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Transmission problem in thermoelasticity with symmetry

ALFREDO MARZOCCHI†
Dipartimento di Matematica, Università degli Studi di Brescia, Via Valotti 9, I - 25133 Brescia, Italia

JAIME E. MUÑOZ RIVERA‡
National Laboratory for Scientific Computation, Rua Getulio Vargas 333, Quitadinha-Petrópolis 25651-070, Rio de Janeiro, RJ, Brazil

AND

MARIA GRAZIA NASO§
Dipartimento di Matematica, Università degli Studi di Brescia, Via Valotti 9, I - 25133 Brescia, Italia

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In this paper we show the existence, uniqueness and regularity of the solutions to the thermoelastic transmission problem. Moreover, when the solutions are symmetrical we show that the energy decays exponentially as time goes to infinity, no matter how small is the size of the thermoelastic part.

Keywords: transmission problem; *n*-d-thermoelasticity; radial symmetry; exponential decay; simultaneous stabilization.

1. Introduction

We consider a model describing oscillations of a body which is composed of two different materials, one of them is a thermoelastic material while the other is indifferent to thermal effects. We assume that the density as well as the elastic coefficients are different constants in each component. Therefore, we have a transmission problem where the damping effect given by the difference of temperature is effective only in a part of the material.

Concerning thermoelastic systems we have the work of Dafermos (1968), where it is proved that the solution of the *n*-dimensional anisotropic thermoelastic material is asymptotically stable as time tends to infinity, and the decay of the displacement is not to zero but to an undamped oscillation. On the other hand the difference of temperature as well as the divergence of the displacement always tend to zero as time tends to infinity. For one-dimensional models it was proved that the dissipation given by difference of temperature is strong enough to produce uniform rate of decay of the solution; see (Kim, 1992; Marzocchi *et al.*, 2002; Muñoz Rivera, 1992). The situation is different in two and three dimensions because the displacement vector field has two or three degrees of freedom

†Email: amarzocc@ing.unibs.it
‡Email: rivera@lncc.br
§Email: naso@ing.unibs.it

while the difference of temperature, which produces the dissipative effect of the system, has only one degree of freedom. For this reason the dissipation is weaker than in the one-dimensional case and uniform rate of decay is not expected any more as was shown by Henry (1987). From this point of view we have not in general a uniform rate of decay for the solution in two- or three-dimensional space. The exception is for symmetrical solutions, as was shown in (Jiang *et al.*, 1998).

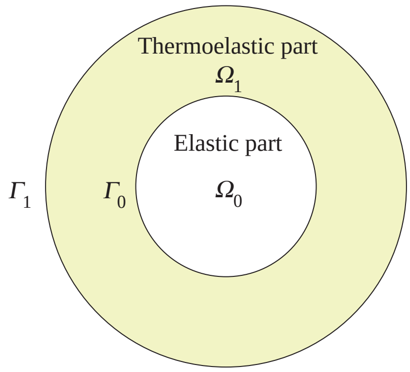
In this paper we consider the transmission problem between a thermoelastic and an elastic material. That is, the dissipation given by the thermal effect is effective only in a part of the material. The main question is whether the localized thermal effect is strong enough to produce uniform rate of decay for symmetrical solutions. The aim of this paper is to prove that no matter how small is the thermoelastic part, the dissipation, produced by the thermal effects, yields uniform rate of decay for the solution.

Let us consider an n-dimensional body which is configured in $\Omega \subset \mathbb{R}^n$. The thermoelastic part is given by Ω_1 and the elastic part by Ω_0, that is,

$$\Omega_0 = \{x \in \mathbb{R}^n \,:\, |x| < r_0\},$$

$$\Omega_1 = \{x \in \mathbb{R}^n \,:\, r_0 < |x| < r_1\},$$

with $0 < r_0 < r_1$ and $n \geqslant 2$. We denote the boundary of Ω_1 as $\partial\Omega_1 = \Gamma_0 \cup \Gamma_1$ and the boundary of Ω_0 as $\partial\Omega_0 = \Gamma_0$.



Let $\boldsymbol{u} = (u_1, \ldots, u_n)^T$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^T$ be the displacements in the thermoelastic and elastic parts, respectively (^{T} denotes transposition). We denote by θ the difference of temperature between the actual state and a reference temperature. Then the system that models the above setting is given by

$$\rho_1 \boldsymbol{u}_{tt} - \mu_1 \Delta\boldsymbol{u} - (\mu_1 + \lambda_1)\nabla \mathrm{div}\, \boldsymbol{u} + \alpha \nabla\theta = 0 \quad \text{in} \quad \Omega_1 \times]0, \infty[, \tag{1.1}$$

$$\theta_t - \kappa \Delta\theta + \beta\, \mathrm{div}\, \boldsymbol{u}_t = 0 \quad \text{in} \quad \Omega_1 \times]0\infty[, \tag{1.2}$$

$$\rho_0 \boldsymbol{v}_{tt} - \mu_0 \Delta\boldsymbol{v} - (\mu_0 + \lambda_0)\nabla \mathrm{div}\, \boldsymbol{v} = 0 \quad \text{in} \quad \Omega_0 \times]0\infty[, \tag{1.3}$$

where μ_1, λ_1, μ_0, λ_0 are the Lamé moduli satisfying $\mu_1, \mu_0 > 0$ and $n\lambda_1 + 2\mu_1 > 0$, $n\lambda_0 + 2\mu_0 > 0$, $\alpha, \beta > 0$, $\kappa > 0$ are given constants depending on the material properties.

The system is subjected to the following boundary conditions:

$$\boldsymbol{u}(x,t) = \theta(x,t) = 0 \text{ on } \Sigma_1, \tag{1.4}$$

$$\boldsymbol{u}(x,t) = \boldsymbol{v}(x,t) \text{ on } \Sigma_0, \tag{1.5}$$

$$\mu_1 \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} + [(\mu_1+\lambda_1)\operatorname{div}\boldsymbol{u} - \alpha\,\theta]\,\boldsymbol{\nu} = \mu_0 \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} + (\mu_0+\lambda_0)(\operatorname{div}\boldsymbol{v})\boldsymbol{\nu} \text{ on } \Sigma_0, \tag{1.6}$$

$$\frac{\partial \theta}{\partial \boldsymbol{\nu}} = 0 \text{ on } \Sigma_0, \tag{1.7}$$

where $\Sigma_1 = \Gamma_1\times]0,\infty[$, $\Sigma_0 = \Gamma_0\times]0,\infty[$. The initial conditions are given by

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad \boldsymbol{u}_t(x,0) = \boldsymbol{u}_1(x), \qquad x \in \Omega_1, \tag{1.8}$$

$$\theta(x,0) = \theta_0(x), \qquad x \in \Omega_1, \tag{1.9}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad \boldsymbol{v}_t(x,0) = \boldsymbol{v}_1(x), \qquad x \in \Omega_0. \tag{1.10}$$

Our main result is that the solution of the symmetrical transmission problem decays exponentially as time tends to infinity, no matter how small is the size of the thermoelastic part. The main problem in showing the exponential stability is to deal with the boundary terms in the interface of the material. We overcome such difficulty using an observability result of the elastic wave equations together with the fact that the solution is radially symmetric. Our method allows us to find a Lyapunov functional $\mathcal{L}$ equivalent to the second-order energy for which we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -\gamma\mathcal{L}(t).$$

That is, we give a direct proof of the exponential stability, which means that our method can be applied to nonlinear problems.

The remaining part of this paper is divided as follows. In Section 3 we show the existence and regularity of radially symmetric solutions to the transmission problem. In Section 4 we show the exponential decay of the solutions.

2. Functional setting and notation

We now introduce the notation used throughout this paper. Let Ω be a domain in $\mathbb{R}^n$. By $W^{m,p}(\Omega)$ ($m \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$) we mean the usual Sobolev space defined over Ω with norm $\|\cdot\|_{W^{m,p}}$ (see, for example, Adams, 1975); $W^{m,2}(\Omega) \equiv H^m(\Omega)$ with norm $\|\cdot\|_{H^m}$, $W^{0,p}(\Omega) \equiv L^p(\Omega)$ with norm $\|\cdot\|_{L^p}$. We write $C^L(I,B)$ (resp. $L^2(I,B)$) for the space of B-valued functions which are L-times continuously differentiable (resp. square integrable) in I, $I \subset \mathbb{R}$ an interval, B a Banach space, L a non-negative integer. We denote by $\mathrm{O}(n)$ the set of orthogonal $n\times n$ real matrices and by $\mathrm{SO}(n)$ the set of matrices in $\mathrm{O}(n)$ which have determinant 1. Then $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the inner product in $\mathbb{R}^n$. For a vector-valued function $f = (f_1,\cdots,f_m)^T$ and a normed space X with the norm $|||\cdot|||$, writing $f \in X$ means that each component of f is in X; we put $|||f||| := |||f_1||| + \cdots + |||f_m|||$.

The same letter C will denote various positive constants which in particular do not depend on t and initial data.

3. Existence and uniqueness

First, we define what we mean by a weak solution of the problem (1.1)–(1.10). In this section, we set $I = [0,\ T]$, with $T > 0$. We introduce the following spaces:

$$H^1_{\Gamma_1} = \{w \in H^1(\Omega_1) : w(x,t) = 0 \quad \text{on } \Gamma_1 \times I\},$$

$$V = \{(h,k) \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_0) : h(x,t) = k(x,t) \quad \text{on } \Gamma_0 \times I\}.$$

DEFINITION 3.1 We say that $(\boldsymbol{u}, \theta, \boldsymbol{v})$ is a *weak solution* of (1.1)–(1.10) when

$$\boldsymbol{u} \in W^{1,\infty}(I, L^2(\Omega_1)) \cap L^\infty(I, H^1_{\Gamma_1}(\Omega_1)),$$

$$\theta \in L^\infty(I, L^2(\Omega_1)) \cap L^2(I, H^1_{\Gamma_1}(\Omega_1)),$$

$$\boldsymbol{v} \in W^{1,\infty}(I, L^2(\Omega_0)) \cap L^\infty(I, H^1(\Omega_0)),$$

satisfying the identities

$$\int_0^T\int_{\Omega_1}\big[\rho_1\, \boldsymbol{u}\cdot\boldsymbol{\phi}_{tt} + \mu_1\nabla\boldsymbol{u} : \nabla\boldsymbol{\phi} + (\mu_1+\lambda_1)\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{\phi} - \alpha\theta\,\mathrm{div}\,\boldsymbol{\phi}\big]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega_0}[\rho_0\,\boldsymbol{v}\cdot\boldsymbol{\omega}_{tt} + \mu_0\nabla\boldsymbol{v} : \nabla\boldsymbol{\omega} + (\mu_0+\lambda_0)\mathrm{div}\,\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{\omega}]\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_1}\big\{\rho_1\big[\boldsymbol{u}_1\,\boldsymbol{\phi}(0) - \boldsymbol{u}_0\,\boldsymbol{\phi}_t(0)\big]\big\}\,\mathrm{d}x + \int_{\Omega_0}\{\rho_0\,[\boldsymbol{v}_1\,\boldsymbol{\omega}(0) - \boldsymbol{v}_0\,\boldsymbol{\omega}_t(0)]\}\,\mathrm{d}x;$$

$$\int_0^T\int_{\Omega_1}(-\theta\zeta_t + \kappa\nabla\theta\cdot\nabla\zeta + \beta\mathrm{div}\,\boldsymbol{u}_t\,\zeta)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_1}\theta_0\zeta(0)\,\mathrm{d}x$$

for all $(\boldsymbol{\phi}, \boldsymbol{\omega}) \in C^2(I, V)$, $\zeta \in C^1(I, H^1_{\Gamma_1})$ and almost every $t \in I$ such that

$$\boldsymbol{\phi}(T) = \boldsymbol{\phi}_t(T) = \boldsymbol{\omega}(T) = \boldsymbol{\omega}_t(T) = 0 \quad \text{and} \quad \zeta(T) = 0.$$

The existence of solutions to system (1.1)–(1.10) is given in the following theorem.

THEOREM 3.1 Let us take the initial data satisfying

$$(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0) \in H^1_{\Gamma_1} \times L^2 \times L^2 \quad \text{and} \quad (\boldsymbol{v}_0, \boldsymbol{v}_1) \in H^1 \times L^2.$$

Then there exists only one solution $(\boldsymbol{u}, \theta, \boldsymbol{v})$ of the system (1.1)–(1.10) satisfying

$$\boldsymbol{u} \in W^{1,\infty}\left(I, L^2\right) \cap L^\infty\left(I, H^1_{\Gamma_1}\right),$$

$$\theta \in L^\infty\left(I, L^2\right) \cap L^2\left(I, H^1_{\Gamma_1}\right),$$

$$\boldsymbol{v} \in W^{1,\infty}\left(I, L^2\right) \cap L^\infty\left(I, H^1\right).$$

In addition, if

$$(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0) \in \left(H^2 \cap H^1_{\Gamma_1}\right) \times H^1_{\Gamma_1} \times \left(H^2 \cap H^1_{\Gamma_1}\right)$$

and

$$(\boldsymbol{v}_0, \boldsymbol{v}_1) \in H^2 \times H^1,$$

verifying the compatibility conditions

$$\boldsymbol{u}_0(x) = \boldsymbol{0}; \quad \theta_0(x) = 0 \quad \text{on} \quad \Gamma_1,$$

$$\boldsymbol{u}_0(x) = \boldsymbol{v}_0(x) \quad \text{on} \quad \Gamma_0,$$

$$\mu_1 \frac{\partial \boldsymbol{u}_0}{\partial \boldsymbol{\nu}} + [(\mu_1 + \lambda_1)\text{div}\, \boldsymbol{u}_0 - \alpha\, \theta_0]\, \boldsymbol{\nu} = \mu_0 \frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{\nu}} + (\mu_0 + \lambda_0)(\text{div}\, \boldsymbol{v}_0)\boldsymbol{\nu} \quad \text{on} \quad \Gamma_0,$$

$$\frac{\partial \theta_0}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma_0,$$

then the solution satisfies

$$\boldsymbol{u} \in L^\infty\left(I, H^2\right) \cap W^{1,\infty}\left(I, H^1\right) \cap W^{2,\infty}\left(I, L^2\right),$$

$$\theta \in L^\infty\left(I, H^2\right) \cap W^{1,\infty}\left(I, L^2\right),$$

$$\boldsymbol{v} \in L^\infty\left(I, H^2\right) \cap W^{1,\infty}\left(I, H^1\right) \cap W^{2,\infty}\left(I, L^2\right).$$

Proof. The proof follows using the standard Galerkin approximation and the elliptic regularity for transmission problem given in (Ladyzhenskaya, 1968; Athanasiadis & Stratis, 1996).

Step 1 (Faedo–Galerkin scheme). Given $\nu \in \mathbb{N}$, denote by P_ν and Q_ν the projections on the subspaces span$\{(\boldsymbol{\phi}_i, \boldsymbol{\omega}_i)\}$, span$\{\zeta_i\}$ $(i = 1, \ldots, \nu)$ of V and $H^1_{\Gamma_1}$ respectively. Let us write

$$(\boldsymbol{u}^\nu, \boldsymbol{v}^\nu) = \sum_{i=1}^{\nu} a_i(t)(\boldsymbol{\phi}_i, \boldsymbol{\omega}_i), \qquad \theta^\nu = \sum_{i=1}^{\nu} b_i(t)\zeta_i,$$

where $\boldsymbol{u}^\nu$ and $\boldsymbol{v}^\nu$ satisfy

$$\begin{aligned} &\int_{\Omega_1} \left[\boldsymbol{u}^\nu_{tt} \cdot \boldsymbol{\phi}_i + \mu_1 \nabla \boldsymbol{u}^\nu : \nabla \boldsymbol{\phi}_i + (\mu_1 + \lambda_1)\text{div}\, \boldsymbol{u}^\nu\, \text{div}\, \boldsymbol{\phi}_i - \alpha\theta^\nu\, \text{div}\, \boldsymbol{\phi}_i\right] \mathrm{d}x \\ &\quad + \int_{\Omega_0} \left[\boldsymbol{v}^\nu_{tt} \cdot \boldsymbol{\omega}_i + \mu_0 \nabla \boldsymbol{v}^\nu : \nabla \boldsymbol{\omega}_i + (\mu_0 + \lambda_0)\text{div}\, \boldsymbol{v}^\nu\, \text{div}\, \boldsymbol{\omega}_i\right] \mathrm{d}x = 0, \\ &\int_{\Omega_1} \left(\theta^\nu \zeta_i + \kappa \nabla\theta^\nu \cdot \nabla\zeta_i + \beta \text{div}\, \boldsymbol{u}^\nu_t\, \zeta_i\right) \mathrm{d}x = 0, \end{aligned} \tag{3.1}$$

$$(\boldsymbol{u}^\nu(0), \boldsymbol{v}^\nu(0)) = (\boldsymbol{u}_0, \boldsymbol{v}_0), \quad (\boldsymbol{u}^\nu_t(0), \boldsymbol{v}^\nu_t(0)) = (\boldsymbol{u}_1, \boldsymbol{v}_1), \quad \theta^\nu(0) = \theta_0,$$

for almost all $t \leqslant T$, where ϕ_0, ω_0 and ζ_0 are the zero vectors in the respective spaces. Recasting exactly the classical Faedo–Galerkin scheme, we get a system of ordinary differential equations (ODEs) in the variables $a_i(t)$ and $b_i(t)$. According to standard existence theory for ODEs there exists a continuous solution of this system, on some interval $(0, T_n)$. The a priori estimates that follow imply that in fact $T_n = +\infty$.

Step 2 (energy estimates). Multiplying $(3.1)_{1,2}$ by $a_i'(t)$, integrating by parts and summing up over i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\nu(t) + \frac{\alpha}{\beta}\int_{\Omega_1} \nabla\theta^\nu \cdot \boldsymbol{u}_t^\nu \,\mathrm{d}x = 0,$$

where

$$\mathcal{E}^\nu(t) = \frac{1}{2}\int_{\Omega_1}\left[\rho_1|\boldsymbol{u}_t^\nu|^2 + \mu_1|\nabla\boldsymbol{u}^\nu|^2 + (\mu_1+\lambda_1)|\mathrm{div}\,\boldsymbol{u}^\nu|^2\right]\mathrm{d}x$$
$$+\frac{1}{2}\int_{\Omega_0}\left[\rho_0|\boldsymbol{v}_t^\nu|^2 + \mu_0|\nabla\boldsymbol{v}^\nu|^2 + (\mu_0+\lambda_0)|\mathrm{div}\,\boldsymbol{v}^\nu|^2\right]\mathrm{d}x.$$

Multiplying $(3.1)_2$ by $b_i(t)$, integrating by parts and summing up over i, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1}|\theta^\nu|^2\,\mathrm{d}x + \kappa\int_{\Omega_1}|\nabla\theta^\nu|^2\,\mathrm{d}x - \beta\int_{\Omega_1}\nabla\theta^\nu\cdot\boldsymbol{u}_t^\nu\,\mathrm{d}x = 0.$$

Summing the two identities above we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}^\nu(t) = -\kappa\frac{\alpha}{\beta}\int_{\Omega_1}|\nabla\theta^\nu|^2\,\mathrm{d}x,$$

where

$$\widehat{\mathcal{E}}^\nu(t) = \mathcal{E}^\nu(t) + \frac{1}{2}\frac{\alpha}{\beta}\int_{\Omega_1}|\theta^\nu|^2\,\mathrm{d}x.$$

After an integration over $(0,t)$, $t\in(0,T)$, we have that

$$\widehat{\mathcal{E}}^\nu(t) + \frac{\kappa\alpha}{\beta}\int_0^t\int_{\Omega_1}|\nabla\theta^\nu(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \widehat{\mathcal{E}}^\nu(0).$$

Thus, we conclude that

$$(\boldsymbol{u}^\nu, \boldsymbol{u}_t^\nu, \theta^\nu) \text{ is bounded in } L^\infty(I,H^1)\times L^\infty(I,L^2)\times L^\infty(I,L^2),$$
$$(\boldsymbol{v}^\nu, \boldsymbol{v}_t^\nu) \text{ is bounded in } L^\infty(I,H^1)\times L^\infty(I,L^2),$$

which implies that

$$\boldsymbol{u}^\nu \rightharpoonup \boldsymbol{u} \ \text{ weak* in } L^\infty(I,H^1),$$
$$\theta^\nu \rightharpoonup \theta \ \text{ weak* in } L^\infty(I,L^2),$$
$$\boldsymbol{v}^\nu \rightharpoonup \boldsymbol{v} \ \text{ weak* in } L^\infty(I,H^1),$$
$$\boldsymbol{u}_t^\nu \rightharpoonup \boldsymbol{u}_t \text{ weak* in } L^\infty(I,L^2),$$
$$\boldsymbol{v}_t^\nu \rightharpoonup \boldsymbol{v}_t \text{ weak* in } L^\infty(I,L^2).$$

In particular,

$$\boldsymbol{u}^\nu \to \boldsymbol{u} \ \text{ strongly in } L^2(I, L^2),$$

whence it follows that

$$\boldsymbol{u}^\nu \to u \quad \text{a.e. in} \quad \Omega_1.$$

The rest of the proof of the existence of weak solution is a matter of routine. Next we show uniqueness. Let us suppose that there exist two solutions $(\boldsymbol{u}^{\mathbf{1}}, \boldsymbol{v}^{\mathbf{1}}, \theta^1)$ and $(\boldsymbol{u}^{\mathbf{2}}, \boldsymbol{v}^{\mathbf{2}}d, \theta^2)$ and let us denote by

$$\boldsymbol{U} = \boldsymbol{u}^{\mathbf{1}} - \boldsymbol{u}^{\mathbf{2}}, \quad \boldsymbol{V} = \boldsymbol{v}^{\mathbf{1}} - \boldsymbol{v}^{\mathbf{2}}, \quad \Theta = \theta^1 - \theta^2.$$

Taking

$$\boldsymbol{u} = \int_0^t \boldsymbol{U}\, \mathrm{d}\tau, \quad \boldsymbol{v} = \int_0^t \boldsymbol{V}\, \mathrm{d}\tau, \quad \theta = \int_0^t \Theta\, \mathrm{d}\tau$$

it is not difficult to see that $(\boldsymbol{u}, \boldsymbol{v}, \theta)$ satisfies

$$\rho_1\, \boldsymbol{u}_{tt} - \mu_1\, \Delta \boldsymbol{u} - (\mu_1 + \lambda_1)\nabla \mathrm{div}\, \boldsymbol{u} + \alpha\, \nabla\theta = 0 \quad \text{in} \quad \Omega_1 \times]0, \infty[, \tag{3.2}$$

$$\theta_t - \kappa\, \Delta\theta + \beta\, \mathrm{div}\, \boldsymbol{u}_t = 0 \quad \text{in} \quad \Omega_1 \times]0, \infty[, \tag{3.3}$$

$$\rho_0\, \boldsymbol{v}_{tt} - \mu_0\, \Delta \boldsymbol{v} - (\mu_0 + \lambda_0)\nabla \mathrm{div}\, \boldsymbol{v} = 0 \quad \text{in} \quad \Omega_0 \times]0, \infty[. \tag{3.4}$$

Since $(\boldsymbol{u}^{\mathbf{1}}, \boldsymbol{v}^{\mathbf{1}}, \theta^1)$, $(\boldsymbol{u}^{\mathbf{2}}, \boldsymbol{v}^{\mathbf{2}}, \theta^2)$ are weak solutions of the system we have that $(\boldsymbol{u}, \boldsymbol{v}, \theta)$ satisfies

$$\boldsymbol{u} \in L^\infty(0, T; H^1_{\Gamma_1}), \quad \boldsymbol{u}_t \in L^\infty(0, T; H^1_{\Gamma_1}), \quad \boldsymbol{u}_{tt} \in L^\infty(0, T; L^2),$$

$$\boldsymbol{v} \in L^\infty(0, T; H^1), \quad \boldsymbol{v}_t \in L^\infty(0, T; H^1), \quad \boldsymbol{v}_{tt} \in L^\infty(0, T; L^2),$$

$$\theta \in L^2(0, T; H^1_{\Gamma_1}), \quad \theta_t \in L^2(0, T; H^1_{\Gamma_1}) \quad \Rightarrow \quad \theta \in L^\infty(0, T; H^1_{\Gamma_1}).$$

Using the elliptic regularity for the elliptic transmission problem (see Athanasiadis & Stratis, 1996) we conclude that

$$\boldsymbol{u} \in L^\infty(0, T; H^1_{\Gamma_1} \cap H^2), \quad \boldsymbol{v} \in L^\infty(0, T; H^1 \cap H^2), \quad \theta \in L^\infty(0, T; H^1_{\Gamma_1} \cap H^2).$$

Thus $(\boldsymbol{u}, \boldsymbol{v}, \theta)$ satisfies (3.2)–(3.4) in the strong sense. Multiplying equation (3.2) by $\boldsymbol{u}_t$, (3.3) by θ and (3.4) by $\boldsymbol{v}_t$ we conclude, using similar arguments as above, that

$$E(t) = 0,$$

where

$$E(t) = \frac{1}{2}\int_{\Omega_1} \Big[\rho_1 |\boldsymbol{u}_t|^2 + \mu_1 |\nabla \boldsymbol{u}|^2 + (\mu_1 + \lambda_1)|\mathrm{div}\, \boldsymbol{u}|^2\Big]\, \mathrm{d}x$$
$$+ \frac{1}{2}\int_{\Omega_0} \Big[\rho_0 |\boldsymbol{v}_t|^2 + \mu_0 |\nabla \boldsymbol{v}|^2 + (\mu_0 + \lambda_0)|\mathrm{div}\, \boldsymbol{v}|^2\Big]\, \mathrm{d}x$$
$$+ \frac{\alpha}{2\beta}\int_{\Omega_1} |\nabla\theta|^2\, \mathrm{d}x$$

which implies that $\boldsymbol{u}^{\mathbf{1}} = \boldsymbol{u}^{\mathbf{2}}$, $\boldsymbol{v}^{\mathbf{1}} = \boldsymbol{v}^{\mathbf{2}}$, $\theta^1 = \theta^2$. From this uniqueness follows.

To show the regularity result, we differentiate the approximate system (3.1), then we multiply the resulting system by $a_i''(t)$ and $b_i'(t)$, and as before we get

$$\mathcal{E}_2^\nu(t) \leqslant C\,\mathcal{E}_2^\nu(0),$$

where

$$\mathcal{E}_2^\nu(t) = \frac{1}{2}\int_{\Omega_1}\left[\rho_1|\boldsymbol{u}_{tt}^\nu|^2 + \mu_1|\nabla\boldsymbol{u}_t^\nu|^2 + (\mu_1+\lambda_1)|\mathrm{div}\,\boldsymbol{u}_t^\nu|^2 + \frac{\alpha}{\beta}|\theta_t^\nu|^2\right]\mathrm{d}x$$
$$+\frac{1}{2}\int_{\Omega_0}\left[\rho_0|\boldsymbol{v}_{tt}^\nu|^2 + \mu_0|\nabla\boldsymbol{v}_t^\nu|^2 + (\mu_0+\lambda_0)|\mathrm{div}\,\boldsymbol{v}_t^\nu|^2\right]\mathrm{d}x.$$

Therefore, we find that

$$\begin{aligned}
\boldsymbol{u}_{tt}^\nu,\ \theta_t^\nu &\quad \text{are bounded in } L^\infty(0,T;L^2(\Omega)),\\
\boldsymbol{v}_{tt}^\nu &\quad \text{is bounded in } L^\infty(0,T;L^2(\Omega)),\\
\boldsymbol{v}_{tt}^\nu &\quad \text{is bounded in } L^\infty(0,T;L^2(\Omega)),\\
\boldsymbol{u}_{t}^\nu &\quad \text{is bounded in } L^\infty(0,T;H^1(\Omega)),\\
\theta_t^\nu &\quad \text{is bounded in } L^\infty(0,T;H^1(\Omega)).
\end{aligned}$$

Finally, our conclusion will follow on using the regularity result for the elliptic transmission problem (see Athanasiadis & Stratis, 1996). □

REMARK 3.1 We can extend the above theorem to higher regularity by introducing the following definition. We will say that the initial data $(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0)$ is *k-regular* ($k \geqslant 2$) if

$$\boldsymbol{u}_j \in H^{k-j}\cap H^1_{\Gamma_1}, \qquad j = 0,\cdots k-1; \quad \boldsymbol{u}_k \in L^2,$$

$$\theta_j \in H^{k-j}\cap H^1_{\Gamma_1}, \qquad j = 0,\cdots k-1; \quad \theta_k \in L^2,$$

where the values of $\boldsymbol{u}_j$ and θ_j are given by

$$\boldsymbol{u}_{j+2} - \mu_1\Delta\boldsymbol{u}_j - (\mu_1+\lambda_1)\nabla\mathrm{div}\,\boldsymbol{u}_j + \alpha\nabla\theta = 0 \quad \text{in} \quad \Omega_1\times]0,\infty[,$$

$$\theta_{j+1} - \kappa\Delta\theta_j + \beta\mathrm{div}\,\boldsymbol{u}_{j+1} = 0 \quad \text{in} \quad \Omega_1\times]0,\infty[,$$

$$\boldsymbol{v}_{j+2} - \mu_0\Delta\boldsymbol{v}_j - (\mu_0+\lambda_0)\nabla\mathrm{div}\,\boldsymbol{v}_j = 0 \quad \text{in} \quad \Omega_0\times]0,\infty[,$$

satisfying the following compatibility conditions:

$$\boldsymbol{u}_j(x) = \boldsymbol{0}; \quad \theta_j(x) = 0 \quad \text{on} \quad \Gamma_1,$$

$$\boldsymbol{u}_j(x) = \boldsymbol{v}_j(x) \quad \text{on} \quad \Gamma_0,$$

$$\mu_1\frac{\partial\boldsymbol{u}_j}{\partial\boldsymbol{\nu}} + \left[(\mu_1+\lambda_1)\mathrm{div}\,\boldsymbol{u}_j - \alpha\,\theta_j\right]\boldsymbol{\nu} = \mu_0\frac{\partial\boldsymbol{v}_j}{\partial\boldsymbol{\nu}} + (\mu_0+\lambda_0)(\mathrm{div}\,\boldsymbol{v}_j)\boldsymbol{\nu} \quad \text{on} \quad \Gamma_0,$$

$$\frac{\partial\theta_j}{\partial\boldsymbol{\nu}} = 0 \quad \text{on} \quad \Gamma_0$$

for $j = 0, \dots, k-1$. Using the above notation we say that, if the initial data is k-regular, then we have that the solution satisfies

$$\boldsymbol{u} \in \bigcap_{j=0}^{k} W^{j,\infty}(0,T;H^{k-j} \cap H^1_{\Gamma_1}), \qquad \boldsymbol{v} \in \bigcap_{j=0}^{k} W^{j,\infty}(0,T;H^{k-j}),$$

$$\theta \in \bigcap_{j=0}^{k} W^{j,\infty}(0,T;H^{k-j} \cap H^1_{\Gamma_1}).$$

The proof follows the same arguments as Theorem 3.1.

Let us denote by O(n) the set of orthogonal $n \times n$ real matrices and by SO(n) the set of matrices in O(n) which have determinant 1.

LEMMA 3.1 Assume that $\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0, \boldsymbol{v}_0, \boldsymbol{v}_1$, satisfy

$$\begin{aligned} &\boldsymbol{u}_0(G\,x) = G\,\boldsymbol{u}_0(x),\ \boldsymbol{u}_1(G\,x) = G\,\boldsymbol{u}_1(x),\ \theta_0(G\,x) = G\,\theta_0(x), \quad \forall\, x \in \bar{\Omega}_1,\\ &\boldsymbol{v}_0(G\,x) = G\,\boldsymbol{v}_0(x),\ \boldsymbol{v}_1(G\,x) = G\,\boldsymbol{v}_1(x), \qquad \forall\, x \in \bar{\Omega}_0,\\ &G \in \mathrm{O}(2) \quad \text{if} \quad n=2 \quad \text{or} \quad G \in \mathrm{SO}(n) \quad \text{if} \quad n \geqslant 3\cdot \end{aligned} \tag{3.5}$$

Then the solution $(\boldsymbol{u}, \theta, \boldsymbol{v})$ of (1.1)–(1.10) has the form

$$\begin{aligned} &u_i(x,t) = x_i\,\phi(r,t), && \forall\, x \in \bar{\Omega}_1, \quad t \geqslant 0,\\ &\theta(x,t) = \psi(r,t), && \forall\, x \in \bar{\Omega}_1, \quad t \geqslant 0,\\ &v_i(x,t) = x_i\,\eta(r,t), \quad v_i(0,t) = 0,\ i = 1,\dots,n, && \forall\, x \in \bar{\Omega}_0, \quad t \geqslant 0, \end{aligned} \tag{3.6}$$

where $r = |x|$, for some functions ϕ, ψ, η.

Proof. We first prove that under (3.5) the solution of system (1.1)–(1.3) $(\boldsymbol{u}, \theta, \boldsymbol{v})$ is radial, that is,

$$\begin{aligned} &\boldsymbol{u}(G\,x,t) = G\,\boldsymbol{u}(x,t), \quad \theta(x,t) = \theta(G\,x,t) && \text{for any } x \in \bar{\Omega}_1, \quad t \geqslant 0,\\ &\boldsymbol{v}(G\,x,t) = G\,\boldsymbol{v}(x,t), && \text{for any } x \in \bar{\Omega}_0, \quad t \geqslant 0,\\ &G \in \mathrm{O}(2) \quad \text{if} \quad n=2 \quad \text{or} \quad G \in \mathrm{SO}(n) \quad \text{if} \quad n \geqslant 3. \end{aligned} \tag{3.7}$$

Let $G = (G_{ij})_{n\times n} \in \mathrm{O}(2)$ for $n = 2$ or $\in \mathrm{SO}(n)$ for $n \geqslant 3$ be arbitary but fixed, and define $\boldsymbol{U}(x,t) := G^T\,\boldsymbol{u}(G\,x,t)$, $\Theta(x,t) := \theta(G\,x,t)$, $\boldsymbol{V}(x,t) := G^T\,\boldsymbol{v}(G\,x,t)$. After a straightforward calculation we get

$$\begin{aligned} &\boldsymbol{U}_{tt} = G^T\,\boldsymbol{u}_{tt}(G\,x,t), && \Delta\boldsymbol{U}(x,t) = G^T(\Delta\boldsymbol{u})(G\,x,t),\\ &\operatorname{div}\boldsymbol{U}_t(x,t) = (\operatorname{div}\boldsymbol{u}_t)(G\,x,t), && \nabla\operatorname{div}\boldsymbol{U} = G^T(\nabla\operatorname{div}\boldsymbol{u})(G\,x,t),\\ &\nabla\Theta(x,t) = G^T(\nabla\theta)(G\,x,t), && \Delta\Theta(x,t) = (\Delta\theta)(G\,x,t),\\ &\boldsymbol{V}_{tt} = G^T\,\boldsymbol{v}_{tt}(G\,x,t), && \Delta\boldsymbol{V}(x,t) = G^T(\Delta\boldsymbol{v})(G\,x,t),\\ &\operatorname{div}\boldsymbol{V}_t(x,t) = (\operatorname{div}\boldsymbol{v}_t)(G\,x,t), && \nabla\operatorname{div}\boldsymbol{V} = G^T(\nabla\operatorname{div}\boldsymbol{v})(G\,x,t),\\ &\frac{\partial\boldsymbol{U}}{\partial\boldsymbol{\nu}}(x,t) = \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}}(G\,x,t), && \frac{\partial\boldsymbol{V}}{\partial\boldsymbol{\nu}}(x,t) = \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\nu}}(G\,x,t). \end{aligned} \tag{3.8}$$

In view of (3.5) and (3.8) we see that $\boldsymbol{U}(\boldsymbol{x}, t)$, $\Theta(x, t)$ and $\boldsymbol{V}(x, t)$ satisfy the equations (1.1)–(1.10). From the uniqueness of solutions to (1.1)–(1.10) we get $\boldsymbol{u}(x, t) \equiv \boldsymbol{U}(x, t)$, $\theta(x, t) \equiv \Theta(x, t)$, $\boldsymbol{v}(x, t) \equiv \boldsymbol{V}(x, t)$, which gives (3.7). Next we show that (3.7) implies (3.6).

Case I: $n = 2$. Let $x = (x_1, x_2)^T \in \bar{\Omega}_1$ be arbitrary but fixed and let $G := \begin{pmatrix} x_1/r & x_2/r \\ -x_2/r & x_1/r \end{pmatrix} \in \mathrm{O}(2)$. From (3.7) it follows that

$$\boldsymbol{u}(x, t) = \begin{pmatrix} x_1/r & -x_2/r \\ x_2/r & x_1/r \end{pmatrix} \begin{pmatrix} u_1(r\boldsymbol{e}_1, t) \\ u_2(r\boldsymbol{e}_1, t) \end{pmatrix}, \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.9}$$

Taking $G := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{O}(2)$, using (3.9), we obtain by (3.7) that $u_2(re_1, t) = 0$, which together with (3.9) gives $\boldsymbol{u}(x, t) = x\phi(r, t)$ with $\phi(r, t) := u_1(r\boldsymbol{e}_1, t)/r$.

Case II: $n \geqslant 3$. For $x \in \Omega_1$, let $\tilde{\mathrm{S}}\mathrm{O}_x(n) := \{G \in \mathrm{SO}(n) |\ G\,x = x\}$ denote the set of all rotations about the x-direction. By (3.7) we have

$$\boldsymbol{u}(x, t) = \boldsymbol{u}(\tilde{G}\,x, t) = \tilde{G}\,\boldsymbol{u}(x, t) \qquad \text{for any} \quad \tilde{G} \in \tilde{\mathrm{S}}\mathrm{O}_x(n)\cdot \tag{3.10}$$

By (3.10) we conclude that there is $a(x, t) \in \mathbb{R}$ such that $\boldsymbol{u}(x, t) = a(x, t)x$. Obviously

$$a(x, t) = \frac{\langle \boldsymbol{u}(x), x\rangle_{\mathbb{R}^n}}{|x|^2}$$

for $x \in \mathbb{R}^n$, $x \neq 0$. It follows from (3.7) that for any $G \in \mathrm{SO}(n)$

$$a(G\,x, t) = \frac{\langle \boldsymbol{u}(G\,x, t), G\,x\rangle_{\mathbb{R}^n}}{|G\,x|^2} = \frac{\langle G\,\boldsymbol{u}(x, t), G\,x\rangle_{\mathbb{R}^n}}{|x|^2}$$

$$= \frac{\langle \boldsymbol{u}(x, t), x\rangle_{\mathbb{R}^n}}{|x|^2} = a(x, t), \quad x \in \bar{\Omega}_1,\ t \geqslant 0,$$

which implies that $a(x, t) \equiv \phi(|x|, t) := a(|x|\boldsymbol{e}_1, t)$. Therefore $\boldsymbol{u}(x, t) = x\,\phi(|x|, t)$ for $x \in \bar{\Omega}_1$, $t \geqslant 0$.

It is easy to see that (3.7) implies that $\theta(x, t) =: \psi(|x|, t)$ for $x \in \bar{\Omega}_1$ and $\boldsymbol{v}(x, t) = x\,\eta(|x|, t)$ for $x \in \bar{\Omega}_0$, $x \neq 0$, $t \geqslant 0$. The proof is complete. □

As a consequence of Lemma 3.1 we have the following.

LEMMA 3.2 Let us suppose that $\boldsymbol{u} : \Omega_1 \to \mathbb{R}^n$ is a radially symmetric function satisfying $\boldsymbol{u}|_{\Gamma_1} = 0$. Then there exists a positive constant c such that

$$\|\nabla \boldsymbol{u}(t)\|_{L^2(\Omega_1)} \leqslant c\,\|\mathrm{div}\,\boldsymbol{u}(t)\|_{L^2(\Omega_1)}, \quad t \geqslant 0. \tag{3.11}$$

Moreover we have the following identity at the boundary:

$$|\nabla \boldsymbol{u}(t)|^2 = \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}\right|^2 + \frac{n-1}{r_0}|\boldsymbol{u}|^2 \quad \text{on} \quad \Gamma_0. \tag{3.12}$$

Proof. By hypotheses we can suppose that

$$\boldsymbol{u}(x,t) = x\,\phi(r,t) \quad \text{with} \quad r = |x|,$$

and we find that

$$|\mathrm{div}\,\boldsymbol{u}|^2 = n^2\phi^2(r) + 2nr\,\phi(r)\,\phi'(r) + r^2\,\phi'^2(r).$$

So we obtain

$$\begin{aligned} r^2\phi'^2(r) &= |\mathrm{div}\,\boldsymbol{u}|^2 - n^2\phi^2(r) - 2nr\,\phi(r)\,\phi'(r) \\ &\leqslant |\mathrm{div}\,\boldsymbol{u}|^2 - 2nr\,\phi(r)\,\phi'(r) \\ &\leqslant |\mathrm{div}\,\boldsymbol{u}|^2 + 2n^2\,\phi^2(r) + \tfrac{1}{2}r^2\phi'(r) \end{aligned}$$

which implies that

$$\tfrac{1}{2}r^2\phi'^2(r) \leqslant |\mathrm{div}\,\boldsymbol{u}|^2 + 2n^2\phi^2(r). \tag{3.13}$$

On the other hand, since $r \neq 0$, we can write

$$\phi'(r) + \frac{n}{r}\,\phi(r) = \frac{\mathrm{div}\,\boldsymbol{u}}{r}. \tag{3.14}$$

Multiplying the above equation by $\exp\{\int_{r_1}^r n/r\,\mathrm{d}r\}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\exp\left\{\int_{r_1}^r n/r\,\mathrm{d}r\right\}\,\phi(r) = \exp\left\{\int_{r_1}^r n/r\,\mathrm{d}r\right\}\,\frac{\mathrm{div}\,\boldsymbol{u}}{r},$$

which is equivalent to

$$r^n\,\phi(r) = \int_{r_1}^r r^n\,\frac{\mathrm{div}\,\boldsymbol{u}}{r}\,\mathrm{d}r.$$

Then, by application of Fubini's theorem, there exists a positive constant c such that

$$\int_{r_0}^{r_1}\phi^2(r)\,\mathrm{d}r \leqslant c\int_{\Omega_1}|\mathrm{div}\,\boldsymbol{u}|^2\,\mathrm{d}x. \tag{3.15}$$

From the hypotheses on $\boldsymbol{u}$, (3.13) and (3.15), we find that

$$\begin{aligned} \|\nabla\boldsymbol{u}(t)\|_{L^2(\Omega_1)} &= \omega_n\int_{r_0}^{r_1} r^{n-1}\left[n\,\phi^2(r) + 2r\,\phi(r)\,\phi'(r) + r^2\,\phi'^2(r)\right]\mathrm{d}r \\ &\leqslant \omega_n\int_{r_0}^{r_1} r^{n-1}\left[(n+1)\,\phi^2(r) + 2r^2\,\phi'^2(r)\right]\mathrm{d}r \\ &\leqslant c\,\|\mathrm{div}\,\boldsymbol{u}(t)\|_{L^2(\Omega_1)}, \end{aligned}$$

which proves (3.11).

To show the second part of this lemma, let us note that

$$\frac{\partial \boldsymbol{u}}{\partial x_i}(x,t) = \boldsymbol{e}_i\,\phi(r,t) + x\phi'(r,t)\,\frac{x_i}{r},$$

$$\left|\frac{\partial \boldsymbol{u}}{\partial x_i}(x,t)\right|^2 = \phi^2(r,t) + \frac{2x_i^2}{r}\,\phi(r,t)\,\phi'(r,t) + x_i^2\phi'^2(r,t).$$

Therefore we have

$$|\nabla \boldsymbol{u}(x,t)|^2 = n\,\phi^2(r,t) + 2r\,\phi(r,t)\,\phi'(r,t) + r^2\phi'^2(r,t). \tag{3.16}$$

It is not difficult to see that

$$\frac{\partial \boldsymbol{u}}{\partial x_i}(x,t)\nu_i = \boldsymbol{e}_i\,\phi(r,t)\,\frac{x_i}{|x|} + \frac{x_i^2}{r^2}\,x\,\phi'(r,t).$$

Whence it follows that

$$\frac{\partial \boldsymbol{u}}{\partial \nu}(x,t) = \phi(r,t)\frac{x}{|x|} + x\,\phi'(r,t)$$

which implies that

$$\left|\frac{\partial \boldsymbol{u}}{\partial \nu}(x,t)\right|^2 = \phi^2(r,t) + 2r\,\phi(r,t)\,\phi'(r,t) + r^2\phi'^2(r,t). \tag{3.17}$$

Then (3.16) and (3.17) yield

$$|\nabla \boldsymbol{u}(x,t)|^2 = \left|\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}(x,t)\right|^2 + (n-1)|\phi(r,t)|^2,$$

whence our conclusion follows.

4. Exponential stability

Let us introduce the functionals

$$\mathcal{E}_1(t) = \mathcal{E}_1(\boldsymbol{u},\theta,\boldsymbol{v},t) = \frac{1}{2}\int_{\Omega_1}\left[\rho_1|\boldsymbol{u}_t|^2 + \mu_1|\nabla\boldsymbol{u}|^2 + (\mu_1+\lambda_1)|\mathrm{div}\,\boldsymbol{u}|^2 + \frac{\alpha}{\beta}|\theta|^2\right]\mathrm{d}x$$

$$+\frac{1}{2}\int_{\Omega_0}\left[\rho_0|\boldsymbol{v}_t|^2 + \mu_0|\nabla\boldsymbol{v}|^2 + (\mu_0+\lambda_0)|\mathrm{div}\,\boldsymbol{v}|^2\right]\mathrm{d}x$$

$$\mathcal{E}_2(t) = \mathcal{E}_1(\boldsymbol{u}_t,\theta_t,\boldsymbol{v}_t,t).$$

In the next lemmas we show the dissipative properties of the system (1.1)–(1.10).

LEMMA 4.1 Let us suppose that the initial data $(\boldsymbol{u}_0,\theta_0,\boldsymbol{v}_0)$ is 3-regular; then the corresponding solution of the system (1.1)–(1.10) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) = -\kappa\frac{\alpha}{\beta}\int_{\Omega_1}|\nabla\,\theta|^2\,\mathrm{d}x, \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t) = -\kappa\frac{\alpha}{\beta}\int_{\Omega_1}|\nabla\,\theta_t|^2\,\mathrm{d}x. \tag{4.2}$$

Proof. Multiplying equation (1.1) by $\boldsymbol{u}_t$, equation (1.2) by θ and equation (1.3) by $\boldsymbol{v}_t$, and integrating over the respective intervals, yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_1}\left[\rho_1|\boldsymbol{u}_t|^2+\mu_1|\nabla\boldsymbol{u}|^2+(\mu_1+\lambda_1)|\mathrm{div}\,\boldsymbol{u}|^2\right]\mathrm{d}x\right\}+\alpha\int_{\Omega_1}\nabla\theta\cdot\boldsymbol{u}_t\,\mathrm{d}x$$
$$=\int_{\partial\Omega_1}\left[\mu_1\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}}+(\mu_1+\lambda_1)(\mathrm{div}\,\boldsymbol{u})\boldsymbol{\nu}\right]\cdot\boldsymbol{u}_t\,\mathrm{d}\Gamma \tag{4.3}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_1}|\theta|^2\,\mathrm{d}x\right)+\kappa\int_{\Omega_1}|\nabla\theta|^2\,\mathrm{d}x+\beta\int_{\Omega_1}\theta\,\mathrm{div}\,\boldsymbol{u}_t\,\mathrm{d}x=0 \tag{4.4}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega_0}\left[\rho_0|\boldsymbol{v}_t|^2+\mu_0|\nabla\boldsymbol{v}|^2+(\mu_0+\lambda_0)|\mathrm{div}\,\boldsymbol{v}|^2\right]\mathrm{d}x\right\}$$
$$=\int_{\partial\Omega_0}\left[\mu_0\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\nu}}+(\mu_0+\lambda_0)(\mathrm{div}\,\boldsymbol{v})\boldsymbol{\nu}\right]\cdot\boldsymbol{v}_t\,\mathrm{d}\Gamma. \tag{4.5}$$

Summing up identities (4.3), (4.4) and (4.5) we find (4.1). Differentiating the system with respect to the time t and using the same procedure as for (4.1), we get (4.2). □

Let us define

$$\mathcal{E}_3(t)=(2\mu_1+\lambda_1)E_3(t)+\frac{\rho_1}{\rho_0}(2\mu_0+\lambda_0)\tilde{E}_3(t)$$
$$E_3(t)=\frac{1}{2}\int_{\Omega_1}\left[\rho_1|\nabla\boldsymbol{u}_t|^2+(2\mu_1+\lambda_1)|\Delta\boldsymbol{u}|^2+\frac{\alpha}{\beta}|\nabla\theta|^2\right]\mathrm{d}x$$
$$\tilde{E}_3(t)=\frac{1}{2}\int_{\Omega_0}\left[\rho_0|\nabla\boldsymbol{v}_t|^2+(2\mu_0+\lambda_0)|\Delta\boldsymbol{v}|^2\right]\mathrm{d}x.$$

LEMMA 4.2 Under the same hypotheses as Lemma 4.1 we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_3(t)\leqslant-\frac{\kappa}{2}\frac{\alpha}{\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x$$
$$+\frac{C}{\epsilon^3}\int_{\Omega_1}|\nabla\theta|^2\,\mathrm{d}x+\epsilon\int_{\Gamma_1}|\mathrm{div}\,\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma+\rho_1\,\alpha\int_{\Gamma_0}\theta_t\,\boldsymbol{u}_{tt}\cdot\boldsymbol{\nu}\,\mathrm{d}\Gamma, \tag{4.6}$$

where ϵ and $C=C(\alpha,\mu_1,\lambda_1)$ are positive constants.

Proof. Note that, by virtue of (3.6), $\nabla\mathrm{div}\,\boldsymbol{u}=\Delta\boldsymbol{u}$. Multiplying equation (1.1) by $-(2\mu_1+\lambda_1)\Delta\boldsymbol{u}_t$, equation (1.2) by $-(2\mu_1+\lambda_1)\Delta\theta$ and equation (1.3) by $-(\rho_1/\rho_0)(2\mu_1+\lambda_1)\Delta\boldsymbol{v}_t$, integrating over the respective intervals and summing the product results, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_3(t)=-\kappa\frac{\alpha}{\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x$$
$$-\alpha(2\mu_1+\lambda_1)\int_{\Gamma_1}\frac{\partial\theta}{\partial\boldsymbol{\nu}}\,\mathrm{div}\,\boldsymbol{u}_t\,\mathrm{d}\Gamma+\rho_1\,\alpha\int_{\Gamma_0}\theta_t\,\boldsymbol{u}_{tt}\cdot\boldsymbol{\nu}\,\mathrm{d}\Gamma. \tag{4.7}$$

Using the Cauchy inequality we have

$$\int_{\Gamma_1} \frac{\partial \theta}{\partial \boldsymbol{\nu}} \operatorname{div} \boldsymbol{u}_t \, \mathrm{d}\Gamma \leqslant \frac{1}{4\epsilon} \int_{\Gamma_1} \left| \frac{\partial \theta}{\partial \boldsymbol{\nu}} \right|^2 \mathrm{d}\Gamma + \epsilon \int_{\Gamma_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}\Gamma,$$

and, from trace and interpolation inequalities we obtain

$$\int_{\Gamma_1} \frac{\partial \theta}{\partial \boldsymbol{\nu}} \operatorname{div} \boldsymbol{u}_t \, \mathrm{d}\Gamma \leqslant \frac{C_1}{4\epsilon} \left[\int_{\Omega_1} |\nabla \theta|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} \left[\int_{\Omega_1} |\Delta \theta|^2 \, \mathrm{d}x \right]^{\frac{1}{2}} + \epsilon \int_{\Gamma_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}\Gamma$$
$$\leqslant \frac{C}{\epsilon^3} \int_{\Omega_1} |\nabla \theta|^2 \, \mathrm{d}x + \epsilon \int_{\Omega_1} |\Delta \theta|^2 \, \mathrm{d}x + \epsilon \int_{\Gamma_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}\Gamma.$$

Inserting the above inequality into (4.7), our conclusion follows. □

Define the quantity

$$\mathcal{K}(t) = \int_{\Omega_1} (\theta \operatorname{div} \boldsymbol{u}_t - \delta \, \boldsymbol{u}_t \cdot \Delta \boldsymbol{u}) \, \mathrm{d}x,$$

where $\delta > 0$. Now we have the following.

LEMMA 4.3 With the same hypotheses as Lemma 4.1 the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(t) \leqslant -\frac{\beta}{3} \int_{\Omega_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}x - \frac{\delta(2\mu_1 + \lambda_1)}{4\rho} \int_{\Omega_1} |\Delta \boldsymbol{u}|^2 \, \mathrm{d}x$$
$$+ \left(\delta C_\epsilon + \frac{\alpha}{\rho_1} + \frac{2\mu_1 + \lambda_1}{\delta \, \rho_1} \right) \int_{\Omega_1} |\nabla \theta|^2 \, \mathrm{d}x + \frac{k^2}{2\beta} \int_{\Omega_1} |\Delta \theta|^2 \, \mathrm{d}x$$
$$+ \int_{\Gamma_0} \boldsymbol{u}_{tt} \theta \cdot \boldsymbol{\nu} \, \mathrm{d}\Gamma - \delta \int_{\Gamma_0} \frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}} \cdot \boldsymbol{u}_t \, \mathrm{d}\Gamma, \tag{4.8}$$

where δ and C_ϵ are positive constants.

Proof. Multiplying equation (1.2) by $\operatorname{div} \boldsymbol{u}_t$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_1} \theta \operatorname{div} \boldsymbol{u}_t \, \mathrm{d}x = \int_{\Omega_1} \theta_t \operatorname{div} \boldsymbol{u}_t \, \mathrm{d}x + \int_{\Omega_1} \theta \operatorname{div} \boldsymbol{u}_{tt} \, \mathrm{d}x$$
$$= \kappa \int_{\Omega_1} \Delta \theta \operatorname{div} \boldsymbol{u}_t \, \mathrm{d}x - \beta \int_{\Omega_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}x$$
$$- \int_{\Omega_1} \nabla \theta \cdot \boldsymbol{u}_{tt} \, \mathrm{d}x + \int_{\partial \Omega_1} \theta \, \boldsymbol{u}_{tt} \cdot \boldsymbol{\nu} \, \mathrm{d}\Gamma$$
$$\leqslant \frac{\kappa^2}{2\beta} \int_{\Omega_1} |\Delta \theta|^2 \mathrm{d}x - \frac{\beta}{2} \int_{\Omega_1} |\operatorname{div} \boldsymbol{u}_t|^2 \, \mathrm{d}x + \int_{\Gamma_0} \theta \boldsymbol{u}_{tt} \cdot \boldsymbol{\nu} \, \mathrm{d}\Gamma$$
$$- \frac{2\mu_1 + \lambda_1}{\rho_1} \int_{\Omega_1} \nabla \theta \cdot \Delta \boldsymbol{u} \, \mathrm{d}x + \frac{\alpha}{\rho_1} \int_{\Omega_1} |\nabla \theta|^2 \, \mathrm{d}x. \tag{4.9}$$

Multiplying equation (1.1) by $-\Delta \boldsymbol{u}$ we find that

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_1} \boldsymbol{u}_t\, \Delta \boldsymbol{u}\,\mathrm{d}x &= -\int_{\Omega_1} \boldsymbol{u}_{tt}\cdot \Delta \boldsymbol{u}\,\mathrm{d}x - \int_{\Omega_1} \boldsymbol{u}_t \cdot \Delta \boldsymbol{u}_t\,\mathrm{d}x \\
&= -\frac{2\mu_1+\lambda_1}{\rho_1}\int_{\Omega_1} |\Delta \boldsymbol{u}|^2\,\mathrm{d}x + \frac{\alpha}{\rho_1}\int_{\Omega_1} \nabla\theta\cdot \Delta \boldsymbol{u}\,\mathrm{d}x \\
&\quad + \int_{\Omega_1} |\nabla \boldsymbol{u}_t|^2\,\mathrm{d}x - \int_{\Gamma_0} \frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\cdot \boldsymbol{u}_t\,\mathrm{d}\Gamma \\
&\leqslant -\frac{2\mu_1+\lambda_1}{2\rho_1}\int_{\Omega_1} |\Delta \boldsymbol{u}|^2\,\mathrm{d}x + \frac{\alpha^2}{2\rho_1(2\mu_1+\lambda_1)}\int_{\Omega_1} |\nabla\theta|^2\,\mathrm{d}x \\
&\quad + \int_{\Omega_1} |\nabla \boldsymbol{u}_t|^2\,\mathrm{d}x - \int_{\Gamma_0} \frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\cdot \boldsymbol{u}_t\,\mathrm{d}\Gamma. \qquad (4.10)
\end{aligned}
$$

From Lemma 3.2, there exists a positive constant c such that

$$
\int_{\Omega_1} |\nabla \boldsymbol{u}_t|^2\,\mathrm{d}x \leqslant c\int_{\Omega_1} |\operatorname{div} \boldsymbol{u}_t|^2\,\mathrm{d}x.
$$

Therefore, from (4.9) and (4.10), using the Cauchy inequality, our assertion follows. □

Let us introduce the following functional:

$$
\Phi(t;\boldsymbol{w}) = \int_{\omega} \rho\, \boldsymbol{w}_t \cdot (\boldsymbol{h}\cdot\nabla)\,\boldsymbol{w}\,\mathrm{d}x,
$$

where ω is a symmetric set of $\mathbb{R}^n$.

LEMMA 4.4 Let ω be a radially symmetric set of $\mathbb{R}^n$. Suppose that $\boldsymbol{f} \in H^1(I, L^2(\omega))$ and $\boldsymbol{h} \in [C^2(\bar{\omega})]^3$. Then for any function $\boldsymbol{w} \in H^2(I, L^2(\omega)) \cap L^2(I, H^2(\omega))$ satisfying

$$
\rho\, \boldsymbol{w}_{tt} - b\,\Delta \boldsymbol{w} = \boldsymbol{f}, \qquad (4.11)
$$

where ρ and b are positive constants, we have that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t;\boldsymbol{w}) &= b\int_{\partial\omega} \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}\Gamma + \frac{\rho}{2}\int_{\partial\omega}\sum_{i=1}^{n} h_i\,\nu_i\,|\boldsymbol{w}_t|^2\,\mathrm{d}\Gamma \\
&\quad -\frac{b}{2}\int_{\partial\omega}\sum_{i=1}^{n} h_i\,\nu_i\,|\nabla \boldsymbol{w}|^2\,\mathrm{d}\Gamma - \frac{1}{2}\int_{\omega}\sum_{i=1}^{n}\frac{\partial h_i}{\partial x_i}\left(\rho|\boldsymbol{w}_t|^2 - b|\nabla \boldsymbol{w}|^2\right)\mathrm{d}x \\
&\quad -b\int_{\omega}\nabla \boldsymbol{w}\,\sum_{i=1}^{n}\nabla h_i\,\frac{\partial \boldsymbol{w}}{\partial x_i}\,\mathrm{d}x + \int_{\omega}\boldsymbol{f}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x. \qquad (4.12)
\end{aligned}
$$

Proof. Consider

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi(t;\boldsymbol{w}) &= \rho\int_\omega \boldsymbol{w}_{tt}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x + \rho\int_\omega \boldsymbol{w}_t\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}_t\,\mathrm{d}x \\
&= b\int_\omega \Delta\boldsymbol{w}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x + \int_\omega \boldsymbol{f}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x \\
&\quad +\frac{\rho}{2}\int_{\partial\omega}\sum_{i=1}^n h_i\,\nu_i\,|\boldsymbol{w}_t|^2\,\mathrm{d}\Gamma - \frac{\rho}{2}\int_\omega\sum_{i=1}^n \frac{\partial h_i}{\partial x_i}\,|\boldsymbol{w}_t|^2\,\mathrm{d}x\,\mathrm{d}x \\
&= b\,I_1 + \int_\omega \boldsymbol{f}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x + \frac{\rho}{2}\int_{\partial\omega}\sum_{i=1}^n h_i\,\nu_i\,|\boldsymbol{w}_t|^2\,\mathrm{d}\Gamma \\
&\quad -\frac{\rho}{2}\int_\omega\sum_{i=1}^n \frac{\partial h_i}{\partial x_i}\,|\boldsymbol{w}_t|^2\,\mathrm{d}x\,\mathrm{d}x. \qquad (4.13)
\end{aligned}$$

We find that

$$\begin{aligned}
I_1 &= \int_\omega \Delta\boldsymbol{w}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x = \int_{\partial\omega}\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{\nu}}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}\Gamma - \int_\omega \nabla\boldsymbol{w}\,\nabla(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}x \\
&= \int_{\partial\omega}\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{\nu}}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{w}\,\mathrm{d}\Gamma + I_2.
\end{aligned}$$

We recall that

$$\begin{aligned}
(\boldsymbol{h}\cdot\nabla)\boldsymbol{w} &= \sum_{i=1}^n h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i}, \\
\nabla(\boldsymbol{h}\cdot\nabla)\boldsymbol{w} &= \sum_{i=1}^n\left(\nabla h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i} + h_i\,\frac{\partial\nabla\boldsymbol{w}}{\partial x_i}\right), \\
\nabla\boldsymbol{w}\,\nabla(\boldsymbol{h}\cdot\nabla)\boldsymbol{w} &= \nabla\boldsymbol{w}\,\sum_{i=1}^n\nabla h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n h_i\,\frac{\partial|\nabla\boldsymbol{w}|^2}{\partial x_i}.
\end{aligned}$$

Then

$$\begin{aligned}
I_2 &= -\int_\omega\left[\nabla\boldsymbol{w}\,\sum_{i=1}^n\nabla h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n h_i\,\frac{\partial|\nabla\boldsymbol{w}|^2}{\partial x_i}\right]\mathrm{d}x \\
&= -\int_\omega\nabla\boldsymbol{w}\,\sum_{i=1}^n\nabla h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i}\,\mathrm{d}x - \frac{1}{2}\int_\omega\sum_{i=1}^n h_i\,\frac{\partial|\nabla\boldsymbol{w}|^2}{\partial x_i}\,\mathrm{d}x \\
&= -\int_\omega\nabla\boldsymbol{w}\,\sum_{i=1}^n\nabla h_i\,\frac{\partial\boldsymbol{w}}{\partial x_i}\,\mathrm{d}x + \frac{1}{2}\int_\omega\sum_{i=1}^n \frac{\partial h_i}{\partial x_i}\,|\nabla\boldsymbol{w}|^2\,\mathrm{d}x \\
&\quad -\frac{1}{2}\int_{\partial\omega}\sum_{i=1}^n h_i\,\nu_i\,|\nabla\boldsymbol{w}|^2\,\mathrm{d}\Gamma,
\end{aligned}$$

whence our conclusion follows. □

Let us now introduce the following integrals:

$$\mathcal{J}_1(t) = \int_{\Omega_1} \rho_1\, \boldsymbol{u}_{tt} \cdot (\boldsymbol{q} \cdot \nabla)\, \boldsymbol{u}_t \,\mathrm{d}x,$$

$$\mathcal{J}_2(t) = \int_{\Omega_1} \rho_1\, \boldsymbol{u}_{tt} \cdot (\boldsymbol{h} \cdot \nabla)\, \boldsymbol{u}_t \,\mathrm{d}x,$$

where

$$\boldsymbol{q} \in [C^2(\bar{\Omega}_0 \cup \bar{\Omega}_1)]^3 \quad \text{and} \quad \boldsymbol{q}(x) = \begin{cases} \boldsymbol{\nu} & \text{if } x \in \Gamma_1, \\ 0 & \text{if } x \in \Omega_0 \cup \Omega_1, \end{cases}$$

and

$$\boldsymbol{h} \in [C^2(\bar{\Omega}_0 \cup \bar{\Omega}_1)]^3 \quad \text{and} \quad \boldsymbol{h}(x) = \begin{cases} x & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \in \Omega_1 \backslash \Omega_2, \end{cases} \tag{4.14}$$

and $\Omega_2 := \Omega_0 \cup \left[\cup_{x\in\Gamma_0} B_\varepsilon(x)\right]$, where $B_\varepsilon(x)$ is a ball with centre x and radius ε.

COROLLARY 4.1 With the same hypotheses as Lemma 4.1 the following inequalities hold:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_1(t) \leqslant -k_0 \int_{\Gamma_1} \left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2 \mathrm{d}\Gamma + C_{k_0} \int_{\Omega_1} \left(|\nabla \boldsymbol{u}_t|^2 + |\Delta \boldsymbol{u}|^2 + |\nabla \theta_t|^2\right) \mathrm{d}x, \tag{4.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_2(t) \leqslant -\frac{r_0}{2} \int_{\Gamma_0} \left[(2\mu_1 + \lambda_1)\left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2 + \rho_1 |\boldsymbol{u}_{tt}|^2\right] \mathrm{d}\Gamma + \frac{(2\mu_1 + \lambda_1)}{2r_0} \int_{\Gamma_0} |\boldsymbol{u}_t|^2 \,\mathrm{d}\Gamma + c \int_{\Omega_1} \left[|\nabla \boldsymbol{u}_t|^2 + |\Delta \boldsymbol{u}|^2 + |\nabla \theta_t|^2\right] \mathrm{d}x, \tag{4.16}$$

where k_0, C_{k_0}, c are positive constants, $r_0 = |x|$, $x \in \Gamma_0$.

Proof. We prove (4.16), the other is similar. Using Lemma 4.4, and taking $\boldsymbol{h}$ as in (4.14),

$\boldsymbol{w} = \boldsymbol{u}_t$, $\boldsymbol{f} = \alpha\,\nabla\theta_t$, $\omega = \Omega_1$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_2(t) = (2\mu_1+\lambda_1)\int_\Gamma \frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{u}_t\,\mathrm{d}\Gamma + \frac{\rho_1}{2}\int_\Gamma\sum_{i=1}^n h_i\,\nu_i\,|\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma +$$

$$-\frac{2\mu_1+\lambda_1}{2}\int_\Gamma\sum_{i=1}^n h_i\,\nu_i\,|\nabla\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma$$

$$-\frac{1}{2}\int_{\Omega_1}\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}\left[\rho_1|\boldsymbol{u}_{tt}|^2-(2\mu_1+\lambda_1)|\nabla\boldsymbol{u}_t|^2\right]\mathrm{d}x$$

$$-(2\mu_1+\lambda_1)\int_{\Omega_1}\nabla\boldsymbol{u}_t\,\sum_{i=1}^n\nabla h_i\,\frac{\partial\boldsymbol{u}_t}{\partial x_i}\,\mathrm{d}x - \alpha\int_{\Omega_1}\nabla\theta_t\cdot(\boldsymbol{h}\cdot\nabla)\boldsymbol{u}_t\,\mathrm{d}x.$$

Applying the hypothesis on $\boldsymbol{h}$ and since

$$\boldsymbol{h} = -r_0\,\boldsymbol{\nu},\quad r_0 = |x|,\quad \forall x\in\Gamma_0,$$
$$\boldsymbol{h} = 0,\qquad\qquad \forall x\in\Gamma_1,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_2(t) \leqslant -\frac{r_0}{2}\int_{\Gamma_0}\left[(2\mu_1+\lambda_1)\left|\frac{\partial\boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\right|^2+\rho_1|\boldsymbol{u}_{tt}|^2\right]\mathrm{d}\Gamma$$

$$+\frac{(n-1)(2\mu_1+\lambda_1)}{2r_0}\int_{\Gamma_0}|\boldsymbol{u}_t|^2\,\mathrm{d}x$$

$$+c\int_{\Omega_1}\left[|\boldsymbol{u}_{tt}|^2+|\nabla\boldsymbol{u}_t|^2+|\nabla\theta_t|^2\right]\mathrm{d}x,$$

where we have used Lemma 3.2. Finally, considering equation (1.1), and applying the trace theorem yields

$$\int_{\Gamma_0}|\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma \leqslant C\int_{\Omega_1}|\nabla\boldsymbol{u}_t|^2\mathrm{d}x,$$

with $C > 0$, there exists a positive constant c which proves (4.16). □

Let us now introduce the integrals

$$\mathcal{J}_3(t) = \int_{\Omega_0}\rho_0\boldsymbol{v}_{tt}\cdot(x\cdot\nabla)\,\boldsymbol{v}_t\,\mathrm{d}x$$

$$\Psi(t) = \mathcal{J}_3(t) + \frac{n-1}{2}\int_{\Omega_0}\rho_0\boldsymbol{v}_{tt}\cdot\boldsymbol{v}_t\,\mathrm{d}x.$$

COROLLARY 4.2 Let $\Omega_0 \subset \mathbb{R}^n$, with the same hypotheses as Lemma 4.1. Then the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi(t) = \frac{r_0}{2}\int_{\Gamma_0}\left[(2\mu_0+\lambda_0)\left|\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\right|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}\Gamma$$
$$-\frac{(n-1)(2\mu_0+\lambda_0)}{2r_0}\int_{\Gamma_0}|\boldsymbol{v}_t|^2\,\mathrm{d}x$$
$$+\frac{n-1}{2}(2\mu_0+\lambda_0)\int_{\Gamma_0}\boldsymbol{v}_t\cdot\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\,\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla \boldsymbol{v}_t|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x, \tag{4.17}$$

where $r_0 = |x|$, $x \in \Gamma_0$.

Proof. Differentiating equation (1.3) with respect to t, we have

$$\rho_0\,\boldsymbol{v}_{ttt} = (2\mu_0+\lambda_0)\Delta \boldsymbol{v}_t,$$

and we find that

$$\rho_0\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0}\boldsymbol{v}_t\cdot\boldsymbol{v}_{tt}\,\mathrm{d}x = \rho_0\int_{\Omega_0}|\boldsymbol{v}_{tt}|^2\,\mathrm{d}x + \rho_0\int_{\Omega_0}\boldsymbol{v}_t\cdot\boldsymbol{v}_{ttt}\,\mathrm{d}x$$
$$= \rho_0\int_{\Omega_0}|\boldsymbol{v}_{tt}|^2\,\mathrm{d}x + (2\mu_0+\lambda_0)\int_{\Omega_0}\boldsymbol{v}_t\cdot\Delta\boldsymbol{v}_t\,\mathrm{d}x$$
$$= \rho_0\int_{\Omega_0}|\boldsymbol{v}_{tt}|^2\,\mathrm{d}x + (2\mu_0+\lambda_0)\int_{\Gamma_0}\boldsymbol{v}_t\cdot\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\,\mathrm{d}\Gamma$$
$$-(2\mu_0+\lambda_0)\int_{\Omega_0}|\nabla\boldsymbol{v}_t|^2\,\mathrm{d}x. \tag{4.18}$$

Using Lemma 4.4, and taking $\boldsymbol{h} = x$, $\boldsymbol{w} = \boldsymbol{v}_t$, $\boldsymbol{f} = 0$, $\omega = \Omega_0$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_3(t) = \frac{r_0}{2}\int_{\Gamma_0}\left[(2\mu_0+\lambda_0)\left|\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\right|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}\Gamma$$
$$-\frac{(n-1)(2\mu_0+\lambda_0)}{2r_0}\int_{\Gamma_0}|\boldsymbol{v}_t|^2\,\mathrm{d}x$$
$$+\frac{n-1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla \boldsymbol{v}_t|^2 - \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x$$
$$-\frac{1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla \boldsymbol{v}_t|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x.$$

Therefore, recalling the definition of Ψ, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = \frac{r_0}{2}\int_{\Gamma_0}\left[(2\mu_0+\lambda_0)\left|\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\right|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}\Gamma \\
-\frac{(n-1)(2\mu_0+\lambda_0)}{2r_0}\int_{\Gamma_0}|\boldsymbol{v}_t|^2\,\mathrm{d}x \\
+\frac{n-1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla\boldsymbol{v}_t|^2 - \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x \\
-\frac{1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla\boldsymbol{v}_t|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x \\
+\frac{n-1}{2}\rho_0\int_{\Omega_0}|\boldsymbol{v}_{tt}|^2\,\mathrm{d}x + \frac{n-1}{2}(2\mu_0+\lambda_0)\int_{\Gamma_0}\boldsymbol{v}_t\cdot\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\,\mathrm{d}\Gamma \\
-\frac{n-1}{2}(2\mu_0+\lambda_0)\int_{\Omega_0}|\nabla\boldsymbol{v}_t|^2\,\mathrm{d}x \\
= \frac{r_0}{2}\int_{\Gamma_0}\left[(2\mu_0+\lambda_0)\left|\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\right|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}\Gamma \\
+\frac{n-1}{2}(2\mu_0+\lambda_0)\int_{\Gamma_0}\boldsymbol{v}_t\cdot\frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{\nu}}\,\mathrm{d}\Gamma - \frac{(n-1)(2\mu_0+\lambda_0)}{2r_0}\int_{\Gamma_0}|\boldsymbol{v}_t|^2\,\mathrm{d}x \\
-\frac{1}{2}\int_{\Omega_0}\left[(2\mu_0+\lambda_0)|\nabla\boldsymbol{v}_t|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right]\mathrm{d}x.
\end{aligned}
$$

Hence, our conclusion follows. □

Let us now introduce the integral

$$
\mathcal{Y}(t) = \mathcal{E}_3(t) + \frac{\alpha(2\mu_1+\lambda_1)}{2\kappa}\mathcal{K}(t) + \delta_1\mathcal{J}_1(t) + \delta_2\mathcal{J}_2(t),
$$

where δ_1 and δ_2 are positive constants.

LEMMA 4.5 With the same hypotheses as in Lemma 4.1 the solution of the system

(1.1)–(1.10) satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y}(t) \leqslant& -\frac{\kappa\alpha}{4\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x\\
&-\left(k_1-\delta_1 C_{k_0}-\delta_2 c-\frac{\delta C_P c}{2}\right)\int_{\Omega_1}|\nabla \boldsymbol{u}_t|^2\,\mathrm{d}x-\frac{\delta\alpha(2\mu_1+\lambda_1)^2}{32\kappa\rho}\int_{\Omega_1}|\Delta \boldsymbol{u}|^2\,\mathrm{d}x\\
&-\left[\frac{\delta_2 r_0(2\mu_1+\lambda_1)}{2}-\frac{\delta c}{2}\right]\int_{\Gamma_0}\left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2\mathrm{d}\Gamma-\frac{\delta_1 k_0}{4}\int_{\Gamma_1}\left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2\mathrm{d}\Gamma\\
&-\frac{r_0\rho}{4}\int_{\Gamma_0}|\boldsymbol{u}_{tt}|^2\,\mathrm{d}\Gamma+\frac{\delta_2(2\mu_1+\lambda_1)}{2r_0}\int_{\Gamma_1}|\boldsymbol{u}_t|^2\,\mathrm{d}x+C_\epsilon\int_{\Omega_1}\left(|\nabla\theta|^2+|\nabla\theta_t|^2\right)\mathrm{d}x.
\end{aligned}
\tag{4.19}
$$

Proof. From (4.6) and (4.8) and by the Cauchy inequality, we find that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{E}_3(t)+\frac{\alpha(2\mu_1+\lambda_1)}{2\kappa}\mathcal{K}(t)\right] \leqslant& -\frac{\kappa\alpha}{4\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x\\
&-\frac{\alpha\beta(2\mu_1+\lambda_1)}{6\kappa}\int_{\Omega_1}|\operatorname{div}\boldsymbol{u}_t|^2\,\mathrm{d}x\\
&-\frac{\delta\alpha(2\mu_1+\lambda_1)^2}{8\kappa\rho}\int_{\Omega_1}|\Delta\boldsymbol{u}|^2\,\mathrm{d}x\\
&+C_\epsilon\int_{\Omega_1}\left(|\nabla\theta|^2+|\nabla\theta_t|^2\right)\mathrm{d}x\\
&+\epsilon\int_{\Gamma_0}|\boldsymbol{u}_{tt}|^2\,\mathrm{d}\Gamma+\varepsilon\int_{\Gamma_1}|\operatorname{div}\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma\\
&-\delta\frac{\alpha(2\mu_1+\lambda_1)}{2\kappa}\int_{\Gamma_0}\frac{\partial \boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\cdot\boldsymbol{u}_t\,\mathrm{d}\Gamma,
\end{aligned}
\tag{4.20}
$$

where ϵ and C_ϵ are positive constants. By Lemma 3.2, there exists a positive constant k_1 such that

$$
-\frac{\alpha\beta(2\mu_1+\lambda_1)}{6\kappa}\int_{\Omega_1}|\operatorname{div}\boldsymbol{u}_t|^2\,\mathrm{d}x \leqslant -k_1\int_{\Omega_1}|\nabla\boldsymbol{u}_t|^2\,\mathrm{d}x.
$$

Taking δ_2 and ϵ such that

$$
\delta_2 c \leqslant \frac{\delta\alpha(2\mu_1+\lambda_1)^2}{16\kappa\rho} \quad \text{and} \quad \epsilon < \frac{r_0\rho}{2},
$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{E}_3(t)+\frac{\alpha(2\mu_1+\lambda_1)}{2\kappa}\mathcal{K}(t)+\delta_2\mathcal{J}_2(t)\right]\leqslant-\frac{\kappa\alpha}{4\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x$$
$$-\left(k_1-\delta_2\,c-\frac{\delta\,C_p\,c}{2}\right)\int_{\Omega_1}|\nabla\boldsymbol{u}_t|^2\,\mathrm{d}x-\frac{\delta\alpha(2\mu_1+\lambda_1)^2}{16\kappa\rho}\int_{\Omega_1}|\Delta\boldsymbol{u}|^2\,\mathrm{d}x$$
$$-\left[\frac{\delta_2\,r_0(2\mu_1+\lambda_1)}{2}-\frac{\delta\,c}{2}\right]\int_{\Gamma_0}\left|\frac{\partial\boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\right|^2\,\mathrm{d}\Gamma-\frac{r_0\,\rho}{4}\int_{\Gamma_0}|\boldsymbol{u}_{tt}|^2\,\mathrm{d}\Gamma$$
$$+\frac{\delta_2(2\mu_1+\lambda_1)}{2r_0}\int_{\Gamma_1}|\boldsymbol{u}_t|^2\,\mathrm{d}x+\varepsilon\int_{\Gamma_1}|\mathrm{div}\,\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma+C_\epsilon\int_{\Omega_1}\left(|\nabla\theta|^2+|\nabla\theta_t|^2\right)\,\mathrm{d}x,\tag{4.21}$$

where we have used

$$\int_{\Gamma_0}|\boldsymbol{u}_t|^2\,\mathrm{d}\Gamma\leqslant C_P\int_{\Omega_1}|\nabla\boldsymbol{u}_t|^2\,\mathrm{d}x,$$

with $C_P>0$. Since $\boldsymbol{u}(x,t)=0$ on $\Gamma_1\times]0,\infty[$, we have

$$\int_{\Gamma_1}\left|\frac{\partial\boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\right|^2\,\mathrm{d}\Gamma=\int_{\Gamma_1}|\mathrm{div}\,\boldsymbol{u}_t|^2\,\mathrm{d}x.$$

We find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{E}_3(t)+\frac{\alpha(2\mu_1+\lambda_1)}{2\kappa}\mathcal{K}(t)+\delta_1\mathcal{J}_1(t)+\delta_2\mathcal{J}_2(t)\right]$$
$$\leqslant-\frac{\kappa\alpha}{4\beta}(2\mu_1+\lambda_1)\int_{\Omega_1}|\Delta\theta|^2\,\mathrm{d}x$$
$$-\left(k_1-\delta_1\,C_{k_0}-\delta_2\,c-\frac{\delta\,C_P\,c}{2}\right)\int_{\Omega_1}|\nabla\boldsymbol{u}_t|^2\,\mathrm{d}x$$
$$-\frac{\delta\alpha(2\mu_1+\lambda_1)^2}{32\kappa\rho}\int_{\Omega_1}|\Delta\boldsymbol{u}|^2\,\mathrm{d}x$$
$$-\left[\frac{\delta_2\,r_0(2\mu_1+\lambda_1)}{2}-\frac{\delta\,c}{2}\right]\int_{\Gamma_0}\left|\frac{\partial\boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\right|^2\,\mathrm{d}\Gamma$$
$$-\frac{r_0\,\rho}{4}\int_{\Gamma_0}|\boldsymbol{u}_{tt}|^2\,\mathrm{d}\Gamma-\frac{\delta_1k_0}{4}\int_{\Gamma_1}\left|\frac{\partial\boldsymbol{u}_t}{\partial\boldsymbol{\nu}}\right|^2\,\mathrm{d}\Gamma$$
$$+\frac{\delta_2(2\mu_1+\lambda_1)}{2r_0}\int_{\Gamma_1}|\boldsymbol{u}_t|^2\,\mathrm{d}x+C_\epsilon\int_{\Omega_1}\left(|\nabla\theta|^2+|\nabla\theta_t|^2\right)\,\mathrm{d}x,\tag{4.22}$$

where $\varepsilon<\frac{1}{4}\delta_1k_0$.

Finally, let us introduce the functional

$$\mathcal{L}(t) = N\mathcal{E}_1(t) + N\mathcal{E}_2(t) + \mathcal{Y}(t) + \varepsilon_0\, \Psi(t),$$

where N and ε_0 are positive constants.

THEOREM 4.1 Let us suppose that $(\boldsymbol{u}, \theta, \boldsymbol{v})$ is a strong solution to the system (1.1)–(1.10). Then there exist positive constants c_0 and γ such that

$$\mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t) \leqslant c_0\, \{\mathcal{E}_1(0) + \mathcal{E}_2(0) + \mathcal{E}_3(0)\}\, \mathrm{e}^{-\gamma t}.$$

Proof. We will assume that the initial data is 3-regular. Our conclusion will follow by standard density arguments. Using Lemmas 4.2 and 4.5, considering boundary conditions (1.6), we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{Y}(t) + \varepsilon_0\, \Psi(t)\big] \leqslant{}& -\frac{\kappa\alpha}{4\beta}(2\mu_1 + \lambda_1)\int_{\Omega_1} |\Delta\theta|^2\, \mathrm{d}x \\
&- \left[k_1 - \delta_1\, C_{k_0} - \delta_2\, c - \frac{(\delta + \varepsilon_0)\, C_P\, c}{2}\right]\int_{\Omega_1} |\nabla \boldsymbol{u}_t|^2\, \mathrm{d}x \\
&- \frac{\delta\alpha(2\mu_1 + \lambda_1)^2}{32\kappa\rho}\int_{\Omega_1} |\Delta \boldsymbol{u}|^2\, \mathrm{d}x \\
&- \left[\frac{\delta_2\, r_0(2\mu_1 + \lambda_1)}{2} - \frac{\delta\, c}{2} - \varepsilon_0\right]\int_{\Gamma_0}\left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2 \mathrm{d}\Gamma - \frac{\delta_1 k_0}{4}\int_{\Gamma_1}\left|\frac{\partial \boldsymbol{u}_t}{\partial \boldsymbol{\nu}}\right|^2 \mathrm{d}\Gamma \\
&- \frac{r_0\, \rho}{4}\int_{\Gamma_0} |\boldsymbol{u}_{tt}|^2\, \mathrm{d}\Gamma - \frac{\varepsilon_0}{2}\int_{\Omega_0}\left[(2\mu_0 + \lambda_0)|\nabla \boldsymbol{v}_t|^2 + \rho_0|\boldsymbol{v}_{tt}|^2\right] \mathrm{d}x \\
&+ \frac{\delta_2(2\mu_1 + \lambda_1)}{2r_0}\int_{\Gamma_1} |\boldsymbol{u}_t|^2\, \mathrm{d}x - \frac{\varepsilon_0(n-1)(2\mu_0 + \lambda_0)}{2r_0}\int_{\Gamma_0} |\boldsymbol{v}_t|^2\, \mathrm{d}x \\
&+ C_\epsilon\int_{\Omega_1}\left(|\nabla\theta|^2 + |\nabla\theta_t|^2\right) \mathrm{d}x.\end{aligned} \tag{4.23}$$

From (4.1), (4.2) and (4.23), there exists a positive constant c_0 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -c_0\, \mathcal{N}(t),$$

where

$$\begin{aligned}\mathcal{N}(t) ={}& \int_{\Omega_1}\left(|\boldsymbol{u}_{tt}|^2 + |\nabla \boldsymbol{u}_t|^2 + |\Delta \boldsymbol{u}|^2 + |\nabla\theta|^2 + |\nabla\theta_t|^2\right) \mathrm{d}x \\
&+ \int_{\Omega_0}\left(|\nabla \boldsymbol{v}_t|^2 + |\boldsymbol{v}_{tt}|^2\right) \mathrm{d}x.\end{aligned}$$

Recalling the definition of $\mathcal{L}$ and using the Cauchy inequality, we see that there exists a positive constant c_1 such that

$$\mathcal{L}(t) \leqslant c_1 \mathcal{N}(t).$$

It is not difficult to see that there exists $\gamma > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leqslant -\gamma \mathcal{L}(t)$$

whence

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)\mathrm{e}^{-\gamma t}.$$

Note that for N large enough we have that

$$C_1 \{\mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t)\} \leqslant \mathcal{L}(t) \leqslant C_2 \{\mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t)\}.$$

From the above two inequalities our conclusion follows. □

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