, and d_i ($i = 1, 2, 3$) are coupled coefficients. Especially, when $d_1 \neq 0, d_2 = d_3 = 0$, then the coupled system (4) is x -coupled. Similarly, $d_2 \neq 0, d_1 = d_3 = 0$ is y -coupled, and $d_3 \neq 0, d_1 = d_2 = 0$ is z -coupled.

In general, when d_i ($i = 1, 2, 3$) satisfy $F(d_1, d_2, d_3) < 0$, where $F(d_1, d_2, d_3) < 0$ is a condition of coupling coefficients, two oscillators will operate independently in their own different chaotic orbits separately, i.e., they are not synchronization. However, if there is a suitable coupling structure and a fittest coupling current intensity, for example, satisfying $F(d_1, d_2, d_3) > 0$, then the two identical oscillators will run in the same chaotic orbit simultaneously. That is to say, the two identical Lorenz systems with linear coupling will approach accurate synchronization,

$$x_i(t) = y_i(t) = s_i(t) \quad (i = 1, 2, 3). \tag{5}$$

In fact, for any coupling coefficients d_i ($i = 1, 2, 3$), the synchronization solution (5) is always the solution of coupled system (4), and locating the invariant subspace $\{(x, y) | \mathbf{x}(t) = \mathbf{y}(t)\}$. However, the synchronization solution (5) is unstable under the condition $F(d_1, d_2, d_3) < 0$. And this instability is caused by the maximum horizontal Lyapunov exponent becoming a positive number.

Define a difference system $\xi(t), \eta(t), \zeta(t)$ with

$$\begin{cases} \dot{\xi}(t) = x_1(t) - y_1(t), \\ \dot{\eta}(t) = x_2(t) - y_2(t), \\ \dot{\zeta}(t) = x_3(t) - y_3(t). \end{cases} \tag{6}$$

Then we have

$$\begin{cases} -x_1x_3 + y_1y_3 = -x_3(\xi + y_1) + y_1(x_3 - \zeta) = -x_3\xi - y_1\zeta, \\ x_1x_2 - y_1y_2 = x_2(\xi + y_1) - y_1(x_2 - \eta) = x_2\xi + y_1\eta. \end{cases} \tag{7}$$

According to Eqs. (4)–(7), we get the following system:

$$\begin{cases} \dot{\xi} = -(a + 2d_1)\xi + a\eta, \\ \dot{\eta} = (c - x_3)\xi - (1 + 2d_2)\eta - y_1\zeta, \\ \dot{\zeta} = x_2\xi + y_1\eta - (b + 2d_3)\zeta, \end{cases} \tag{8}$$

where x_2, x_3, y_1 are status variables of system (4).

The coefficient matrix of system (8) is

$$A(t) = \begin{pmatrix} -(a + 2d_1) & a & 0 \\ c - x_3 & -(1 + 2d_2) & -y_1 \\ x_2 & y_1 & -(b + 2d_3) \end{pmatrix}, \tag{9}$$

then we have the following symmetric matrix:

$$\frac{(A(t) + A^T(t))}{2} = \begin{pmatrix} -(a + 2d_1) & (c + a - x_3)/2 & x_2/2 \\ (c + a - x_3)/2 & -(1 + 2d_2) & 0 \\ x_2/2 & 0 & -(b + 2d_3) \end{pmatrix}. \tag{10}$$

Let $\alpha(t)$, $\beta(t)$ be the minimum and maximum eigenvalues of matrix $(A(t) + A^T(t))/2 = B(t)$. From the theorem in [26], we get the following lemma.

Lemma 1. Assume that the differential equation $\dot{X} = AX$ has a solution $X(t)$, thus we have

$$\|X(0)\| \exp \left\{ \int_0^t \alpha(s) ds \right\} \leq \|X(t)\| \leq \|X(0)\| \exp \left\{ \int_0^t \beta(s) ds \right\}. \tag{11}$$

In fact, it is easily attained by the following equation:

$$\frac{d\|X\|^2}{dt} = \frac{d(X^T X)}{dt} = X^T \left(\frac{A + A^T}{2} \right) X.$$

Therefore, if $\exists \varepsilon > 0$, such that $\beta(t) < -\varepsilon$, then for any initial value $X(0)$, we have $X(t) \rightarrow 0$ with exponential rate.

Note that the matrix $B(t)$ is a symmetric matrix, and all eigenvalues of $B(t)$ are real numbers. Let the eigenvalues be λ_i , $i = 1, 2, 3$, and satisfying $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

It is very interesting that, for two identical Lorenz systems, if the initial value $(x_1(0), x_2(0), x_3(0)) \neq (y_1(0), y_2(0), y_3(0))$, then the trajectories of two identical Lorenz systems will quickly separate each other and become irrelevant. However, if the coupling coefficients satisfy certain condition, then the two coupled Lorenz systems will approach global synchronization for any initial value.

In the following, assume that $d_i > 0$ ($i = 1, 2, 3$), then we have Theorem 1.

Theorem 1. If the coupled coefficients d_1, d_2, d_3 satisfy the condition:

$$r_0 = (a + 2d_1)(1 + 2d_2)(b + 2d_3) - \frac{b^2(a + c)^2}{16(b - 1)} M > 0,$$

$$s_0 = (a + 2d_1)(1 + 2d_2)(1 + a + 2d_1 + 2d_2) + (b + 2d_3)(1 + a + 2d_1 + 2d_2)(1 + a + b + 2(d_1 + d_2 + d_3)) - \frac{b^2(a + c)^2}{16(b - 1)}(a + 2d_1 + M) > 0,$$

then for any initial value $(x_1(0), x_2(0), x_3(0), y_1(0), y_2(0), y_3(0))$, the two coupled Lorenz systems will approach global synchronization as $t \rightarrow +\infty$, provided that the orbit is close enough to the basin of attraction, where $M = \max\{1 + 2d_2, b + 2d_3\}$.

Proof. The characteristic equation of matrix (10) is

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \tag{12}$$

where

$$p = 1 + a + b + 2(d_1 + d_2 + d_3), \tag{13}$$

$$q = (a + 2d_1)(1 + 2d_2) + (1 + a + 2d_1 + 2d_2)(b + 2d_3) - \frac{1}{4}x_2^2 - \frac{1}{4}(x_3 - c - a)^2, \tag{14}$$

$$r = (a + 2d_1)(1 + 2d_2)(b + 2d_3) - \frac{1}{4}(1 + 2d_2)x_2^2 - \frac{1}{4}(b + 2d_3)(x_3 - c - a)^2. \tag{15}$$

Therefore, we have

$$\begin{aligned}
 s &= pq - r \\
 &= (a + 2d_1)(1 + 2d_2)(1 + a + 2d_1 + 2d_2) + (b + 2d_3)(1 + a + 2d_1 + 2d_2)(1 + a + b + 2(d_1 + d_2 + d_3)) \\
 &\quad - \frac{1}{4}(a + b + 2d_1 + 2d_3)x_2^2 - \frac{1}{4}(1 + a + 2d_1 + 2d_2)(x_3 - c - a)^2.
 \end{aligned} \tag{16}$$

Furthermore, it has been shown that there is a bounded region $\Gamma \subset \mathbb{R}^3$ containing the whole attractor such that every orbit of (1) never leaves it. As shown by Leonov et al. [27] enclosure of the attractor is given:

$$\Gamma = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z - a - c)^2 = C \right\}, \tag{17}$$

where

$$C = \frac{b^2(a + c)^2}{4(b - 1)}.$$

Thus we can get the boundary of $y^2 + (z - a - c)^2$ about status variables y, z :

$$y^2 + (z - a - c)^2 \leq \frac{b^2(a + c)^2}{4(b - 1)}. \tag{18}$$

Since the orbit is close enough to the basin of attraction, we can get the following approximate estimate:

$$x_2^2 + (x_3 - a - c)^2 \leq \frac{b^2(a + c)^2}{4(b - 1)}, \tag{19}$$

then we have

$$\begin{aligned}
 r &= (a + 2d_1)(1 + 2d_2)(b + 2d_3) - \frac{1}{4}(1 + 2d_2)x_2^2 - \frac{1}{4}(b + 2d_3)(x_3 - c - a)^2, \\
 &\geq (a + 2d_1)(1 + 2d_2)(b + 2d_3) - \frac{M}{4} \frac{b^2(a + c)^2}{4(b - 1)} = r_0 > 0, \\
 s &\geq (a + 2d_1)(1 + 2d_2)(1 + a + 2d_1 + 2d_2) + (b + 2d_3)(1 + a + 2d_1 + 2d_2)(1 + a + b + 2(d_1 + d_2 + d_3)) \\
 &\quad - \frac{1}{4}(a + 2d_1 + M) \frac{b^2(a + c)^2}{4(b - 1)} = s_0 > 0,
 \end{aligned}$$

where $M = \max\{1 + 2d_2, b + 2d_3\}$.

Obviously, we have

$$p = 1 + a + b + 2(d_1 + d_2 + d_3) > 0,$$

$$q = \frac{pq}{p} = \frac{r + s}{p} > 0.$$

From the Routh–Hurwitz conditions, we know that all characteristic roots λ_i ($i = 1, 2, 3$) of Eq. (12) are negative, i.e. $\lambda_i < 0$ ($i = 1, 2, 3$).

Moreover, since $\lambda_1 + \lambda_2 + \lambda_3 = -p$ and $\lambda_1\lambda_2\lambda_3 = -r$, then $\exists \varepsilon > 0$, such that $\hat{\lambda}_i(t) < -\varepsilon$ ($i = 1, 2, 3$). According to Lemma 1, Theorem 2 holds. And the proof is completed. \square

Remarks.

1. Theorem 1 gives only sufficient conditions for global synchronization of linearly coupled Lorenz system; in other words if the coupling coefficients do not satisfy the above conditions, it does not mean that the coupled Lorenz system (4) cannot realize chaos synchronization. In fact, numerical simulation reveals that some coupling coefficients can make the coupled systems be synchronization but not satisfy the above sufficient conditions.
2. Similarly, we can attain the sufficient conditions for linearly coupled Chen system, Lü system.
3. We can get a better estimate than (19) for the boundary of $F(x_2, x_3) = x_2^2 + (x_3 - a - c)^2$. In fact, we can seek the maximum value of $F(x_2, x_3)$ under the condition of satisfying Eq. (4). We can use the polar coordinate to compute the accurate maximum value of $F(x_2, x_3)$, but it is very complicated, since the coupled system (4) is a six-dimensional

autonomous system. By example, we can easily calculate the maximum value of $F(y, z) = y^2 + (z - c)^2$ satisfying the Lorenz system (1).

When function $F(y, z)$ attains the maximum value, we have

$$\dot{F} = 2y\dot{y} + 2(z - c)\dot{z} = 2 \left[-y^2 - b \left(z - \frac{c}{2} \right)^2 + \frac{bc^2}{4} \right] = 0, \tag{20}$$

that is,

$$y^2 + b \left(z - \frac{c}{2} \right)^2 = \frac{bc^2}{4}. \tag{21}$$

Then we get the following parameter equations:

$$\begin{cases} x = x, \\ y = \frac{c\sqrt{b}}{2} \cos t, \\ z = \frac{c(1 + \sin t)}{2}. \end{cases} \tag{22}$$

Substituting Eq. (22) into Eq. (1), we have

$$\begin{cases} \dot{x} = a \left(\frac{c\sqrt{b}}{2} \cos t - x \right), \\ -\frac{c\sqrt{b}}{2} \sin t = cx - x \frac{c(1 + \sin t)}{2} - \frac{c\sqrt{b}}{2} \cos t, \\ \frac{c}{2} \cos t = x \frac{c\sqrt{b}}{2} \cos t - \frac{bc(1 + \sin t)}{2}. \end{cases} \tag{23}$$

By complicated algebraical operation, we get the solution of Eq. (23)

$$\sin t = -\frac{1}{b-1}, \quad \cos t = \frac{\sqrt{b^2 - 2b}}{b-1}, \quad x = \frac{1}{\sqrt{b}} + \sqrt{b-2}, \tag{24}$$

and it is easily verified that $\ddot{F}(t) < 0$, thus the maximum value of function $F(y, z)$ satisfying Eq. (1) is

$$F(y, z) = \frac{b^2c^2}{4(b-1)}. \tag{25}$$

In fact, this result concurs with the theoretical value of Zylinder [27].

4. The stability of synchronization solution

In this section, we will introduce a new method for analyzing the stability of synchronization solution.

Let the differential equation be

$$\dot{X} = F(X), \tag{26}$$

where $X \in \mathbb{R}^n$. Suppose that Eq. (26) has a solution $X = \phi(t)$.

Then we consider the following general linearly coupled system

$$\begin{cases} \dot{X} = F(X) - D(X - Y), \\ \dot{Y} = F(Y) - D(Y - X), \end{cases} \tag{27}$$

where $D = (d_{ij})_{n \times n}$ is the coupling matrix. Obviously, Eq. (27) has a solution $\Phi(t) = (\phi(t), \phi(t))^T$. In the following, we will investigate the stability of synchronization solution $\Phi(t)$.

For convenience, we denote the coupled system (27) by

$$\frac{dZ}{dt} = G(Z), \tag{28}$$

where $Z = (X, Y)^T$.

Consider the variational equation of coupled system (28) at synchronization solution $\Phi(t)$

$$\frac{dW}{dt} = DG(\Phi(t))W = \begin{pmatrix} A - D & D \\ D & A - D \end{pmatrix} W, \tag{29}$$

where

$$A = \left. \frac{\partial F(X)}{\partial X} \right|_{X=\phi(t)}.$$

Define a permutation operator $\omega: \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n$,

$$\omega \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} Y \\ X \end{pmatrix}. \tag{30}$$

Obviously, it satisfies:

$$\omega G(Z) = G(\omega Z), \tag{31}$$

and $\Phi(t)$ is a fixed point of it. According to Eq. (30), we know that the operator ω and derivative operator $\theta(t) = DG(\Phi(t))$ are exchangeable, that is,

$$\omega \theta = \theta \omega. \tag{32}$$

Let S be representational matrix of ω on $\mathbb{R}^n \times \mathbb{R}^n$ and then

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is n -order unit matrix.

S has two invariant subspaces:

$$S_1 = \left\langle \begin{pmatrix} x \\ x \end{pmatrix} \right\rangle, \quad \dim(S_1) = n,$$

and

$$S_2 = \left\langle \begin{pmatrix} x \\ -x \end{pmatrix} \right\rangle, \quad \dim(S_2) = n.$$

Since ω and θ are exchangeable, S_1 and S_2 are also invariant subspaces of θ , respectively. The action of θ on S_1 and S_2 , however, can be expressed as, separately:

- (a) $\theta(t)|_{S_1} \sim A(t)$,
- (b) $\theta(t)|_{S_2} \sim A(t) - 2D$.

In fact, let

$$P = \begin{pmatrix} I & I \\ -I & I \end{pmatrix},$$

then

$$P^{-1} = \frac{1}{2} \begin{pmatrix} I & -I \\ I & I \end{pmatrix},$$

where $I_{n \times n}$ is a unit matrix.

Substituting the linear transformation $W = PU$ into variable equation (29), we have

$$\begin{aligned} \frac{dU}{dt} &= P^{-1} \begin{pmatrix} A - D & D \\ D & A - D \end{pmatrix} PU, \\ &= \begin{pmatrix} A - 2D & 0 \\ 0 & A \end{pmatrix} U. \end{aligned} \tag{33}$$

Therefore, the variational equation (29) is decomposed into two independent equations:

$$\frac{dW_1}{dt} = A(t)W_1 \tag{34}$$

and

$$\frac{dW_2}{dt} = (A(t) - 2D)W_2. \tag{35}$$

This means that the space of the synchronization solution of coupled system (27) consists of that of Eq. (34) and that of Eq. (35).

It is noticed that for any coupled matrix D , the variational Eq. (29) can be decomposed into two independent Eqs. (34) and (35). Furthermore, when the coupled matrix D changes, it only influences Eq. (35), and Eq. (34) is a invariant. Numerical simulation reveals that the synchronization solution $\Phi(t)$ of coupled system (27) is stable for some coupled matrix D . Therefore, we will focus on the solution of Eq. (35), and seek some suitable conditions for coupled matrix D .

In fact, the invariant subspaces S_1 and S_2 are corresponding to Eqs. (34) and (35) separately. Furthermore, S_1 is corresponding to the movement of chaos synchronization solution $\phi(t)$, and all its Lyapunov exponents are corresponding to those of single chaos oscillator (26), and the maximum Lyapunov exponent must be a positive number. At the same time, all Lyapunov exponents of S_2 are the horizontal Lyapunov exponents of invariant subspace S_1 . Therefore, the synchronization solution is stable if and only if all horizontal Lyapunov exponents are negative. For convenience, we call S_1 the *synchronization subspace* and S_2 the *associate subspace*. In the following, we will give a sufficient condition of coupling matrix for the stability of synchronization solution from rigorously mathematical theory. It is noticed that the sufficient condition does not compute Lyapunov exponent, and it is only the function of coefficient matrix A .

From Lemma 1, we have

$$\|W_2(0)\| \exp \left\{ \int_0^t \alpha(s) ds \right\} \leq \|W_2(t)\| \leq \|W_2(0)\| \exp \left\{ \int_0^t \beta(s) ds \right\}, \tag{36}$$

where $\alpha(t)$ and $\beta(t)$ are the minimum and maximum characteristic roots of matrix $B(t) = (1/2)(A(t) + A^T(t)) - 2D$.

Let Eq. (26) be the Lorenz system and coupling matrix

$$D = d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

Then we have the symmetric matrix

$$B(t) = \begin{pmatrix} -a - 2d & \frac{a + c - z}{2} & \frac{y}{2} \\ \frac{a + c - z}{2} & -1 & 0 \\ \frac{y}{2} & 0 & -b \end{pmatrix}, \tag{38}$$

and its characteristic equation is

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \tag{39}$$

where

$$\begin{aligned} p &= 1 + a + b + 2d, \\ q &= b + (a + 2d)(b + 1) - \frac{1}{4}y^2 - \frac{1}{4}(z - c - a)^2, \\ r &= b(a + 2d) - \frac{1}{4}y^2 - \frac{b}{4}(z - c - a)^2, \end{aligned}$$

then

$$s = pq - r = b(1 + b) + (1 + b)(a + 2d)(1 + a + b + 2d) - \frac{1}{4}y^2(a + b + 2d) - \frac{1}{4}(z - c - a)^2(1 + a + 2d).$$

According to Eqs. (18), we have

$$r > b(a + 2d) - \frac{b^3(a + c)^2}{16(b - 1)} \tag{40}$$

and

$$s > b(1 + b) + (1 + b)(a + 2d)(1 + a + b + 2d) - \frac{b^2(a + c)^2}{16(b - 1)}(a + b + 2d). \tag{41}$$

We assume

$$a + 2d > \frac{b^2(a + c)^2}{16(b - 1)} = r_0. \tag{42}$$

According to Eq. (41), we consider the following equation:

$$f(x) = x^2 + x \left(1 + b - \frac{b^2(a + c)^2}{16(b^2 - 1)} \right) + b - \frac{b^3(a + c)^2}{16(b^2 - 1)} = 0. \tag{43}$$

Let r_1 and r_2 ($r_1 \leq r_2$) be the characteristic roots of Eq. (43). Since system (1) is chaos, then we have

$$r_1 r_2 = b \left(1 - \frac{b^2(a + c)^2}{16(b^2 - 1)} \right) < 0,$$

that is, $r_1 < 0 < r_2$.

Substituting r_0 into Eq. (43), we get

$$f(r_0) = \frac{b^5(a + c)^4}{256(b - 1)^2(b + 1)} + \frac{b^2(a + c)^2(1 + b + b^2)}{16(b^2 - 1)} + b > 0, \tag{44}$$

this means that $r_0 > r_2$ (see Fig. 2) and implies $s > 0$.

From Eq. (42), we have

$$d > \frac{b^2(a + c)^2}{32(b - 1)} - \frac{a}{2} = d_0. \tag{45}$$

When $d > d_0$, from Eq. (39), we get

$$p = 1 + a + b + 2d > 0,$$

$$q = \frac{pq}{p} = \frac{r + s}{p} > 0.$$

Therefore, according to Routh–Hurwitz conditions, if $d > d_0$, then all characteristic roots of Eq. (39) are negative. Moreover, since $\lambda_1 + \lambda_2 + \lambda_3 = -p$, $\lambda_1 \lambda_2 \lambda_3 = -r$, then $\exists \varepsilon > 0$, such that $\lambda_i < -\varepsilon$ ($i = 1, 2, 3$). Thus we have Theorem 2.

Theorem 2. *If $d > ((b^2(a + c)^2/32(b - 1)) - (a/2))$, then the synchronization solution $\Phi(t)$ of the coupled Lorenz system (27) is stable.*

In fact, the Poincarés mapping of Eq. (35) is compressible. Also, when $t \rightarrow +\infty$, the solution $W_2(t)$ of Eq. (35) approaches 0 with exponential rate.

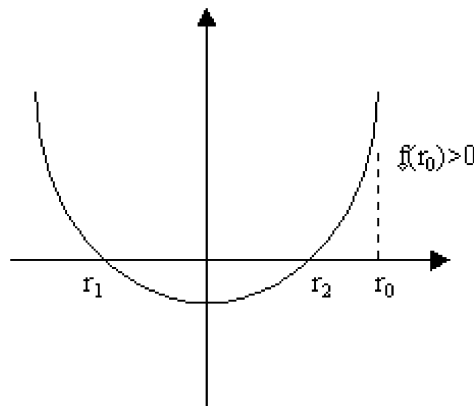


Fig. 2. The graph of curve $f(x)$.

Remarks.

1. Theorem 2 gives only sufficient condition for the stability of the synchronization solution $\Phi(t)$ of linear coupled Lorenz system; in other words if the coupling coefficients do not satisfy the above condition, it does not mean that the synchronization solution $\Phi(t)$ of coupled Lorenz system (27) is not stable.
2. This method is easily applied to some similar linear coupled systems and can be extended to multi-coupled chaotic systems. In fact, we have introduced a new method for analyzing the stability of synchronization solution. At first, we can decompose the coupled system into two lower-dimensional systems by using a linear transform, that is, the synchronization subsystem and associate subsystem. Then we can estimate the maximum characteristic root of the associate subsystem using an approximate formula. Finally, we can get the conditions of the stability for synchronization solution. We must point out that this in-phase solution decomposition method need not compute the Lyapunov exponents, and it is a rigorously mathematical method.

5. The influence of coupling coefficients on chaos synchronization

In this section, we will further investigate the impact of coupling coefficients on chaos synchronization for the coupled Lorenz system, coupled Chen system, and coupled Lü system, respectively.

Fig. 3 shows the maximum horizontal Lyapunov exponents of coupled Lorenz system: (a) x -coupled, that is $D = \text{diag}(1, 0, 0)$; (b) y -coupled, that is $D = \text{diag}(0, 1, 0)$; (c) z -coupled, that is $D = \text{diag}(0, 0, 1)$; (d) x, y, z -coupled, that is $D = \text{diag}(1, 1, 1)$.

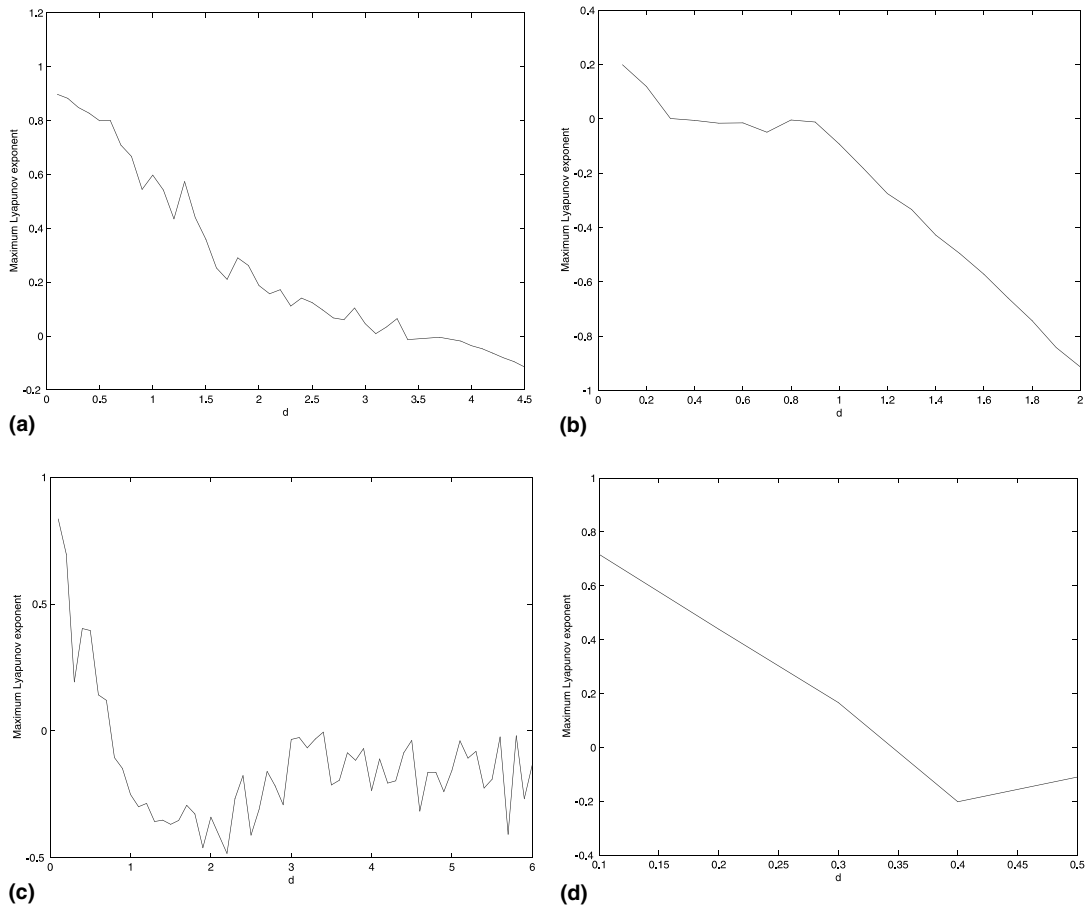


Fig. 3. The maximum horizontal Lyapunov exponents of coupled Lorenz system. (a) x -coupled; (b) y -coupled; (c) z -coupled, (d) x, y, z -coupled.

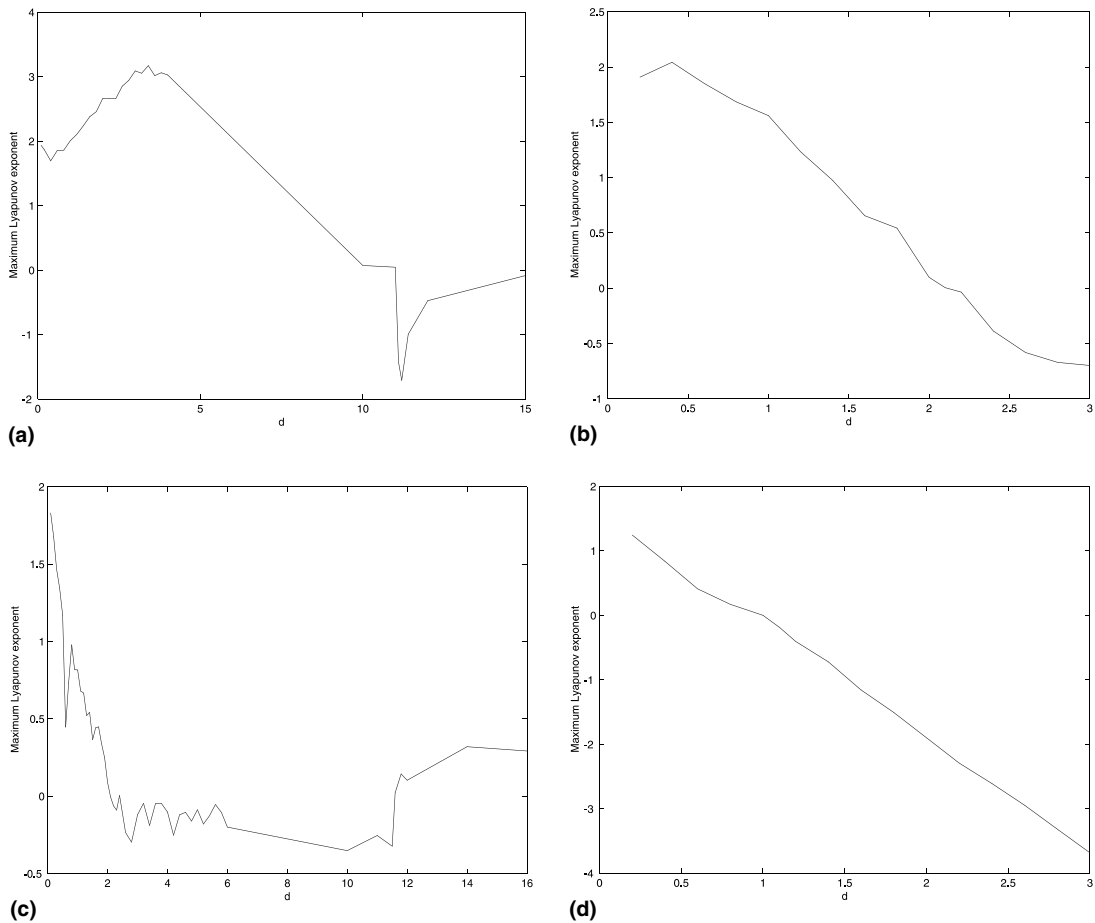


Fig. 4. The maximum horizontal Lyapunov exponents of coupled Chen system. (a) x -coupled; (b) y -coupled; (c) z -coupled, (d) x, y, z -coupled.

Fig. 4 displays the maximum horizontal Lyapunov exponents of coupled Chen system. (a) x -coupled; (b) y -coupled; (c) z -coupled, (d) x, y, z -coupled.

Fig. 5 shows the maximum horizontal Lyapunov exponents of coupled Lü system. (a) x -coupled; (b) y -coupled; (c) z -coupled, (d) x, y, z -coupled.

From Theorem 2, when $d > (b^2(a + c)^2/32(b - 1)) - (a/2)$, that is $d > 187.5333$, the synchronization solution of coupled Lorenz system (27) is stable from our theory. However, numerical simulation shows that when $d > 3.3$, the synchronization solution is stable. This means that theoretical result concurs with the numerical result, but the theoretical condition deserves improving in the future.

For comparison, we summarize the stability domain of coupling coefficients for synchronization solution in Table 1.

Remarks.

1. For x -coupled, the three systems have similar stability domains of coupling coefficients. That is, when coupling coefficient d oversteps some critical value, the synchronization solution becomes stable.
2. For y -coupled, the three systems have analogous stability domains of coupling coefficients. Furthermore, the system that has more complicated topological structure will need larger coupling coefficients to realize chaos synchronization. Since the Chen system has the most complicated topological structure, then Chen system needs largest coupling coefficients to reach chaos synchronization. It is noticed that the stability domain of coupled Lorenz system concerns the result in [6].
3. For z -coupled, the three systems have analogous type of stability domains of coupling coefficients. It is noticed that the stability domain of coupled Chen system is very narrow, but the stability domain of coupled Lü system is very broad.

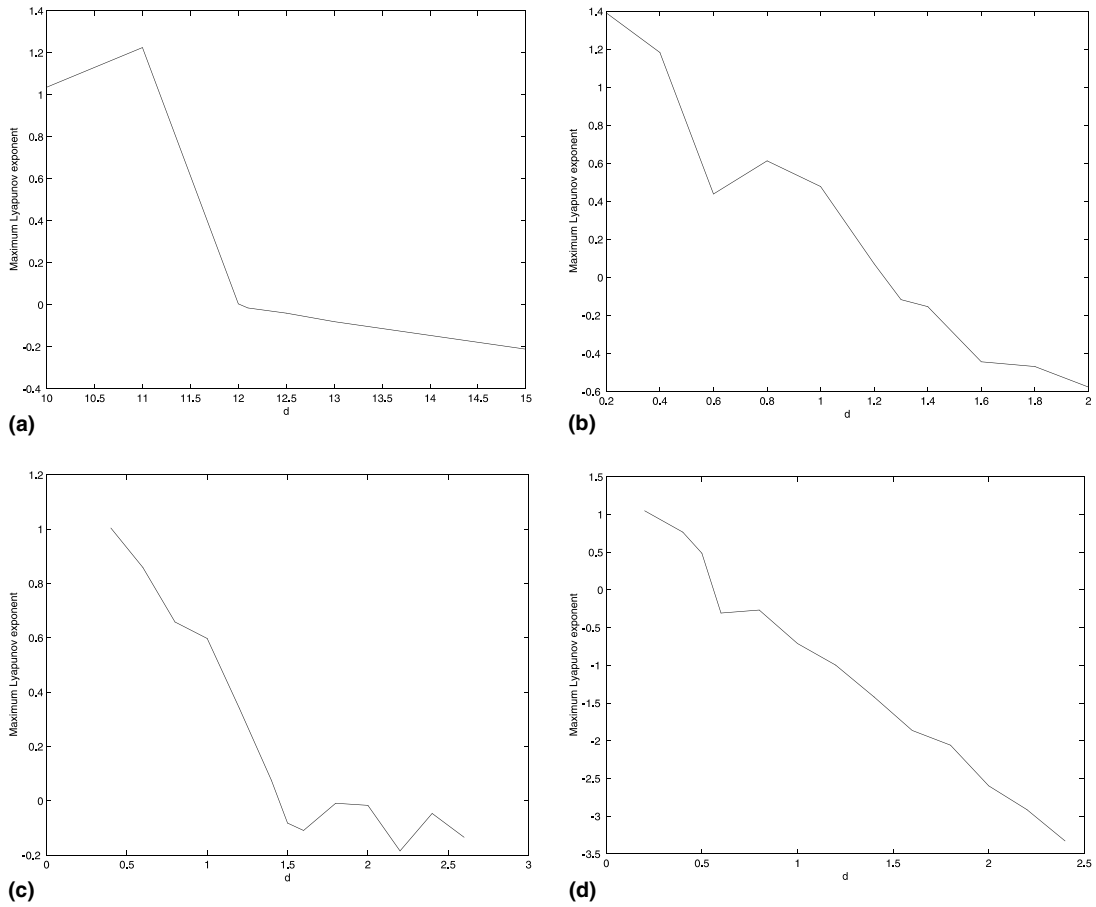


Fig. 5. The maximum horizontal Lyapunov exponents of coupled Lü system. (a) x -coupled; (b) y -coupled; (c) z -coupled, (d) x, y, z -coupled.

Table 1
Stability domain of coupling coefficients for synchronization solution

Coupled system	x -coupled	y -coupled	z -coupled	x, y, z -coupled
Lorenz system	[3.4, $+\infty$]	[0.35, $+\infty$]	[0.8, 999]	[0.4, $+\infty$]
Chen system	[11.1, $+\infty$]	[2.2, $+\infty$]	[2.1, 11.5]	[1.1, $+\infty$]
Lü system	[12.1, $+\infty$]	[1.3, $+\infty$]	[1.5, 10000]	[0.6, $+\infty$]

4. For x, y, z -coupled, the three systems have similar stability domains of coupling coefficients. Moreover, the system that has more complicated topological structure needs larger coupling coefficients to realize chaos synchronization.
5. According to Table 1, we can see that the different coupling structures, such as x -coupled and z -coupled, have different types of stability domains of coupling coefficients. Also, the more connection of coupling signal the system has, the easier the system realizes chaos synchronization. For x -coupled and x, y, z -coupled, when $d > 0.4$, the x, y, z -coupled Lorenz system can reach chaos synchronization; however, when $d > 3.4$, the x -coupled Lorenz system can reach chaos synchronization.

6. Conclusion

In this paper, we study the chaos synchronization between two linearly coupled systems. We attain some sufficient conditions for global synchronization using rigorously mathematical theory. Also, we introduce a new method for

analyzing the stability of synchronization solution of coupled system. By using this method, we investigate the stability of synchronization solution for the classical Lorenz system. For comparison, the impact of coupling coefficients on chaos synchronization is further explored for Lorenz system, Chen system, and newly found Lü system. We notice that the global synchronization and stability are rather complex, and they deserve further study in the near future.

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