On the Asymptotic Continuum Analysis of Quasistatic Elastic-Plastic Crack Growth and Related Problems

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The present work provides rigorous substantiation of certain crucial asymptotic expressions which constitute the basis of a number of previous near-tip analyses of quasistatic elastic-plastic crack growth. This is accomplished as part of an investigation of the general features of two-dimensional near-tip continuum fields for quasistatic elastic-plastic crack growth, under general unsteady conditions, for a broad class of constitutive behavior and crack loading conditions. The approach employed and results obtained are also applicable to a number of geometrically similar problems, such as the plane strain analysis of the continuum fields near the leading and trailing edges of a quasistatically moving distributed surface load.

1 Introduction

The complete elastic-plastic analysis of the stress and deformation fields in a body containing a quasistatically growing crack is of sufficient mathematical complexity to prohibit (thus far) closed-form solutions. To make these problems analytically tractable, attention is focused on the region very close to the moving crack tip, where the general continuum mechanical governing equations can be approximated by simplified asymptotic forms. Another reason for focusing on the near-tip fields is that they would be expected, on physical grounds, to play a crucial role in the behavior of a cracked solid.

In view of the complexity of the general (nonlinear) elastic-plastic continuum governing equations, it is important to ensure that their reduced forms, which are employed in asymptotic analyses, are correct. A principal goal of the present paper is to attempt to provide a rigorous basis for the asymptotic forms employed in the growing crack analyses of Rice et al. [1], Rice [2], Drugan et al. [3], and other researchers (whose work is reviewed in [2, 3]). In addition, I will sharpen the asymptotic expressions used in these studies, as well as demonstrate their validity for a wider class of problems than those thus far analyzed.

The results presented herein have already facilitated a higher-order analysis (Drugan, [4]) of the plane-strain crack growth problem studied by Drugan et al. [3], this higher-order analysis differing from typical perturbation analyses in that the higher-order structure of the continuum fields is derived rather than assumed. Furthermore, I anticipate that the present results may be suggestive, if not directly applicable, in the analysis of a number of yet unsolved problems (e.g., crack growth for more general constitutive models).

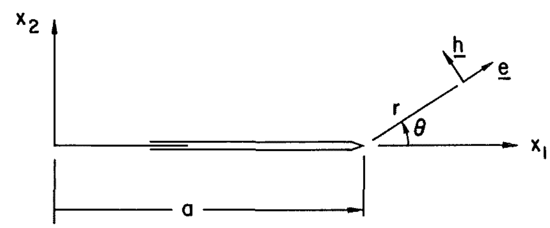

Fig. 1 Cartesian coordinates x_1, x_2, x_3 are fixed in the body; polar coordinates r, θ are centered at the tip and move with it through the material as the crack grows

The geometry of the crack problems considered is as depicted in Fig. 1: a Cartesian coordinate system x_1, x_2, x_3 is fixed in the body, with x_1 pointing in the direction of crack growth, a being the measure of crack length, and x_3 lying parallel to the crack front (which is presumed to be straight). A polar coordinate system r, θ lies in the x_1, x_2 plane, is centered at the crack tip, and moves with it through the material as the crack grows; θ is measured from the line ahead of the crack. The unit vectors **e** and **h** correspond to the radial and angular directions, respectively, of this translating polar coordinate system. Therefore

$$\partial r/\partial x_i = e_i, \quad \partial \theta/\partial x_i = h_i/r \tag{1.1}$$

where

$$e_1 = h_2 = \cos\theta, \quad e_2 = -h_1 = \sin\theta, \quad e_3 = h_3 = 0. \tag{1.2}$$

Here and throughout the paper, Latin indices i, j, k, and l have range 1, 2, 3, obey the Einstein summation convention, and indicate Cartesian components of tensors.

The two order symbols $0[w(r)]$ and $o[w(r)]$ will be used, where here $w(r)$ is an arbitrary gauge function chosen for purposes of illustration. A function $d(r)$ can be characterized as

Contributed by the Applied Mechanics Division for publication in the Journal of Applied Mechanics.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by ASME Applied Mechanics Division, February, 1984; final revision, September, 1984.

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$$d(r) = 0[w(r)] \quad \text{as } r \to 0$$

if

$$\lim_{r \to 0} \left| \frac{d(r)}{w(r)} \right| < \infty;$$

$d(r)$ can be characterized as

$$d(r) = \text{o}[w(r)] \quad \text{as } r \to 0$$

if

$$\lim_{r \to 0} \left| \frac{d(r)}{w(r)} \right| = 0.$$

A clear discussion of the use of order symbols and gauge functions is given by Van Dyke [5].

2 Assumptions

2.1 Geometry. The analysis to follow will treat two-dimensional crack problems. In particular, the crack front is assumed to be straight and, with reference to Fig. 1, the continuum fields of stress, strain, body force, etc., are assumed independent of x_3. Thus, the stress field near an advancing crack tip will be regarded as

$$\sigma_{ij} = \sigma_{ij}(r,\theta,t). \tag{2.1}$$

In addition, I will neglect the effects of geometry changes due to ongoing deformation on the formulation of stresses and stress rates, and on the form of the equilibrium equations. This is equivalent to the usual "small strain" (i.e., small displacement gradient) assumption.

2.2 Material. The principal constitutive restrictions on the class of solids considered are that *deviatoric* components of stress are assumed to be bounded, that the elastic part of total strain is linearly related (although with arbitrary anisotropy) to the stress state, that stress is derivable from a positive-definite elastic strain energy function, and that inelastic deformation proceeds in accordance with the principle of maximum plastic resistance (discussed in the following). The latter three constitutive assumptions, together with the requirements of equilibrium and a continuous displacement field, are sufficient to prove that the stress field near a quasistatically growing crack tip must be *fully continuous* (Drugan and Rice, [6]). This property of the stress field renders more plausible the assumption, employed throughout the present paper, that $\sigma_{ij}(r,\theta,t)$ are differentiable with respect to each independent variable for small $r > 0$. Since the study of Drugan and Rice [6] assumes continuity of all displacement components, their proof of full stress continuity does not apply for the special case of plane stress, where the thickness direction displacement component need not be continuous (Hill, [7]). Thus the derivations to follow are valid for the cases of plane strain, antiplane strain, and general combinations of these two. The special case of plane stress is discussed separately in Section 6.

The principle of maximum plastic resistance may be stated as

$$(\sigma_{ij} - \sigma_{ij}^0)d\epsilon_{ij}^p \geq 0 \tag{2.2}$$

where σ_{ij} is the stress state (at yield) corresponding to the plastic strain increment $d\epsilon_{ij}^p$, and σ_{ij}^0 is any other stress state that is at or below yield. As discussed by Rice [8, 9], this principle, especially in its small strain form as employed here, results from a number of different viewpoints and is thus believed to be a sensible characterizer of metal plasticity.

Note that the constitutive assumptions delineated in the foregoing, and hence the resulting requirement of full stress continuity near a quasistatically advancing crack tip, restrict but do not exclude inelastic behavior characterized by strain hardening, as well as inelastic behavior that is time-dependent. The latter is true since, as shown by Rice [8], the principle of maximum plastic resistance is also a sensible restriction for time-*dependent* inelastic deformation in metals if the concept of the yield surface is replaced by that of the inelastic flow potential surface. This means that inequality (2.2) applies if σ_{ij}^0 is understood to represent a stress state lying within or on the current flow potential surface. Drugan and Rice [6] point out that their proof of stress continuity is valid for isotropically hardening solids, and also for a large class of anisotropically hardening solids defined by the requirement that the current yield locus at any step of the deformation incorporate all prior yield loci. I emphasize that time-dependent and strain-hardening material behaviors are included in the present treatment provided that the material model employed disallows unbounded *deviatoric* stresses. Since many established models of time-dependent or strain-hardening behavior permit unbounded derivatoric stresses, the primary application of the results derived here is to solids which are elastic-ideally plastic, but in the generalized sense that some rate sensitivity or strain hardening is permitted before deviatoric stresses saturate to finite levels.

2.3 Equilibrium. This study treats problems of the quasistatic type, meaning that the inertia terms in the equations of motion are negligible. Such processes are governed by the equilibrium

$$\partial\sigma_{ij}/\partial x_j + f_i = 0 \tag{2.3}$$

where $\sigma_{ij} = \sigma_{ji}$ are components of the stress tensor, and f_i are components of the body force vector which are also assumed to be bounded. These equations may be rephrased in terms of the translating polar coordinate system by regarding $\sigma_{ij} = \sigma_{ij}(r,\theta,t)$ in light of the assumptions of Section 2.1, and by using (1.1):

$$(\partial\sigma_{ij}/\partial\theta)(h_j/r) + (\partial\sigma_{ij}/\partial r)e_j + f_i = 0. \tag{2.4}$$

In terms of polar components of stress, these equations are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \tag{2.5a}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0 \tag{2.5b}$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\sigma_{rz}}{r} + f_z = 0 \tag{2.5c}$$

where $z \equiv x_3$.

3 Asymptotic Structure of the Stress Field

Since a principal goal of the study is rigorous derivations of asymptotic forms of general continuum expressions and governing equations, and since these involve first partial derivatives of stress with respect to the independent variables in (2.1), it is necessary to establish the asymptotic order of such first partial derivatives. This will be accomplished by employing the assumptions of Section 2.

3.1 Radial Derivative of Stress Components. To establish the asymptotic order of the terms $\partial\sigma_{ij}/\partial r$, we begin by considering the equilibrium equations (2.5) when multiplied by r. In this form they suggest the plausibility of the assumption that

$$r\partial\sigma_{ij}/\partial r \quad \textit{exist} \text{ in the limit as } r \to 0, \tag{3.1}$$

in light of the boundedness assumptions on f_i and deviatoric stress components. On the basis of (3.1) we may write

$$\lim_{r \to 0}\left[r\partial\sigma_{ij}/\partial r\right] = A_{ij}(\theta,t) \tag{3.2}$$

where $A_{ij}(\theta,t)$ are as yet unknown. It follows that

$$\partial\sigma_{ij}/\partial r = A_{ij}(\theta,t)/r + \text{o}(1/r) \quad \text{as} \quad r \to 0. \tag{3.3}$$

Integrate this with respect to r for r small to obtain

$$\sigma_{ij} = -A_{ij}(\theta,t)ln(R/r) + \text{o}[ln(R/r)] \quad \text{as } r \to 0, \tag{3.4}$$

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where R is an undetermined constant having length dimensions. Divide this equation by $[ln(R/r)]$:

$$\sigma_{ij}/[ln(R/r)] = -A_{ij}(\theta,t) + o(1) \quad \text{as } r \to 0. \tag{3.5}$$

In terms of deviatoric stress components $s_{ij}(\equiv \sigma_{ij} - 1/3\ \delta_{ij}\sigma_{kk})$, (3.5) is

$$s_{ij}/[ln(R/r)] = -A_{ij} + \frac{1}{3}\delta_{ij}A_{kk} + o(1) \quad \text{as} \quad r \to 0. \tag{3.6}$$

Now, take the limit of (3.6) as $r \to 0$, invoking the boundedness of s_{ij} to obtain

$$A_{ij}(\theta,t) - \frac{1}{3}\delta_{ij}A_{kk}(\theta,t) \equiv 0. \tag{3.7}$$

This shows, via (3.2), that

$$\lim_{r \to 0} \left[r\partial s_{ij}/\partial r \right] = 0, \tag{3.8}$$

which now may be employed to elucidate the asymptotic structure of the hydrostatic stress component $\sigma_{kk}/3$.

Asymptotic Boundedness of Hydrostatic Stress Component. To prove that hydrostatic stress is bounded as $r \to 0$, we enforce circumferential equilibrium, equation (2.5*b*), in the rearranged form

$$\partial\sigma_{\theta\theta}/\partial\theta = -2\sigma_{r\theta} - r\partial\sigma_{r\theta}/\partial r - rf_\theta. \tag{3.9}$$

Integrating (3.9) with respect to θ at fixed $r \neq 0$ and t gives

$$\sigma_{\theta\theta} = -2\int_{-\pi}^{\theta} \sigma_{r\theta}d\phi - \int_{-\pi}^{\theta} (r\partial\sigma_{r\theta}/\partial r)d\phi - \int_{-\pi}^{\theta} rf_\theta d\phi, \tag{3.10}$$

assuming traction-free crack faces; if the crack faces were subject to a traction of finite magnitude, (3.10) would contain an additional bounded term which would not affect the proof. Now making use of the bounded body force assumption and the result (3.8), we examine (3.10) as $r \to 0$ to find

$$\sigma_{\theta\theta} = -2\int_{-\pi}^{\theta} \sigma_{r\theta}d\phi + o(1) \quad \text{as} \quad r \to 0, \tag{3.11}$$

which shows that $\sigma_{\theta\theta}$ is bounded as $r \to 0$ since $\sigma_{r\theta}$ is by assumption (Section 2). Thus, coupling (3.11) with the assumption that all *deviatoric* stress components are bounded proves that *the hydrostatic component of stress is bounded in the limit of* $r \to 0$, and therefore that *all* components of stress are bounded in that limit. Applying this result to (3.5) in the limit as $r \to 0$ now shows

$$A_{ij}(\theta,t) \equiv 0 \tag{3.12}$$

so that from (3.2) we obtain the important conclusion

$$\partial\sigma_{ij}/\partial r = o(1/r) \quad \text{as } r \to 0. \tag{3.13}$$

Since many previous studies of near-tip fields depend on the veracity of (3.13), I next provide an alternate proof to the one in the foregoing. Here, instead of (3.1), I assume that *for sufficiently small* $r_o > 0$:

(*i*) $\partial\sigma_{ij}/\partial r$ are defined on the *open* interval $0 < r < r_o$ for each fixed θ, t, and

(*ii*) $|\partial\sigma_{ij}/\partial r|$ are monotonic in r on $0 < r < r_o$.

First, if $|\partial\sigma_{ij}/\partial r|$ are nonincreasing as $r \to 0$, then obviously $r|\partial\sigma_{ij}/\partial r| \to 0$ as $r \to 0$, so we need only consider the case of *increasing* $|\partial\sigma_{ij}/\partial r|$ as $r \to 0$. The stress continuity proof of Drugan and Rice [6] when applied to the radial behavior of σ_{ij} holds for $r > 0$, but since all *deviatoric* components of stress s_{ij} are assumed bounded (Section 2), they are continuous on the *closed* interval $[0, r_o]$. Then for $0 < r < r_o$

$$|s_{ij}(r,\theta,t) - s_{ij}(0,\theta,t)|$$

$$= |r\partial s_{ij}(\xi,\theta,t)/\partial r| \geq |r\partial s_{ij}(r,\theta,t)/\partial r|, \tag{3.14}$$

where the equality is the mean value theorem, with ξ lying somewhere in the *open* interval $(0,r)$ and since therefore $\xi < r$, the inequality in (3.14) follows from Assumption (*ii*) in the foregoing. Since the first term of (3.14) vanishes as $r \to 0$ due to continuity of s_{ij}, so too does the last term, thus proving (3.8). Result (3.8) then permits use of circumferential equilibrium to prove boundedness of hydrostatic stress as $r \to 0$, as in the preceding equations (3.9)–(3.11). This means that hydrostatic stress must also be continuous on the *closed* interval $[0, r_o]$, so that (3.14) may now be written for the hydrostatic stress component, and this combines with the foregoing proof of (3.8) to prove (3.13).

Results that are more strict than (3.13) are obtained by reapplying part of the first proof of (3.13); in particular, the slightly stronger requirement that

$$\partial\sigma_{ij}/\partial r = o\left[\frac{1}{r ln(R/r)}\right] \quad \text{as } r \to 0 \tag{3.15}$$

is a crucial result (e.g., see Drugan, [4]). To prove (3.15), we strengthen (3.1), on the basis of result (3.13), to the assumption that

$$r[ln(R/r)]\partial\sigma_{ij}/\partial r \quad \textit{exist} \text{ in the limit as } r \to 0, \tag{3.16}$$

and hence write

$$\lim_{r \to 0} \{ r[ln(R/r)]\partial\sigma_{ij}/\partial r \} = B_{ij}(\theta,t) \tag{3.17}$$

where $B_{ij}(\theta,t)$ are initially unknown. Now, following steps similar to those from (3.2–3.5), we find that (3.17) leads to

$$\sigma_{ij}/ln[ln(R/r)] = -B_{ij}(\theta,t) + o(1) \quad \text{as } r \to 0. \tag{3.18}$$

Taking the limit of this as $r \to 0$, enforcing the boundedness of σ_{ij}, shows $B_{ij}(\theta,t) \equiv 0$, which via (3.17) demonstrates (3.15).

3.2 Circumferential and Time Derivatives of Stress Components. If in addition to the assumptions of Section 2 I assume that $\partial\sigma_{ij}/\partial\theta$ and $\partial\sigma_{ij}/\partial t$ admit the following asymptotic representations (where here no summation is implied by repeated indices)

$$\partial\sigma_{ij}/\partial\theta = a_{ij}(r)C_{ij}(\theta,t) + o[a_{ij}(r)] \quad \text{as } r \to 0 \tag{3.19}$$

$$\partial\sigma_{ij}/\partial t = b_{ij}(r)D_{ij}(\theta,t) + o[b_{ij}(r)] \quad \text{as } r \to 0 \tag{3.20}$$

where the functions a_{ij}, b_{ij}, C_{ij}, and D_{ij} are initially unspecified except that C_{ij} and D_{ij} are bounded but not identically zero, then the maximum asymptotic orders of these quantities must be

$$\partial\sigma_{ij}/\partial\theta = 0(1) \quad \text{as } r \to 0 \tag{3.21}$$

$$\partial\sigma_{ij}/\partial t = 0(1) \quad \text{as } r \to 0. \tag{3.22}$$

I will prove (3.21) and (3.22) by contradiction. Beginning with (3.22), only the (11) component will be treated since the proof for any other component is identical. Suppose

$$|\partial\sigma_{11}/\partial t| \to \infty \quad \text{as} \quad r \to 0. \tag{3.23}$$

From assumption (3.20),

$$\partial\sigma_{11}/\partial t = b_{11}(r)D_{11}(\theta,t) + o[b_{11}(r)] \quad \text{as} \quad r \to 0, \tag{3.24}$$

which can be integrated with respect to time at fixed r, θ to give

$$\sigma_{11} = b_{11}(r)\int^{t} D_{11}(\theta,\tau)d\tau + F(r,\theta) + o[b_{11}(r)] \quad \text{as } r \to 0, \tag{3.25}$$

where $F(r,\theta)$ is a function of integration. Expression (3.25) could meet the boundedness requirement on σ_{11} in the limit of $r \to 0$ only if $F(r,\theta)$ were to cancel the singularity of the $b_{11}(r)$ term as $r \to 0$, but this is not possible for general t since $b_{11}(r)$ is multiplied by a function of t (which cannot be identically zero since $D_{11}(\theta,t) \not\equiv 0$). Thus (3.25) requires $b_{11}(r)$ to be

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bounded as $r \to 0$, which via (3.24) contradicts our supposition (3.23).

The proof of (3.21) is identical to that given for (3.22) if the roles of θ and t are interchanged and assumption (3.19) is employed in place of (3.20).

4 Asymptotic Form of Equilibrium Equations

The asymptotic bounds on stress derivatives proved in Section 3 will now be applied to show how the general equilibrium equations and other important expressions simplify for small r. In the case of the equilibrium equations, the results (3.15) and (3.21) are employed, showing that equilibrium requires (via(2.4))

$$(\partial\sigma_{ij}/\partial\theta)h_j + o\{[ln(R/r)]^{-1}\} = 0 \quad \text{as } r \to 0 \tag{4.1}$$

or (via (2.5))

$$\partial\sigma_{r\theta}/\partial\theta + \sigma_{rr} - \sigma_{\theta\theta} + o\{[ln(R/r)]^{-1}\} = 0 \quad \text{as } r \to 0 \tag{4.2a}$$

$$\partial\sigma_{\theta\theta}/\partial\theta + 2\sigma_{r\theta} + o\{[ln(R/r)]^{-1}\} = 0 \quad \text{as } r \to 0 \tag{4.2b}$$

$$\partial\sigma_{\theta z}/\partial\theta + \sigma_{rz} + o\{[ln(R/r)]^{-1}\} = 0 \quad \text{as } r \to 0. \tag{4.2c}$$

5 Asymptotic Forms of Stress Rate

Constitutive laws for the types of inelastic materials considered here usually involve the stress rate at a material point, $\dot{\sigma}_{ij}$. Following Rice [2], a convenient expression for this quantity results from applying the chain rule to (2.1):

$$\begin{aligned}\dot{\sigma}_{ij} &= (\partial\sigma_{ij}/\partial\theta)\dot{\theta} + (\partial\sigma_{ij}/\partial r)\dot{r} + \partial\sigma_{ij}/\partial t \\ &= (\partial\sigma_{ij}/\partial\theta)\dot{a}\sin\theta/r - (\partial\sigma_{ij}/\partial r)\dot{a}\cos\theta + \partial\sigma_{ij}/\partial t\end{aligned} \tag{5.1}$$

where a superposed dot denotes time rate at a material point, and we have employed $\dot{\theta} = \dot{a}\sin\theta/r$ and $\dot{r} = -\dot{a}\cos\theta$, which result from the translation of the crack tip polar coordinate system with the growing crack.

Application of the results (3.15), (3.21), and (3.22) to expression (5.1) yield the two useful asymptotic forms

$$\dot{\sigma}_{ij} = (\partial\sigma_{ij}/\partial\theta)\dot{a}\sin\theta/r - (\partial\sigma_{ij}/\partial r)\dot{a}\cos\theta + 0(1) \quad \text{as } r \to 0 \tag{5.2}$$

$$\dot{\sigma}_{ij} = (\partial\sigma_{ij}/\partial\theta)\dot{a}\sin\theta/r + o\{[rln(R/r)]^{-1}\} \quad \text{as } r \to 0. \tag{5.3}$$

Previous analyses of lowest-order *growing* crack-tip fields by Rice et al. [1], Rice [2], and Drugan et al. [3] are based on the asymptotic results (4.2) and (5.3) (although in the weaker forms obtained by using (3.13) instead of (3.15)), which were *assumed* by those authors to be valid on the basis of the stress boundedness requirement of their elastic-ideally plastic constitutive models. A principal objective of this paper was to prove these pivotal hypotheses, while at the same time strengthening them and demonstrating that their applicability extends to the more general material class delineated in Section 2.

6 Applicability to Plane Stress

All preceding results are valid for the general class of materials specified in Section 2.2, where it was mentioned that the derivations of Sections 3–5 exclude the special case of plane stress, to which the Drugan and Rice [6] proof of continuity of all stress components does not apply. However, Pan [10] has proved that the stress field near a *growing plane stress* crack must be fully continuous for an isotropic elastic-ideally plastic material which obeys the Prandtl-Reuss flow rule and Huber-Mises yield condition. Thus, for this specific material type at least, the principal results (3.15), (3.21), (3.22), (4.1), (4.2), (5.2), and (5.3) also apply to the plane stress case.

7 Results for Stationary Crack Tip Analysis

The derivations in Sections 3–5 are valid for the stress and stress rate fields near a quasistatically *growing* crack tip. Their extension to the case of a stationary crack is not automatic, since those results rely on the continuity of the stress field, which is necessary near a growing crack for the constitutive assumptions discussed (Drugan and Rice [6], Pan [10]), but not required near a stationary crack. However, some conclusions relevant to stationary crack tip analysis are still possible from the perspective of the preceding sections. In particular, (3.15) and hence (4.1) and (4.2) must hold if the dependence of σ_{ij} on r is continuous, which would seem to be a reasonable assumption in the limit as the crack tip is approached (since if a stress discontinuity surface were to exist, one would anticipate it to emanate from the tip, and hence to affect only the θ-dependence of σ_{ij} in the limit as $r \to 0$). This supports the assumption made by Rice and Tracey [11] in their analysis of the fields near a stationary crack tip in an isotropic elastic-ideally plastic Prandtl-Reuss-Mises solid, namely that $r\partial\sigma_{ij}/\partial r \to 0$ as $r \to 0$, and hence that the first term of (2.4) is asymptotically dominant.

If the loading applied to a cracked solid changes quasi-statically while the crack remains stationary, a plausible assumption for many materials is that the near-tip stress field changes *continuously* with time. Whenever this is true, assumption (3.20) leads as before to (3.22), and since $\dot{a} = 0$ for a stationary crack (5.2) reduces to

$$\dot{\sigma}_{ij} = 0(1) \quad \text{as } r \to 0. \tag{7.1}$$

For materials that exhibit a linear relationship between the stress state and the elastic part of total strain, (7.1) implies that all *elastic* strain rate components must be bounded at a stationary crack tip given the assumptions stated above.

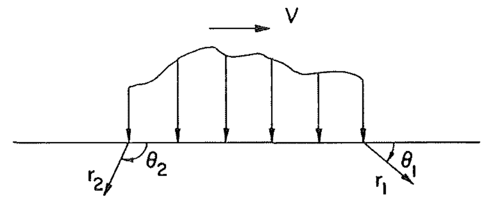

Fig. 2 Distributed load moving quasistatically, with velocity V, across the surface of an elastic-plastic body under plane strain conditions

8 Application to Related Problems

The derivations of Sections 3–5 are applicable to many problems which share a similar feature with the crack growth problems that motivated this work; in particular, problems that involve the quasistatic motion of certain geometric or load inhomogeneities through, or across the surface of, an elastic-plastic body. Just as for the crack problems, these results help elucidate the structure of the stress and stress-rate fields in the neighborhood of such a moving geometric or load inhomogeneity.

I mention just one example that will hopefully be sufficiently suggestive of the nature of related problems for which the results of this work can prove useful. Consider the plane strain problem of the quasistatic motion of a distributed load across the surface of an elastic-plastic body, as illustrated in Fig. 2. For any elastic-plastic material within the class described in Section 2, the structure of the stress and stress-rate fields in the vicinity of the leading and trailing edges of the distributed loading is delineated by the results of Sections 3–5; i.e., the asymptotic equations and expressions derived in those sections apply for r_1, θ_1 and r_2, θ_2 of Fig. 2.

Acknowledgments

Support of this work by the U.S. National Science Foundation, Solid Mechanics Program, under Grant MEA-

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8312348, and by the Department of Engineering Mechanics, University of Wisconsin, is gratefully acknowledged. I am indebted to Professors R. D. James, A. C. Pipkin, and J. R. Rice for interesting and valuable discussions.

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