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# Inequalities among Related Triplets of Fibonacci Numbers

**Sanjay Harne[1], Bijendra Singh[2], Gurbeer Kaur Khanuja[2], Manjeet Singh Teeth[3]**

[1]Government Holkar Science College, Indore, M.P., India

[2]School of Studies in Mathematics,Vikram University, Ujjain, M.P., India

[3]M.B. Khalsa College, Indore, M.P., India



## Abstract

In this paper we consider Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=1}^{n}$ defined by where $n$ is a fixed natural number and $F_1, F_2, F_3, \ldots$ are the ordinary Fibonacci numbers.

**Mathematics Subject Classification:** 11B39

**Keywords:** Fibonacci sequence

## INTRODUCTION

"Fibonacci inequalities" have been studied in a variety of contexts**.** Atanassov [3] considered the Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=0}^{n}$ defined by

$m_r = F_r F_{n+1-r}$, where $n$ is a fixed natural number and $F_1, F_2, F_3, .......$ are the ordinary Fibonacci numbers as defined in [4]. We consider the similar Fibonacci inequalities and relate them through the sequence $\{m_r\}_{r=1}^{n}$ defined by $m_r = F_r F_{n+1-r} F_{n+2-r}$, where $n$ is a fixed natural number and $F_1, F_2, F_3, .......$ are the ordinary Fibonacci numbers.

**Theorem**: For every natural number k, the following inequalities for the elements of the sequence $\{m_k\}_{k=1}^{n}$ are valid :

(a) For n = 4k,

(i) $F_1F_{4k}F_{4k+1} > F_3F_{4k-2}F_{4k-1} > ... > F_{2k+1}F_{2k}F_{2k+1} > F_{2k+2}F_{2k-1}F_{2k} > F_{2k+4}F_{2k-3}F_{2k-2} > ... > F_{4k}F_1F_2$

(ii)

$F_2F_{4k-1}F_{4k} > F_4F_{4k-3}F_{4k-2} > .. > F_{2k}F_{2k+1}F_{2k+2} > F_{2k+1}F_{2k}F_{2k+1} > F_{2k+3}F_{2k-2}F_{2k-1} > .. > F_{4k-1}F_2F_3$

(b) For n=4k+1,

(i)

$F_1F_{4k+1}F_{4k+2} > F_3F_{4k-1}F_{4k} > .. > F_{2k+1}F_{2k+1}F_{2k+2} > F_{2k+2}F_{2k}F_{2k+1} > F_{2k+4}F_{2k-2}F_{2k-1} > .. > F_{4k}F_2F_3$

(ii)

$F_2F_{4k}F_{4k+1} > F_4F_{4k-2}F_{4k-1} > .. > F_{2k}F_{2k+2}F_{2k+3} > F_{2k+1}F_{2k+1}F_{2k+2} > F_{2k+3}F_{2k-1}F_{2k} > .. > F_{4k+1}F_1F_2$

(c) For n = 4k+2,

(i)

$F_1F_{4k+2}F_{4k+3} > F_3F_{4k}F_{4k+1} > .... > F_{2k+1}F_{2k+2}F_{2k+3} > F_{2k+2}F_{2k+1}F_{2k+2} > F_{2k+4}F_{2k-1}F_{2k} > .... > F_{4k+2}F_1F_2$

(ii)

$F_2F_{4k+1}F_{4k+2} > F_4F_{4k-1}F_{4k} > .... > F_{2k}F_{2k+3}F_{2k+4} > F_{2k+1}F_{2k+2}F_{2k+3} > F_{2k+3}F_{2k}F_{2k+1} > .... > F_{4k+1}F_2F_3$

(d) For n = 4k+3,

(i)

$F_1F_{4k+3}F_{4k+4} > F_3F_{4k-1}F_{4k+2} > .... > F_{2k+1}F_{2k+3}F_{2k+4} > F_{2k+2}F_{2k+2}F_{2k+3} > F_{2k+4}F_{2k}F_{2k+1} > .... > F_{4k+2}F_2F_3$

(ii)

$F_2F_{4k+2}F_{4k+3} > F_4F_{4k}F_{4k+1} > .... > F_{2k}F_{2k+4}F_{2k+5} > F_{2k+1}F_{2k+3}F_{2k+4} > F_{2k+3}F_{2k+1}F_{2k+2} > .... > F_{4k+3}F_1F_2$

**Proof :** case (a)

$(i)$ $F_1F_{4k+4}F_{4k+5} - F_3F_{4k+2}F_{4k+3}$

$$= F_{4k+4}F_{4k+3} + F_{4k+4}^2 - 2F_{4k+2}F_{4k+3}$$

$$= F_{4k+3}^2 + F_{4k+4}^2 - F_{4k+2}F_{4k+3}$$

$$= F_{4k+4}^2 + F_{4k+3}(F_{4k+3} - F_{4k+2}) > 0$$

Thus, $F_1 F_{4k+4} F_{4k+5} > F_3 F_{4k+2} F_{4k+3}$.

This shows that the inequality (1.1) is valid for i = 1.

$$F_{2i-1}F_{4k-2i+6}F_{4k-2i+7} > F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} \qquad \text{..........(1.1)}$$

Let us assume that, for some $i$, $1 \le i \le k$, the inequality (1.1) is true.

For desired result, we use induction method and prove that the inequality

$$F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} > F_{2i+3}F_{4k-2i+2}F_{4k-2i+3} \qquad \text{..........(1.2)}$$

is also true.

But, $F_{2i+1}F_{4k-2i+4}F_{4k-2i+5} - F_{2i+3}F_{4k-2i+2}F_{4k-2i+3}$

$$= 2F_{4k-2i+3}^2 F_{2i+1} + 3F_{4k-2i+3}F_{4k-2i+2}F_{2i+1} + F_{4k-2i+2}^2 F_{2i+1} - F_{2i+2}F_{4k-2i+2}F_{4k-2i+3} - F_{2i+1}F_{4k-2i+2}F_{4k-2i+3}$$

$$= 2F_{4k-2i+3}^2 F_{2i+1} + F_{4k-2i+2}^2 F_{2i+1} + (2F_{2i+1} - F_{2i+2})F_{4k-2i+2}F_{4k-2i+3}$$

$$= 2F_{4k-2i+3}^2 F_{2i+1} + F_{4k-2i+2}^2 F_{2i+1} + (F_{2i+1} - F_{2i})F_{4k-2i+2}F_{4k-2i+3} > 0$$

Now, $(ii)$ $F_2 F_{4k+3} F_{4k+4} - F_4 F_{4k+1} F_{4k+2}$

$$= (F_{4k+1} + F_{4k+2})(F_{4k+1} + 2F_{4k+2}) - 3F_{4k+1}F_{4k+2}$$

$$= F_{4k+1}^2 + 2F_{4k+2}^2 > 0$$

Thus, $F_2 F_{4k+3} F_{4k+4} > F_4 F_{4k+1} F_{4k+2}$

This shows that the inequality (1.3) is valid for i = 1.

$$F_{2i}F_{4k-2i+5}F_{4k-2i+6} > F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} \qquad \text{..........(1.3)}$$

Let us assume that for some $i$, $1 \le i \le k$ the inequality (1.3) is true.

Then we must prove that the inequality (1.4) is also true.

$$F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} > F_{2i+4}F_{4k-2i+1}F_{4k-2i+2} \qquad \text{..........(1.4)}$$

But, $F_{2i+2}F_{4k-2i+3}F_{4k-2i+4} - F_{2i+4}F_{4k-2i+1}F_{4k-2i+2}$

$$= F_{2i+2}(F_{4k-2i+1} + F_{4k-2i+2})(2F_{4k-2i+2} + F_{4k-2i+1}) - (F_{2i+1} + 2F_{2i+2})F_{4k-2i+1}F_{4k-2i+2}$$

$$= F_{2i+2}(3F_{4k-2i+1}F_{4k-2i+2} + F_{4k-2i+1}^2 + 2F_{4k-2i+2}^2) - (F_{2i+1} + 2F_{2i+2})F_{4k-2i+1}F_{4k-2i+2} > 0$$

This shows that the inequality (1.4) is true and hence (ii) is true.

Particularly, when k = 3, n becomes 12 which gives the following two results:

1) $F_1F_{12}F_{13} > F_3F_{10}F_{11} > F_5F_8F_9 > F_7F_6F_7 > F_8F_5F_6 > F_{10}F_3F_4 > F_{12}F_1F_2$

2) $F_2F_{11}F_{12} > F_4F_9F_{10} > F_6F_7F_8 > F_7F_6F_7 > F_9\,F_4F_5 > F_{11}F_2F_3$

Similarly other cases can be proved.

**Corollary :** For every natural number n the maximal element[1] and [2] of the sequence $\{m_k\}_{k=1}^{n}$ is $F_1F_nF_{n+1}$ and the minimal element is $F_nF_1F_2$.

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