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Optimal Control and Problem with Integral Boundary Conditions

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Abstract

The problems of optimal control are met in many fields of physics, techniques and mechanics, where functional minimization of solutions of non-linear differential equations with non-local boundary conditions is considered. At this paper problems of optimal control with integral boundary conditions are considered. It is known, that mathematically boundary value problems can be given by different ways:Problems with bounded boundary conditions or with conditions in liner points, and also boundary problems with integral conditions, and etc.

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1 Introduction

The problem of optimal control for non-linear system with integral conditions is considered in the given paper i.e. it is considered minimization of the functional:

$$\text{Minimize} \quad \mathrm{J}(u) = \sum_{i=1}^{N} \varphi(x(t_i)) \tag{1}$$

On solutions of the system

$$\dot{x} = f(t, x, u); \quad t \in [t_0, T] \tag{2}$$

Under Non-liner conditions

$$\int_{t_0}^{T} n(t)x(t)dt = A \tag{3}$$

where

$$u = u(0) \in U = \{u(t) \in L_2^r[t_0, T] : u(t) \in V \subset R^r\} \quad a.e. \quad t \in [t_0, T]. \tag{4}$$

Assumed, that $t_0 \leq t_1 < t_2 < \cdots < t_{N-1} < t_N \leq T$ are fixed moments of time, $x \in R^n$ is a phase variable, $u \in R^r$ are controls, $((n(t) - n) \times n)-$dimensional matrix function,$n(t) \in L_\infty^{n\times n}[t_0, T]$, $\tilde{n}(T) = \int_0^t n(\tau)d\tau$, det$\tilde{n}(T) \neq 0$, A is the given n- dimensional vector.
Denote a norm in R^n (or R^r) by, $|.|$ i.e. $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$, and scalar derivative product by $\langle , \rangle$, i.e. $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$.
Let's denote the norm and scalar space $L_2^r[t_0, T]$ by $\|.\|$ and $(., .)$ respectively, i.e. $\|u\| = (\int_{t_0}^T |u(t)|^2dt)^{\frac{1}{2}}$, $(u, v) = \int_{t_0}^T \langle u(t), v(t)\rangle dt.$
Suppose that elements of matrix $n(t)$ are piece wise continuous and det $\int_{t_0}^T n(t)dt$ $\neq 0$. Note, that if the condition $\|\det \int_{t_0}^T n(t)dt\| < 1$ holds, then matrix $\int_{t_0}^T n(t)dt$ is reversible.

2 Main assumptions

Condition 1 *Let the function $f(t, x, u)$ be continuous by t, x, u for $x \in R^n$, $u \in V$, $t_0 \leq t \leq T$ and derivatives $f(t, x, u)$ with respect to x exist, continuous and bounded i.e. $|\nabla_x f(t, x, u)y| \leq k_1|\overline{y}|$ Here $\nabla_x f(t, x, u)y$ is a matrix of $(n \times n)$ with elements $\frac{\partial f_i(t,x,u)}{\partial x_j}$ (used below notation $\nabla_x\varphi$), y is an arbitrary vector from R^n. It is easy to see, that at the made assumptions the solutions of problems (2), (3) are equivalent to solution of integral equation*

$$x(t) = \tilde{n}^{-1}(T)A - \int_t^T f(\tau, x(\tau), u(\tau))d\tau \\ -\tilde{n}^{-1}(T) \int_{t_0}^T \int_{t_0}^t n(\tau)d\tau f(t, x(t), u(t))dt, \tag{5}$$

we can show with the help of sequential approximations method, that at

$$k_1(T - t_0)\lfloor 1 + \tilde{n}_1(T)|\tilde{n}^{-1}(T)|\rfloor < 1, \tag{6}$$

the integral equation (5) has a unique solution at each fixed $u \in U$, where $\tilde{n}_1(T) = \max_{[0,T]} |\int_0^t n(\tau)d\tau|$. Sequential approximations are constructed by formulae:

$$x^{n+1}(t) = \tilde{n}^{-1}(T)A - \int_t^T f(\tau, x^n(\tau), u(\tau))d\tau \\ -\tilde{n}^{-1}(T) \int_{t_0}^T \int_{t_0}^t n(\tau)d\tau f(t, x^n(t), u(t))dt; \quad n = 0, 1, 2, \ldots, \quad x^0 = \tilde{n}^{-1}(T)A.$$

In the future well's suppose, that the condition(6) is always fulfilled. Let's denote, that at made above assumptions any $x(t)$ solution of (1)-(2) is also bounded

$$|x(t)| \leq \frac{|\tilde{n}^{-1}(T)A| + k_1(T-t_0)\lfloor|\tilde{n}^{-1}(T)|\tilde{n}_1(T)+1\rfloor}{1-k_1(T-t_0)(1+|\tilde{n}^{-1}(T)|\tilde{n}_1(T))},$$

where $|f(t, 0, u)| \leq k$.

Condition 2 *The function* $\varphi(x)$ *is differentiable and* $|\nabla_x \varphi(x)| \leq k_2$.

Condition 3 *The derivatives* $f(t,x,u)$ *with respect to* u *are bounded:*

$$|\nabla_u f(t,x,u)\overline{u}| \leq k_3(\overline{u}).$$

Condition 4 *The derivatives* $\nabla_x \varphi(x)$ *satisfy the Lipcshits condition:*

$$|\varphi(x+\overline{x}) - \varphi(x) - \langle \nabla_x \varphi(x), \overline{x} \rangle| \leq k_4 |\overline{x}|^2.$$

Condition 5 *The derivatives* $f(t,x,u)$ *with respect to* x *and* u *satisfy the lipcshits conditions, i.e.* $(x(t),u(t))$ *and* $(x(t)+\overline{x}(t), u(t)+\overline{u}(t))$

$$s.t: |f(t,x+\overline{x},u+\overline{u}) - f(t,x,u) - \nabla_x f(t,x,u)\overline{x} - \nabla_x f(t,x,u)\overline{u}| \leq k_5|\overline{x}|^2 + k_6|\overline{u}|^2.$$

Condition 6 *The set* V *is close and convex.*

3 Main Theorem

Theorem 3.1 *Let the conditions* $1,2,5$ *and (6) be fulfilled and* $u(t), x(t)$ *and* $u(t)+\bar{u}(t)$, $x(t)+\bar{x}(t)$, *be two solutions of system (2), (3). Then*

$$|\bar{x}(t)| \leq c_1 \|\bar{u}\|,$$

where $c_1 = \frac{k_3 \lfloor 1+|\tilde{n}^{-1}(T)|\tilde{n_1}(T) \rfloor (T-t_0)}{1-k_1(T-t_0)(1+|\tilde{n}^{-1}(T)|\tilde{n_1}(T))}$.

Proof. Clearly, from (5) we get

$$\begin{aligned}&\bar{x}(t) = -\int_t^T [f(\tau, x(\tau), u(\tau)+\bar{u}(\tau)) - f(\tau, x(\tau), u(\tau))]dt \\ &-\tilde{n}^{-1}(T)\int_{t_0}^T \int_{t_0}^t n(\tau)d\tau [f(\tau, x(t)+\bar{x}(t), u(t)+\bar{u}(t)) - f(t, x(t), u(t))]dt.\end{aligned}$$

Using the conditions 3 and the elements of matrix $n(t)$ are continuous on $[t_0, T]$ and $\det\tilde{n}(T) \neq 0$, where $\tilde{n}(T) = \int_0^T n(t)dt$, we get

$$\begin{aligned}&|\bar{x}(t)| \leq k_1 \int_{t_0}^T |\bar{x}(\tau)|d\tau + k_3 \int_t^T |\bar{u}(\tau)|d\tau \\ &+|\tilde{n}^{-1}(T)|\tilde{n_1}(T) \times (k_1 \int_{t_0}^T |\bar{x}(\tau)|d\tau + k_3 \int_{t_0}^T |\bar{u}(\tau)|d\tau).\end{aligned}$$

Taking into account (6) we have

$$\int_{t_0}^T |\bar{x}(t)|dt \leq \frac{k_3(T-t_0)\lfloor 1+|\tilde{n}^{-1}(T)|\tilde{n_1}(T)\rfloor}{1-k_1(T-t_0)(1+|\tilde{n}^{-1}(T)|\tilde{n_1}(T))} \times \int_{t_0}^T |\bar{u}(t)|dt. \tag{7}$$

From (6) and (7) it follows the statement of theorem (3.2).
Let's introduce the system of equations in variations:

$$\dot{y} = \nabla_x f(t, x(t), u(t))y - f(t, x(t), u(t))\bar{u}(t)$$

$$\int_{t_0}^T n(t)y(t)dt = 0. \tag{8}$$

Theorem 3.2 *Let the conditions* $1-6$ *be fulfilled. Then the functional (1) at restrictions (2)-(4) is differentiable and its gradient has the form*

$$J'(u) = (\nabla_x f(t, x(t), u(t)))'\psi(t) \in L_2[t_0, T],$$

where $\psi(t)$ *is solution of the adjoint system*

$$\begin{array}{l} \psi(t) = -\int_{t_0}^{t} \nabla_x H(\tau, x(\tau), u(\tau), \psi(\tau))d\tau \\ +(\tilde{n}^{-1}(T)\int_{t_0}^{t} n(\tau)d\tau)' \int_{t_0}^{T} \nabla_x H(t, x(t), u(t), \psi(t))dt \\ +\sum_{i=1}^{N}[E_x(t-t_i) - \tilde{n}^{-1}(T)\int_{t_0}^{t} n(\tau)d\tau]' \times \nabla_x \psi(x(t_i)). \end{array}$$

Proof. Let $(x(t), u(t))$ and $u(t)+\overline{u}(t)$, $x(t)+\overline{x}(t)$ be two solutions of system (2), (3). At this solutions increment of functional (1) is of the form :

$$\mathrm{J}(u+\overline{u}) - \mathrm{J}(u) = \sum_{i=1}^{N}(\varphi(x(t_i)+\overline{x}(t_i)) - \varphi(x(t_i))) = \sum_{i=1}^{N}\langle \nabla_x \varphi(x(t_i)), y(t_i)\rangle + \tau,$$

where

$$\tau = \sum_{i=1}^{N}[\varphi(x(t_i)+\overline{x}(t_i)) - \varphi(x(t_i)) - \langle \nabla_x \varphi(x_i), \overline{x}(t_i)\rangle] + \sum_{i=1}^{N}\langle \nabla_x \varphi(x(t_i), \overline{x}(t_i)) - y(t_i)\rangle,$$

then

$$\begin{array}{l} \mathrm{J}(u+\overline{u}) - \mathrm{J}(u) = -\int_{t_0}^{T}\langle \nabla_x H(t, x(t), u(t), \psi(t)), y(t)\rangle dt \\ -\int_{t_0}^{T}\langle \nabla_u H(t, x(t), u(t), \psi(t)), y(t)\rangle dt \\ -\int_{t_0}^{T}\langle \nabla_u H(t, x(t), u(t), \psi(t)), \overline{u}\rangle dt \\ +\sum_{i=1}^{N}\langle \nabla_x \varphi(x(t_i)), y(t_i)\rangle + \int_{t_0}^{T}\langle \psi(t), \dot{y}(t)\rangle dt + \tau \end{array} \tag{9}$$

where

$$\begin{array}{c} \dot{y} = \nabla_x(t, x(t), u(t))y - f(t, x(t), u(t))\overline{u}(t), \\ \int_{t_0}^{T} n(t)y(t)dt = 0. \end{array} \tag{10}$$

Then, using the second equality (10) we have:

$$y(t) = \tilde{n}^{-1}(T)\int_{t_0}^{T}\int_{t_0}^{t} n(\tau)d\tau\dot{y}(t)dt, \tag{11}$$

$$y(t_i) = \int_{t_0}^{T}[\tilde{n}^{-1}(T)\int_{t_0}^{t} n(\tau)d\tau - Ex(t-t_i)]\dot{y}(t)dt, \tag{12}$$

where E is $n \times n$-dimensional unique matrix and $x(t-t_i) = \begin{cases} 0 & t \le t_i, \\ 1 & t > t_i. \end{cases}$

Carry out the following equivalent transformation:

$$\begin{array}{c} \int_{t_0}^{T}\langle \nabla_x H(t, x(t), u(t), \psi(t)), y(t)\rangle dt = \\ \langle \int_{t_0}^{T} \nabla_x H(t, x(t), u(t), \psi(t))dt, y(T)\rangle \\ -\int_{t_0}^{T}\langle \int_{t_0}^{t} \nabla_x H(\tau, x(\tau), u(\tau), \varphi(\tau))d\tau, \dot{y}(t)\rangle dt, \end{array} \tag{13}$$

where $H(t, x(t), u(t), \ldots) = \langle \psi(t), f(t, x(t), u(t) \rangle$.
Taking into account (11)-(13) in (9) for increment of functional (1), we get the following expression:

$$\begin{aligned} &\mathrm{J}(u+\overline{u}) - \mathrm{J}(u) = \textstyle\int_{t_0}^{T} \langle \int_{t_0}^{t} \nabla_x H(\tau, x(\tau), \psi(\tau)) d\tau, \dot{y}(t) \rangle dt \\ &-\langle \textstyle\int_{t_0}^{T} \nabla_x H(t, x(t), u(t), \psi(t)) dt, \tilde{n}^{-1}(T) \int_{t_0}^{T} \int_{t_0}^{t} n(\tau) d\tau, \dot{y}(t) dt \rangle \\ &+ \textstyle\sum_{i=1}^{N} \langle \nabla_x \varphi(x(t_i)), \int_{t_0}^{T} [\tilde{n}^{-1}(T) \int_{t_0}^{t} n(\tau) d\tau - E_x(t-t_i)] \dot{y}(t) \rangle dt \\ &+ \textstyle\int_{t_0}^{T} \langle \psi(t), \dot{y}(t) \rangle dt - \int_{t_0}^{T} \langle \nabla_u H(t, x(t), \varphi(t)), \bar{u}(t) \rangle dt + \tau \end{aligned} \tag{14}$$

$$\mathrm{J}(u+\overline{u}) - \mathrm{J}(u) = \int_{t_0}^{T} \langle \nabla_u H(t, x(t), u(t), \psi(t), \overline{u}(t) \rangle dt + \tau$$

It is easy to show, that there exists a finite member $c > 0$

$$|\tau| \leq c \|\overline{u}^2\|.$$

This means, the theorem is proved i.e., that the functional (1) at restrictions (2)-(4) is differentiable.

Theorem 3.3 *Let all conditions of theorem 3.2 be fulfilled. For optimality of the control $u^\star = u^\star(t) \in U$ it is necessary the fulfillment of the equality*

$$\int_{t_0}^{T} \langle \nabla_u H(t, x(t, u), u^\star(t), \psi(t, u^\star)), u(t) - u^\star(t) \rangle dt \leq 0, \tag{15}$$

at all $u(t) \in V$. If $u^\star = u^\star(t)$ is an inner point of the set V, then condition (15) is equivalent to the condition $\nabla_u H(t, x(t, u^\star), u^\star(t), \psi(t, u^\star)) = 0; t_0 \leq t \leq T$. Here $x(t, u^\star)$ and $\psi(t, u^\star)$ are solutions of problem (2)-(3) and adjoint problem corresponding to the control $u^\star \in V$. The proof of this theorem is carried with the help of the method from [1], (see pp. 524)

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