ON IDEALS AND UPPER SETS IN BE-ALGEBRAS

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Abstract. In this paper, the idea of ideals of BE-algebras is introduced and several descriptions of ideals are given in terms of upper sets $A(u,v) = \{z \in X | u*(v*z) = 0\}$ for transitive and for self distributive BE-algebras $(X;*,1)$ of type $(2,0)$.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([8]) introduced the notion of d-algebras which is another generalization of BCK-algebras. Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced the notion of BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x*y=0$ and $y*x=0$ imply $x=y$ for any $x,y \in X$. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. Using the notion of upper sets they provided an equivalent condition describing filters in BE-algebras.

In this paper we continue the study of BE-algebras. In particular we define the proper notion of ideal for this class of algebras. After introducing the notions of transitive and self distributive BE-algebras as rather natural subclasses of interest in terms of an order relation associated with the product-operation of the algebras of this type, we obtain several characterizations of ideals in this setting as unions of special collections of upper sets $A(x,y)$ (defined below) in these algebras.

2. Preliminaries.

We recall some definitions and results (See [5]).

Definition 2.1. An algebra $(X;*,1)$ of type (2,0) is called a BE-*algebra* ([5]) if

(BE1) $x*x=1$ for all $x \in X$;

(BE2) $x*1=1$ for all $x \in X$;

(BE3) $1*x=x$ for all $x \in X$;

(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z \in X$ (*exchange*)

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We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Proposition 2.2. ([5]) *If* $(X; *, 1)$ *is a* BE*-algebra, then* $x * (y * x) = 1$ *for any* $x, y \in X$.

Example 2.3. ([5]) Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.4. A BE-algebra $(X, *, 1)$ is said to be *self distributive* ([5]) if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 2.5. ([5]) Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE-algebra satisfying self distributivity.

Note that the BE-algebra in Example 2.3 is not self distributive, since $d * (a * 0) = d * d = 1$, while $(d * a) * (d * 0) = 1 * a = a$.

3. Main Results.

In what follows let X be denote a BE-algebra unless otherwise specified. We begin by defining the notion of ideals of X.

Definition 3.1. A non-empty subset I of X is called an *ideal* of X if

(I1) $\forall x \in X$ and $\forall a \in I$ imply $x * a \in I$, i.e., $X * I \subseteq I$;

(I2) $\forall x \in X$, $\forall a, b \in I$ imply $(a * (b * x)) * x \in I$.

Example 3.2. In Example 2.3, $\{1, a, b\}$ is an ideal of X, but $\{1, a\}$ is not an ideal of X, since $(a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}$.

Lemma 3.3. *Every ideal of* X *contains* 1.

Proof. Let $I(\neq \emptyset)$ be an ideal of X. There exists $x \in I$. Hence $1 = x * x \in I * I \subseteq X * I \subseteq I$. Thus $1 \in I$. □

Lemma 3.4. *If* I *is an ideal of* X*, then* $(a * x) * x \in I$ *for all* $a \in I$ *and* $x \in X$.

Proof. Let $b := 1$ in (I2). Then $(a * (1 * x)) * x \in I$. Hence $(a * x) * x \in I$. □

Corollary 3.5. *Let I be an ideal of X. If $a \in I$ and $a \leq x$, then $x \in I$.*

Proof. Let $a \in I, x \in X$ with $a \leq x$. Hence $a*x = 1$. Therefore $x = 1*x = (a*x)*x \in I$. Thus $x \in I$. □

Lemma 3.6. *Let I be a subset of X such that*

(I3) $1 \in I$;

(I4) $x*(y*z) \in I$ *and* $y \in I$ *imply* $x*z \in I$ *for all* $x, y, z \in X$.

If $a \in I$ and $a \leq x$, then $x \in I$.

Proof. Let $a \in I$, $a \leq x, x \in X$. Then $1*(a*x) = 1*1 = 1 \in I$. By $a \in I$, (I4), we have $1*x = x \in I$. Thus $x \in I$. □

Definition 3.7. A BE-algebra $(X; *, 1)$ is said to be *transitive* if for any $x, y, z \in X$,

$$y*z \leq (x*y)*(x*z).$$

Example 3.8. Let $X := \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive BE-algebra.

Example 3.9. Let $X := \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	1
c	1	a	c	1

Then X is a BE-algebra. Since $b*a = 1$ and $(c*b)*(c*a) = c*a = a$, X is not transitive.

Proposition 3.10. *If X is a self distributive BE-algebra, then it is transitive.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(y*z)*[(x*y)*(x*z)] &= (y*z)*[x*(y*z)] && \text{[self distributive]} \\ &= x*[(y*z)*(y*z)] && \text{[(BE4)]} \\ &= x*1 && \text{[(BE1)]} \\ &= 1 && \text{[(BE2)]},\end{aligned}$$

proving the proposition. □

The converse Proposition 3.10 need not be true in general. In Example 3.8, X is a transitive BE-algebra, but $a*(a*b) = a*a = 1$, while $(a*a)*(a*b) = 1*a = a$, showing that X is not self distributive.

The following is a characterization of ideals

Theorem 3.11. *Let X be a transitive BE-algebra. A subset $I(\neq \emptyset)$ of X is an ideal of X if and only if it satisfies conditions (I3) and (I4).*

Proof. Let I be an ideal of X. Then $1 \in I$ by Lemma 3.3. Thus (I3) holds. Let $x, y, z \in X$ be such that $x*(y*z) \in I$ and $y \in I$. Using Lemma 3.4, we get $(y*z)*z \in I$. Now, let $\alpha := y*z$, $\beta := z$ in

$$\alpha * \beta \leq (x*\alpha)*(x*\beta).$$

Then $(y*z)*z \leq (x*(y*z))*(x*z)$ and so $[(y*z)*z]*[(x*(y*z))*(x*z)] = 1$. Hence $x*z = 1*(x*z) = [((y*z)*z)*((x*(y*z))*(x*z))]*(x*z) \in I$. Thus $x*z \in I$. Thus (I4) holds.

Conversely, we assume that I satisfies (I3) and (I4). Let $x \in X$, $a \in I$. Then $x*(a*a) = x*1 = 1 \in I$, by (I3). By (I4), $x*a \in I$, i.e., (I1) holds. Let $x \in X, a, b \in I$. Then $(a*x)*(a*x) = 1 \in I$. By (I4), $(a*x)*x \in I$. Now, $((a*x)*x)*((b*(a*x))*(b*x)) = 1$, by Proposition 3.10. Hence $(a*x)*x \leq (b*(a*x))*(b*x)$. Using Lemma 3.6, we have $(b*(a*x))*(b*x) \in I$. Since $b \in I$, by (I4), we obtain $(b*(a*x))*x \in I$. Thus (I2) holds. Therefore I is an ideal of X. □

Corollary 3.12. *Let X be a self distributive BE-algebra. A subset $I(\neq \emptyset)$ of X is an ideal of X if and only if it satisfies conditions (I3) and (I4).*

Proof. The proof follows from Proposition 3.10 and Theorem 3.11 □

For any $u, v \in X$, consider a set

$$A(u,v) := \{z \in X | u*(v*z) = 1\}.$$

We call $A(u,v)$ the *upper set* ([5]) of u and v. In Example 3.2, the set $A(1,a) = \{1, a\}$ is not an ideal of X. Hence we know that $A(u,v)$ may not be an ideal of X in general.

Theorem 3.13. *If X is a self distributive BE-algebra, then $A(u,v)$ is an ideal of X, $\forall u, v \in X$.*

Proof. Let $a, b \in A(u,v)$ and $x \in X$. Then $u*(v*a) = 1$ and $u*(v*b) = 1$. It follows from the self distributivity law that

$$\begin{aligned} u*(v*(x*a)) &= u*[(v*x)*(v*a)] && \text{[self distributive]} \\ &= [u*(v*x)]*[u*(v*a)] && \text{[self distributive]} \\ &= (u*(v*x))*1, && [a \in A(u,v)] \\ &= 1 && \text{[(BE2)]} \end{aligned}$$

whence $x*a \in A(u,v)$. Thus, (I1) holds.

Let $a, b \in A(u, v)$ and $x \in X$. Then $u * (v * a) = 1$ and $u * (v * b) = 1$. It follows from the self distributivity law that

$$\begin{aligned}
u*(v*((a*(b*x))*x)) &= u*[(v*(a*(b*x)))*(v*x)] && \text{[self distributive]}\\
&= [u*(v*(a*(b*x)))]*[u*(v*x)] && \text{[self distributive]}\\
&= [(u*(v*a))*(u*(v*(b*x)))]*[u*(v*x)] && \text{[self distributive]}\\
&= [1*(u*(v*(b*x))]*[u*(v*x)] && [a \in A(u,v)]\\
&= [u*(v*(b*x))]*[u*(v*x)] && \text{[(BE3)]}\\
&= [(u*(v*b))*(u*(v*x)]*[u*(v*x)] && \\
&= (u*(v*x))*(u*(v*x)) && \\
&= 1 && \text{[(BE2)]}
\end{aligned}$$

whence $(a * (b * x)) * x \in A(u, v)$. Thus, (I2) holds. This proves that $A(u, v)$ is an ideal of X. □

Lemma 3.14. *Let X be a BE-algebra. If $y \in X$ satisfies $y * z = 1$ for all $x \in X$, then*

$$A(x, y) = X = A(y, x)$$

for all $x \in X$.

Proof. The proof is straightforward. □

Example 3.15. Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then X is a self distributive BE-algebra. By Lemma 3.14, $A(x, d) = A(d, x) = X$ for all $x \in X$. Furthermore, we have that $A(1, 1) = \{1\}$, $A(1, a) = A(a, 1) = A(a, a) = A(a, b) = \{1, a\}$, $A(1, b) = A(b, 1) = A(b, b) = \{1, b\}$, $A(1, c) = A(a, c) = A(c, 1) = A(c, a) = A(c, c) = \{1, a, c\}$, $A(b, a) = \{1, a, b\}$, and $A(c, b) = X$ are ideals of X.

Using the notion of upper set $A(u, v)$, we given an equivalent condition of the ideal in BE-algebras.

Theorem 3.16. *Let X be a transitive BE-algebra. A subset $I(\neq \emptyset)$ of X is an ideal of X if and only $A(u, v) \subseteq I$, $\forall u, v \in I$.*

Proof. Assume that I is an ideal of X. If $z \in A(u, v)$, then $u * (v * z) = 1$ and so $z = 1 * z = (u * (v * z)) * z \in I$ by (I2). Hence $A(u, v) \subseteq I$.

Conversely, suppose that $A(u, v) \subseteq I$ for all $u, v \in I$. Note that $1 \in A(u, v) \subseteq I$. Let $x, y, z \in I$ with $x*(y*z), y \in I$. Since $(x*(y*z))*(y*(x*z)) = (x*(y*z))*(x*(y*z)) = 1$, we have $x * z \in A(x * (y * z), y) \subseteq I$. By Theorem 3.11, I is an ideal of X. □

Corollary 3.17. *Let X be a self distributive BE-algebra. A subset $I(\neq \emptyset)$ of X is an ideal of X if and only $A(u,v) \subseteq I$, $\forall u, v \in I$.*

Proof. The proof follows from Proposition 3.10 and Theorem 3.16 □

Theorem 3.18. *Let X be a transitive BE-algebra. If I is an ideal of X, then*

$$I = \cup_{u,v\in I} A(u,v).$$

Proof. Let I be an ideal of X and let $x \in I$. Obviously, $x \in A(x,1)$ and so

$$I \subseteq \cup_{x\in I} A(x,1) \subseteq \cup_{u,v\in I} A(u,v).$$

Now, let $y \in \cup_{u,v\in I} A(u,v)$. Then there exist $a, b \in I$ such that $y \in A(a,b) \subseteq I$ by Theorem 3.16. Hence $y \in I$. Therefore $\cup_{u,v\in I} A(u,v) \subseteq I$. This completes the proof. □

Corollary 3.19. *Let X be a self distributive BE-algebra. If I is an ideal of X, then*

$$I = \cup_{u,v\in I} A(u,v).$$

Proof. The proof follows from Proposition 3.10 and Theorem 3.18 □

Corollary 3.20. *Let X be a transitive BE-algebra. If I is an ideal of X, then*

$$I = \cup_{w\in I} A(w,1).$$

Corollary 3.21. *Let X be a self distributive BE-algebra. If I is an ideal of X, then*

$$I = \cup_{w\in I} A(w,1).$$

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