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On global properties of passivity-based control of an inverted pendulum

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SUMMARY

The paper adresses the problem of stabilization of a specific target position of underactuated Lagrangian or Hamiltonian systems. We propose to solve the problem in two steps: first to stabilize a set with the target position being a limit point for all trajectories originating in this set and then to switch to a locally stabilizing controller. We illustrate this approach by the well-known example of inverted pendulum on a cart. Particularly, we design a controller which makes the upright position of the pendulum and zero displacement of the cart a limit point for almost all trajectories. We derive a family of static feedbacks such that any solution of the closed loop system except for those originating on some two-dimensional manifold approaches an arbitrarily small neighbourhood of the target position. The proposed technique is based on the passivity properties of the inverted pendulum. A possible extension to a more general class of underactuated mechanical systems is discussed. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: passivity; dissipativity; stabilization of the inverted pendulum

1. INTRODUCTION

The inverted pendulum is an ubiquitous example of nonlinear control systems analysis and design [1–8]. It is a popular experiment used for educational purposes, and belongs to a class of underactuated mechanical systems. In this paper using this example we demonstrate a general approach to stabilization of such systems.

Mathematical models of mechanical systems are usually described by the Hamiltonian or Euler–Lagrange equations. These models have a structure that makes them very attractive for the design of control algorithms based on energy or passivity consideration. This approach has already proved to be effective in the control design for mechanical and electro-mechanical systems [9]. In this paper this method is utilized to investigate the global properties of the designed controller.

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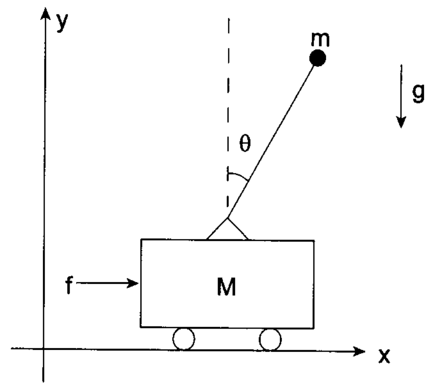

Figure 1. The cart pendulum system.

The problem of the local stabilization of the inverted pendulum on the cart in the upright position was solved a long time ago, see [1]. A possible approach is based on the linearization of the system equation around the desired equilibrium point. However, the problem becomes more ambitious if one wants to investigate non-local behaviour of the closed-loop system. It can be shown that there is no continuous time-invariant controller which can solve the stabilization problem for all initial conditions and hence it is reasonable to look for a globally stabilizing controller in the class of *hybrid systems.*

Such a controller can be designed if one finds a feedback which makes the desired equilibrium point an ω-limit point for all solutions of the closed-loop system. This problem is the subject of the paper. Loosely speaking, our goal is to design a controller such that all (or almost all) solutions of the closed-loop system approach an arbitrary vicinity of the target position (zero displacement of the cart and upright pendulum position(Figure 1)), where such controller can be switched to any locally stabilizing one.

Although the problem considered in this paper is a kind of 'toy example' since any friction and disturbances are neglected, we hope that a proposed solution gives a deep insight of how to perform a nonlocal analysis in the complex mechanical underactuated systems. In Section 4 the possible extension of the main ideas to a class of Lagrange systems, that are invariant with respect to some group action, is discussed.

The paper is organized as follows. Section 2 contains the description of the problem and some known material. The main results of the paper are collected in Sections 3 and 4. Simulation of the obtained results are presented in Section 5 and conclusions are drawn in Section 6.

2. PRELIMINARIES

Under the standard assumptions of a massless rod, point masses, no friction, etc., the equations of the inverted pendulum motion are

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where $q = [x, \theta]^{\mathrm{T}} \in R^1 \times S^1$, x is the horizontal displacement of the cart, θ is the angle between the pendulum rod and the vertical which is zero at the upright position:

$$M(q) = \begin{bmatrix} M + m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix} \tag{2}$$

m, M are the masses of the pendulum and the cart, respectively; l is the length of the rod;

$$C(q, \dot{q}) = \begin{bmatrix} 0 & -ml\sin\theta\,\dot{\theta} \\ 0 & 0 \end{bmatrix} \tag{3}$$

$$G(q) = \begin{bmatrix} 0 \\ -mgl\sin\theta \end{bmatrix}, \quad \tau = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{4}$$

where f is the control input to be defined.

Consider the following problem: *to design a time-invariant static feedback which globally stabilizes the upright equilibrium of the pendulum with zero displacement of the cart.*

It can be shown that this problem has no solution in a class of continuous static feedbacks which are periodic in θ. Indeed, due to time invariance and the periodicity in θ of the feedback the phase space of the closed-loop system remains cylindrical and the closed-loop system always has at least one more equilibrium point additional to the upright position. On the other hand, using the linear approximation of the system around the upright equilibrium it is well known how to construct linear locally stabilizing controller [1]. These facts show that it is natural to devide the problem into parts: (1), to drive any solution to an arbitrary neighbourhood of the upright equilibrium with zero displacement of the cart; (2) to stabilize this point by a local controller.

The first subproblem is the main subject of the paper. The following simple statement gives a foundation for its solution.

Proposition

Consider a subset Ω_0 of the cylindrical phase space defined by equations

$$E(q, \dot{q}) = 0, \quad x = 0, \quad \dot{x} = 0$$

where

$$E(q, \dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q} + mgl(\cos\theta - 1) \tag{5}$$

is the total energy of the unforced (with $f = 0$) inverted pendulum (1). Then for any continuous time-invariant feedback control, which locally stabilizes the set Ω_0, the upright equilibrium of the pendulum with zero displacement of the cart is an ω-limit point of any solution from the basin of attraction of Ω_0.

To prove this fact, one can remark that the constraint $\dot{x} = 0$ obviously implies $\ddot{x} = 0$, and that the last relation uniquely defines, see (1), the value of the control f on the set Ω_0 as follows:

$$f = m\sin\theta\,(g\cos\theta - l\dot{\theta}^2)$$

Substituting this control in the $\ddot{\theta}$-part of Equations (1), one has

$$\ddot{\theta} = \frac{(M+m)mgl\sin\theta - m^2l^2\dot{\theta}^2\sin\theta\cos\theta - ml\cos\theta\cdot f}{ml^2(M+m\sin^2\theta)}$$

$$= \frac{g}{l}\sin\theta$$

This is the equation for the motion of the free pendulum with the fixed pivot.

The constraint $E(q, \dot{q}) = 0$ singles out three special motions of the simple pendulum: the upright equilibrium and two homoclinic curves. All of these solutions have the upright position as the unique ω-limit point. Therefore, any solution of the closed-loop system with initial conditions, lying in the area of attraction of the set Ω_0, has this equilibrium as an ω-limit point (probably not unique).

Thus the solution of the first subproblem (stabilization of the set Ω_0) is important and leads to the solution of the original problem. Introduce the function

$$V(q,\dot{q}) = \frac{k_E}{2}E(q,\dot{q})^2 + \frac{k_v}{2}\dot{x}^2 + \frac{k_x}{2}x^2 \tag{6}$$

where k_E, k_v, k_x are some positive constants. It is obvious that V is non-negative and that the set $\{(q,\dot{q})\colon V(q,\dot{q}) = 0\}$ coincides with the desired goal set Ω_0. Straightforward computations show that the derivative of V along any solution of (1) is

$$\dot{V} = \dot{x}\left[f\left(k_E E(q,\dot{q}) + \frac{k_v}{1+\sin^2\theta}\right) + \frac{k_v\sin\theta(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x x\right] \tag{7}$$

for simplicity it will be assumed that $M = m = l = 1$.

To assure the stabilization of the set Ω_0 the function V should decrease (not increase) along the closed-loop system solutions. Particularly, this is true if the feedback control law f satisfies the relation

$$f(q,\dot{q})\left(k_E E(q,\dot{q}) + \frac{k_v}{1+\sin^2\theta}\right) + \frac{k_v\sin\theta\cdot(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x\cdot x = -\phi(q,\dot{q}) \tag{8}$$

where ϕ is some scalar function forming an acute angle with $\dot{x}$, i.e.

$$\dot{x}\phi(q,\dot{q}) > 0 \quad \forall\, \dot{x} \neq 0, \quad \forall\, x, \theta, \dot{\theta}$$

In this case $\dot{V}$ is negative along the closed-loop system solutions.

It is worth mentioning that Equation (8) means that the closed-loop system is passive from ϕ to $\dot{x}$ and the problem of feedback design in this case is equivalent to the problem of state feedback passification. We focus our attention to C^1-smooth functions $\phi(q,\dot{q})$ depending only on $\dot{x}$, i.e. $\phi(q,\dot{q}) = \phi(\dot{x})$. In particular, one of these regulators, with $\phi(z) = z$, was used in Reference [8]. Denote the set

$$\mathscr{F} = \{(k_E, k_v, k_x, \phi(\cdot))\colon k_E > 0,\ k_v > 0,\ k_x > 0 \text{ and } \phi(z) \text{ is a } C^1\text{-smooth function forming an acute angle with } z \text{ and such that } \dot{\phi}(0) > 0\} \tag{9}$$

Any element in $\mathscr{F}$ corresponds to some feedback controller f.

3. MAIN RESULTS

To establish the global properties of the controller $f(q, \dot{q})$ implicitly defined in (8), one should provide the criterion of solvability of Equation (8) with respect to the control variable f. The next statement contains such a condition.

Proposition 1

Let k_E, k_v be any positive constants. The inequality

$$k_E\, E(x_1, x_2, \theta_1, \theta_2) + \frac{k_v}{1 + \sin^2 \theta_1} \neq 0, \quad \forall\, x_1, x_2, \theta_1, \theta_2 \tag{10}$$

holds if and only if the constants k_E, k_v satisfy the inequality

$$k_v > \rho g k_E \tag{11}$$

where

$$\rho = \frac{34 + 14\sqrt{7}}{27} \tag{12}$$

Moreover, inequality (11) guarantees that the left-hand side of (10) is a positive function.

Proof. Denote

$$\mathscr{E} = k_E\, E(x_1, x_2, \theta_1, \theta_2) + \frac{k_v}{1 + \sin^2 \theta_1}$$

and consider the equation $\mathscr{E} = 0$. This is equivalent to

$$k_v - k_E(1 - \cos\theta_1)(1 + \sin^2\theta_1)\, g + k_E(1 + \sin^2\theta_1)(\alpha_1 + \alpha_2) = 0$$

where

$$\alpha_1 = (x_2 + \tfrac{1}{2}\theta_2 \cos\theta_1)^2, \quad \alpha_2 = \tfrac{1}{2}\theta_2^2(1 - \tfrac{1}{2}\cos^2\theta_1)$$

Define the value of θ_1 where the function

$$F(\theta_1) = (1 - \cos\theta_1)(1 + \sin^2\theta_1)$$

attains its maximal value. Taking the derivative of F, we have

$$F'(\theta_1) = \sin\theta_1(2 + 2\cos\theta_1 - 3\cos^2\theta_1).$$

Solving the equation $F'(\theta_1) = 0$ we find that

$$\theta_1^* = \mathrm{argmax}_\theta F(\theta) = \pm \arccos\left(\frac{2 - \sqrt{28}}{6}\right) \quad \text{and} \quad F(\theta_1^*) = \frac{34 + 14\sqrt{7}}{27}$$

Thus for any x_i, θ_i, $i = 1, 2$

$$\begin{aligned} \mathscr{E} \times (1 + \sin^2\theta_1) &= k_v - k_E\, g\, F(\theta_1) + k_E(1 + \sin^2\theta_1)(\alpha_1 + \alpha_2) \\ &\geqslant k_v - k_E\, g\, F(\theta_1^*) \\ &= k_v - k_E\, g\, \rho \end{aligned}$$

By (11) the last expression is positive. Thus inequality (11) implies (10). Suppose that (11) is not valid. Then it is obvious that there exists values of x_i, θ_i, $i = 1, 2$ which solve the equation $\mathscr{E} = 0$. This ends the proof.

Thus, due to Proposition 1, if inequality (11) holds, Equation (8) is globally solvable and $f(q(t), \dot{q}(t))$ is bounded provided that $(q(t), \dot{q}(t))$ is bounded. Denote $\mathscr{F}_\rho$ as a subset of elements of $\mathscr{F}$ for which inequality (11) holds. We will write $f \in \mathscr{F}_\rho$ having in mind that f is uniquely defined by the controller given by Equation (8), corresponding to some point in $\mathscr{F}_\rho$. The following simple statement contains some qualitative result on the system behaviour even for the case when Equation (8) cannot be solved.

Proposition 2

Let k_E, k_v be any positive constants. Suppose that the equality

$$k_E E(q(t), \dot{q}(t)) + \frac{k_v}{1 + \sin^2 \theta(t)} = 0, \quad \forall\, t \in \mathscr{T} \tag{13}$$

is valid for some time interval $\mathscr{T}$. Then the functions $\dot{x}(t)$, $\dot{\theta}(t)$ are uniformly bounded on $\mathscr{T}$.

Proof. Indeed, equality (13) is equivalent to

$$g(\cos\theta - 1) + \frac{k_v}{k_E(1 + \sin^2\theta)} = -\left((\dot{x} + \tfrac{1}{2}\dot{\theta}\cos\theta)^2 + \tfrac{1}{2}\dot{\theta}^2(1 - \tfrac{1}{2}\cos^2\theta)\right)$$

This implies that the sum of $(\dot{x} + \frac{1}{2}\dot{\theta}\cos\theta)^2$ and $\frac{1}{2}\dot{\theta}^2(1 - \frac{1}{2}\cos^2\theta)$ is uniformly bounded from below and above. But both items are positive so the value of $\dot{\theta}^2(1 - \frac{1}{2}\cos^2\theta)$ is bounded from below and above and, in particular, $|\dot{\theta}|$ is uniformly bounded. Therefore, $|\dot{x}|$ is also uniformly bounded. □

One can easily verify that, under the assumption $M = m = l = 1$, the system (1) can be rewritten in the equivalent form

$$\ddot{x} = \frac{1}{1 + \sin^2\theta}[\sin\theta(\dot{\theta}^2 - g\cos\theta) + f] \tag{14}$$

$$\ddot{\theta} = \frac{1}{1 + \sin^2\theta}[-\dot{\theta}^2\sin\theta\cos\theta + 2g\sin\theta - f\cos\theta] \tag{15}$$

One of the main results of this paper is:

Theorem 3

Consider the controlled inverted pendulum (1). Take any state feedback controller $f \in \mathscr{F}_\rho$ defined by Equation (8) with appropriate parameters $\{k_E, k_v, k_x, \phi(\cdot)\}$. Then

(1) any solution of the closed-loop system is globally well defined and bounded;
(2) for any solution $[q(t), \dot{q}(t)]$ of the closed-loop system, its ω-limit set Ω_* consists of either the equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, 0, 0, 0,] \tag{16}$$

the equilibrium

$$[x, \theta, \dot{x}, \dot{\theta}] = [0, \pi, 0, 0] \tag{17}$$

or the set

$$\Omega_0 = \{[x, \theta, \dot{x}, \dot{\theta}]: \dot{\theta}^2 = g(1 - \cos\theta), \dot{x} = 0, x = 0\}; \tag{18}$$

(3) if, in addition, the initial conditions satisfy the inequality $V(q_0, \dot{q}_0) < V(0, \pi, 0, 0)$, then any solution of the closed-loop system satisfies the following limit relation:

$$\lim_{t \to \infty} V(q(t), \dot{q}(t)) = 0 \tag{19}$$

Proof. Let $[q_0, \dot{q}_0]$ be any point in the state space. Consider the solution $[q(t), \dot{q}(t)]$ of the closed-loop system with $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$. From the assumption $f \in \mathscr{F}_\rho$ and Proposition 1, Equation (8) is solvable with respect to f. Then the derivative of V along the solution $q(t)$ takes the form

$$\dot{V}(q(t), \dot{q}(t)) = -\dot{x}(t)\phi(\dot{x}(t)) \tag{20}$$

Therefore, one can conclude that the solution $q(t) \in R^1 \times S^1$ is bounded, has a non-empty ω-limit set Ω_*, and that $V(q(t), \dot{q}(t))$ tends to some constant value as $t \to +\infty$. Indeed, this follows from the facts that V is proper with respect to $R^3 \times S^1$, nonnegative and nonincreasing along the solution $q(t)$. Moreover, the value of $\dot{x}(t)$ tends to zero as $t \to +\infty$. Indeed, due to (20) the value of the integral

$$\int_0^{+\infty} \dot{x}(\tau)\phi(\dot{x}(\tau))\, d\tau$$

is bounded. Obviously the function $[q(t), \dot{q}(t)]$ is differentiable and its derivative is uniformly bounded. Thus by Barbalat lemma the value of

$$\dot{x}(t)\phi(\dot{x}(t))$$

tends to zero as $t \to +\infty$. This implies that $\dot{x}(t) \to 0$ as $t \to +\infty$.

Thus one can conclude that on the set Ω_*, which is non-empty and consists of the whole trajectories of the closed-loop system, the function V is constant, $\dot{x} = 0$ and

$$f(q, \dot{q})\left(k_E E(q, \dot{q}) + \frac{k_v}{1 + \sin^2\theta}\right) + \frac{k_v \sin\theta(\dot{\theta}^2 - g\cos\theta)}{1 + \sin^2\theta} + k_x x = -\phi(0) = 0 \tag{21}$$

The relation $\dot{x} = 0$ immediately implies that x is some constant and $\ddot{x} = 0$ on Ω_*. The last, due to Equation (14), is equivalent to

$$f(q, \dot{q}) = \sin\theta(g\cos\theta - \dot{\theta}^2) \tag{22}$$

on Ω_*.

Substituting (22) into (15) one obtains that any motion of the closed-loop system subjected to the above-mentioned constraints is: x equals to a constant, θ is some trajectory of the unforced pendulum. Indeed, due to (15) one has

$$\begin{aligned}\ddot{\theta} &= \frac{1}{1+\sin^2\theta}[-\dot{\theta}^2\sin\theta\cos\theta + 2g\sin\theta - f\cos\theta]\\ &= \frac{1}{1+\sin^2\theta}[-\dot{\theta}^2\sin\theta\cos\theta + 2g\sin\theta - \sin\theta(g\cos\theta - \dot{\theta}^2)\cos\theta]\\ &= g\sin\theta. \end{aligned}\tag{23}$$

Substituting the value (22) of f into (21), one has

$$\begin{aligned}0 &= f(q,\dot{q})\left(k_E E(q,\dot{q}) + \frac{k_v}{1+\sin^2\theta}\right) + \frac{k_v\sin\theta(\dot{\theta}^2 - g\cos\theta)}{1+\sin^2\theta} + k_x x\\ &= f(q,\dot{q})k_E E(q,\dot{q}) + k_x x + \frac{k_v(f(q,\dot{q}) + \sin\theta(\dot{\theta}^2 - g\cos\theta))}{1+\sin^2\theta}\\ &= f(q,\dot{q})k_E E(q,\dot{q}) + k_x x\end{aligned}\tag{24}$$

Here x is constant. Moreover, the value of the function $V(q,\dot{q})$ on the set Ω_* is constant. Therefore due to relation (6) and $\dot{x}=0$, the value of $E(q,\dot{q})$ on the set Ω_* is also constant. The last arguments and relation (24) result in two possible cases.

First, for any closed-loop system trajectory $[q_*(t), \dot{q}_*(t)]$ in Ω_* the constant value of $E(q_*(t), \dot{q}_*(t))$ is equal to zero. Moreover, by equality (24) the constant value of x is also zero. Thus, the set Ω_* is set (18).

Second, if for any trajectory $[q_*(t), \dot{q}_*(t)]$ of the closed-loop system in Ω_* the value of $E(q_*(t), \dot{q}_*(t))$ is not zero then due to (24) the following relation

$$f(q_*(t), \dot{q}_*(t)) = \sin\theta_*(t)(g\cos\theta_*(t) - \dot{\theta}_*^2(t)) = \text{const}\tag{25}$$

holds for all $t \geqslant 0$. Now, it is worth to point out that for any trajectory of system (23), except the upright position, there exists a moment of time $T_* \geqslant 0$ such that $\theta_*(T_*) = \pi$. In particular, this means that the constant in (25) is always zero, i.e. $t \geqslant 0$

$$f(q_*(t), \dot{q}_*(t)) = \sin\theta_*(t)(g\cos\theta_*(t) - \dot{\theta}_*^2(t)) = 0.\tag{26}$$

In turn, due to relation (24) and the positiveness of k_x, it immediately implies that $x = 0$.

To complete the proof of part (2) one should determine all motions of system (23), which satisfy constraint (26). System (23) describes the motions of a simple pendulum and has the conserved quantity

$$H_0(\theta, \dot{\theta}) = \tfrac{1}{2}\dot{\theta}^2 + g(\cos\theta - 1)$$

Suppose that there exists a trajectory $[\theta_*(t), \dot{\theta}_*(t)]$ of system (23), which differs from the upright and downward equilibria, $[0, 0]$ and $[\pi, 0]$, and such that relation (26) is valid for all $t \geqslant 0$. Thus there exist $\varepsilon > 0$ and a time interval $\mathscr{T} = (T_* - \varepsilon, T_* + \varepsilon)$ such that $\sin\theta_*(t) \neq 0\ \forall\, t \in \mathscr{T}$. Due to (26) this implies that

$$g\cos\theta_*(t) - \dot{\theta}_*^2(t) = 0, \quad \forall\, t \in \mathscr{T}\tag{27}$$

Then for this special trajectory one has that $\forall\, t \in \mathscr{T}$

$$H_0(\theta_*(0), \dot{\theta}_*(0)) = \tfrac{1}{2}\dot{\theta}_*^2(t) + g\cos\theta_*(t) - g$$
$$= \tfrac{1}{2}\dot{\theta}_*^2(t) + \dot{\theta}_*^2(t) - g \tag{28}$$

In particular, this implies that $\ddot{\theta}_*(t) \equiv 0$ for all $t \in \mathscr{T}$. Coming back to Equation (23) one concludes that $\sin\theta_*(t) \equiv 0$ on the interval $\mathscr{T}$. This obviously corresponds to only one of the equilibriums of pendulum (23) : $[0, 0]$ and $[\pi, 0]$. Thus it is shown that the assumption $E(q_*(t), \dot{q}_*(t)) \neq 0$ for any closed-loop system trajectory lying in Ω_* implies that $[q_*(t), \dot{q}_*(t)]$ is either equilibrium (16) or (17).

To prove part (3) of Theorem 3 let us suppose that $V(q_0, \dot{q}_0) < V(0, \pi, 0, 0)$. Consider the solution $[q(t), \dot{q}(t)]$ of the closed-loop system starting from the point $[q_0, \dot{q}_0]$. As it is shown above, first, along this trajectory the function V is not increasing, and, second, its ω-limit set consists of either equilibrium (16) or equilibrium (17) or is contained in the union of set (18) and equilibrium (17). Then the downward equilibrium (16) is strictly separated from the trajectory $[q(t), \dot{q}(t)]$. Therefore along this solution the limit relation (19) is valid. □

Remark 4

Essentially the proof of Theorem 3 can be divided into two parts. The first part contains standard arguments based on the properties of some appropriate storage function and Barbalat's lemma. To complete the second part of the proof one should analyse *motions of the closed-loop system* subject to some constraints (equalities) with respect to state variables. Such an analysis corresponds to the verification of V-detectability of the system with some suitable output, see [10, Definition 2.2]. In our case such an output is $y(q, \dot{q}) = \dot{x}$.

Remark 5

Theorem 3 is partly reproduced the results developed in Reference [8], where it was shown that if the initial conditions are close enough to the set Ω_0 then for any controller $f \in \mathscr{F}$ the limit relation (19) is valid. The simulations made in the Section 5 show that the controllers introduced in Reference [8] do not work globally and the closed-loop system may have singularities for some initial conditions.

Theorem 3 singles out three sets in the state space of the inverted pendulum, which serve as ω-limit sets for the closed-loop system solutions. But Theorem 3 does not reflect the global properties of the closed-loop system in a sense that it does not say which of these three sets are 'generically' attractive sets, i.e. which of these three sets attract almost all trajectories of the closed-loop system. This problem is discussed in the next statements.

Proposition 6

Consider the controlled inverted pendulum (1). For any controller $f \in \mathscr{F}$ the upright equilibrium (17) is hyperbolic. Moreover, the dimension of its stable manifold is 3 and the dimension of its unstable manifold is 1.

Proposition 7

Consider the controlled inverted pendulum (1). For any controller $f \in \mathscr{F}_\rho$ the downward equilibrium (16) has at least a two-dimensional stable manifold. If the parameters of the controller f satisfy the inequality

$$4g^2 k_E > g k_v + k_x \tag{29}$$

then the downward equilibrium (16) is a hyperbolic stationary point of the closed-loop system. Moreover, the dimension of its stable manifold is 2 and the dimension of its unstable manifold is 2.

The proofs of Propositions 6 and 7 are given in the appendix. These statements make it possible to formulate the next theorem.

Theorem 8

Consider the controlled inverted pendulum (1). Take any state feedback controller $f \in \mathscr{F}_\rho$ defined by (8). Suppose that the parameters of the controller f satisfy inequality (29). Then the set Ω_0, defined by (18), is the 'generic' attractive set, i.e. for all initial conditions $[q_0, \dot{q}_0]$ (except a stable two-dimensional manifold of the downward equilibrium with zero displacement of the cart) the solution $[q(t), \dot{q}(t)]$ of the closed-loop system, starting at this point, tends to the set Ω_0 and satisfies the limit relation

$$\lim_{t \to +\infty} V(q(t), \dot{q}(t)) = 0$$

4. SET STABILIZATION FOR UNDERACTUATED LAGRANGE SYSTEMS WITH CYCLIC CO-ORDINATES

The main ideas of the presented results are applicable for the control of a class of Lagrangian systems being invariant with respect to some group action. Suppose that the system has a configuration space $Q = \Theta \times X$, $q = (\theta, x)$, and the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\dot{\theta}} \mathscr{L} - \nabla_{\theta} \mathscr{L} = 0 \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla_{\dot{x}} \mathscr{L} = u \tag{31}$$

Here $\mathscr{L}$ is the Lagrangian of the unforced system

$$\mathscr{L}(q, \dot{q}) = \tfrac{1}{2} \langle \dot{q}, \dot{q} \rangle - \Pi(q)$$

with $\langle \cdot, \cdot \rangle$ being a Riemannian metric on Q and Π being a potential energy; u is a control action. It is assumed that $\mathscr{L}$ is cyclic in the X-variables, i.e. the Lagrangian does not depend on these co-ordinates. This implies that the corresponding generalized momenta are conserved quantities of the unforced system, see (31). Introduce the total energy $E(q, \dot{q})$ of the unforced system (30)–(31)

$$E(q, \dot{q}) = \dot{q}^i \nabla_{\dot{q}^i} \mathscr{L}(q, \dot{q}) - \mathscr{L}$$

Due to standard assumptions (the compatibility of the Riemannian metric and the geometric connection) the total energy satisfies the passivity relation

$$\frac{\mathrm{d}}{\mathrm{d}t} E(q(t), \dot{q}(t)) = \dot{x}(t)^{\mathrm{T}} u(t)$$

Given a constant E_0 and a vector a, the problem is to define the feedback control such that along the closed-loop system solutions $(q(t), \dot{q}(t))$ the limit relations

$$\lim_{t \to +\infty} E(q(t) \dot{q}(t)) = E_0 \tag{32}$$

$$\lim_{t \to +\infty} x((t)) = a \tag{33}$$

are valid. Such problem includes the main subject of the paper, i.e. the swinging up of the inverted pendulum, with $a = 0$ and E_0 being equal to the energy corresponding to the upright equilibrium. If one changes the value of E_0 it will correspond to the stabilization of the rotations of the pendulum with zero displacement of the cart. Another examples come from the consideration of the spherical pendulum on the cart or the Furuta pendulum.

Consider the storage function

$$V(q, \dot{q}) = \frac{k_E}{2}[E - E_0]^2 + \frac{k_v}{2}|\dot{x}|^2 + \tfrac{1}{2}|x - a|^2$$

its derivative along the solutions of (30) and (31) has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V = \dot{x}^{\mathrm{T}}[k_E[E - E_0]u + k_v\ddot{x} + [x - a]]$$

Using (31), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}V = \dot{x}^{\mathrm{T}}\left[\left(k_E[E - E_0] + k_v[0 \;\; I]M(q)^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\right)u + g(q, \dot{q})\right]$$

where $M(q)$ is a metric tensor associated with the Riemannian metric and $g(q, \dot{q})$ is some function which depends on $\mathscr{L}$ and parameters k_E, k_v, a.

Take any smooth function $\phi(x)$, such that $x^{\mathrm{T}}\phi(x) > 0$ for any $x \neq 0$, and consider the equation

$$\left(k_E[E - E_0] + k_v[0 \;\; I]M(q)^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\right)u + g(q, \dot{q}) = -\phi(\dot{x}) \tag{34}$$

It is worth to mention that we have already seen Equation (34) for the inverted pendulum, (cf. (8)). To solve this equation with respect to u one should invert the matrix

$$k_E[E - E_0]I + k_v[0 \;\; I]M(q)^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} \tag{35}$$

It can be shown that if the energy function E is bounded from below (this is the standard situation), then there exist positive parameters kE, k_v such that this matrix is globally strictly positive definite, i.e. globally invertible. For the case of the inverted pendulum E is bounded from below and the conditions of the invertability of matrix (35) are stated in Proposition 1.

Thus Equation (34) can be solved and for such defined control variable u one has

$$\frac{\mathrm{d}}{\mathrm{d}t}V(q(t), \dot{q}(t)) = -\dot{x}(t)^{\mathrm{T}}\phi(\dot{x}(t)) \leqslant 0$$

To prove the validity of the limit relations (32) and (33) for the closed-loop system one should verify the V-detectability of the set $\{(q, \dot{q}) : E(q, \dot{q}) = E_0, x = a, \dot{x} = 0\}$.

5. SIMULATION RESULTS

In order to observe the performance difference compared with the result in Reference [8], we have done simulations Matlab 5/SIMULINK.[§]

We have considered the system with the same parameters as in Reference [8]: $M = 1$, $m = 1$, $l = 1$, $g = 9.8\ \mathrm{m/s^2}$. For the initial conditions:

$$x(0) = -10, \quad v(0) = 1$$

$$\theta(0) = \frac{\pi}{4}, \quad \omega(0) = 0$$

and the controller parameters $k_v = 10$, $k_x = 0.1$ and $k_E = 1$ we get the same as shown in Reference [8], reproduced in Figure (2).

These controller parameters do not satisfy the conditions for global existence of the controller given in Proposition 1, see (11). This means that there might exist initial conditions for which the

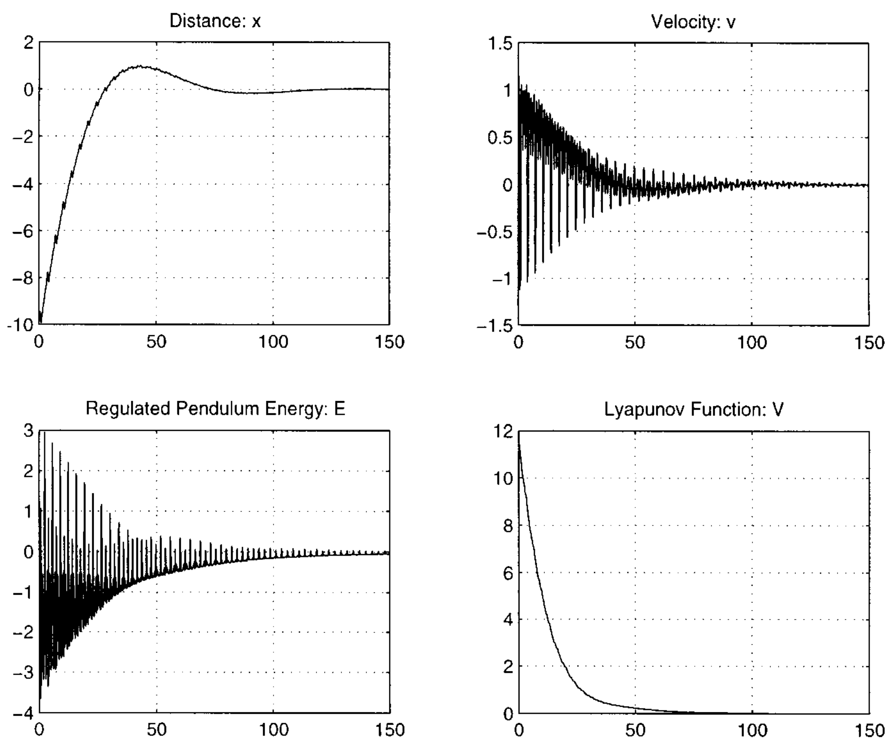

Figure 2. Local asymptotic stability.

[§] Matlab/SIMULINK is a trademark of The Math Works, Inc.

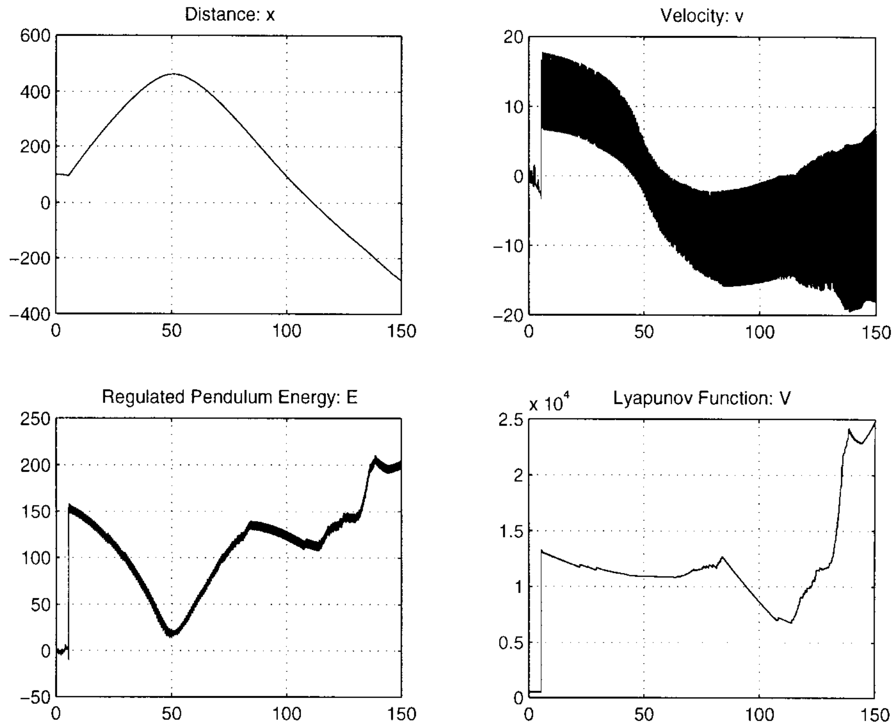

Figure 3. Singularity in the controller.

system does not converge. The initial conditions

$$x(0) = 100, \quad v(0) = 1$$

$$\theta(0) = \frac{\pi}{4}, \quad \omega(0) = 0$$

is outside the estimated region of attraction given in Reference [8], and Figure 3 shows that the controller indeed has a singularity. By changing the controller parameters to $k_v = 30$, $k_x = 0.1$, $k_E = 1$, the controller is globally defined, and Figure 4 shows that the system again converges to the desired set.

Figure 5 shows that for large initial conditions, outside the estimated region given in Reference [8], the system still converges to the desired set.

6. CONCLUSIONS

Using the 'toy example' of the inverted pendulum we have demonstrated a general approach to stabilization of some target positions of undeructuated mechanical systems. The approach is based on the passivity properties of Lagrangian or Hamiltonian systems and allows one to find a family of static feedback controllers. Using a storage function which has the term corresponding

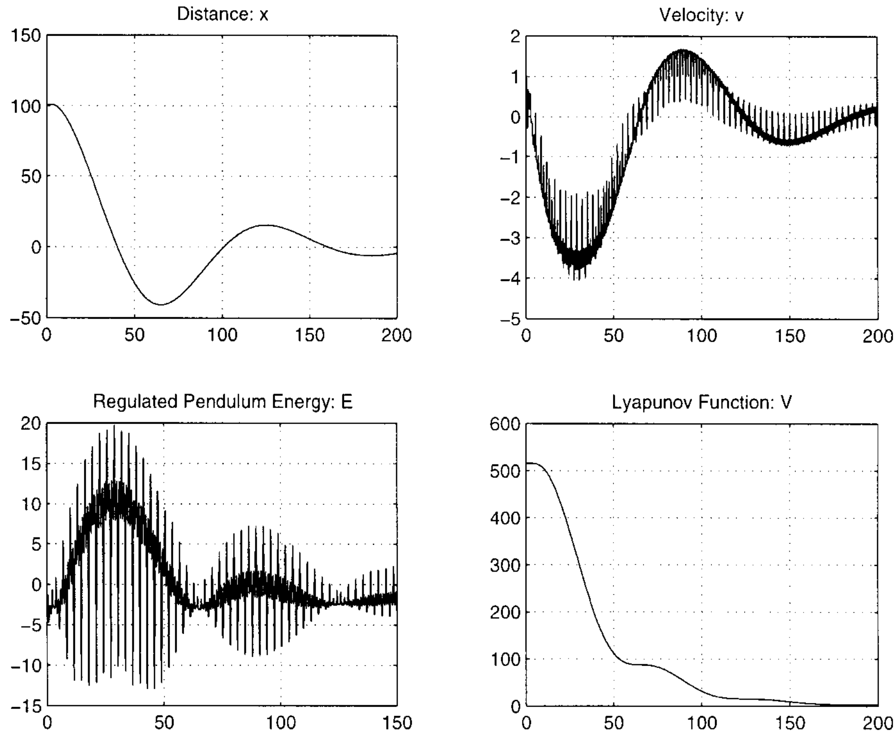

Figure 4. Convergence with new controller parameters.

to the total energy of the system it is possible to stabilize the set corresponding to the desired energy level with a prespecified value of some variables (zero displacement of the cart). For the example considered in this paper it turns out that this set is a union of homoclinic solutions and therefore any solution which starts inside this set tends to the target point. Moreover, using the storage function as a Lyapunov function candidate it is possible to estimate the region of attraction of this set—the set of all initial data for which the corresponding solution has a target position as an ω-limit point. For the inverted pendulum example the region of attraction is the whole phase space except for some two-dimensional manifold.

Thus, we found a family of feedbacks which make the target position a limit point for almost any solution of the closed-loop system. In other words, almost any solution approaches an arbitrarily neighbourhood of the target position. Hence to solve the initial problem of stabilization at the target position it is sufficient to design a locally stabilizing controller with a switching rule between the controllers. In a more general situation we propose to design step-by-step controllers which stabilize the sets corresponding to the desired values of the first integrals of the free system. The main advantage of this approach is that the Lyapunov functions rely on Hamiltonian or Lagrangian structure of the system and therefore have clear physical sense. However, one can see some drawbacks as well: first, all dissipative forces are assumed to be neglectable and, second, one has to carefully design the switching rules between controllers. We will continue our study in this direction.

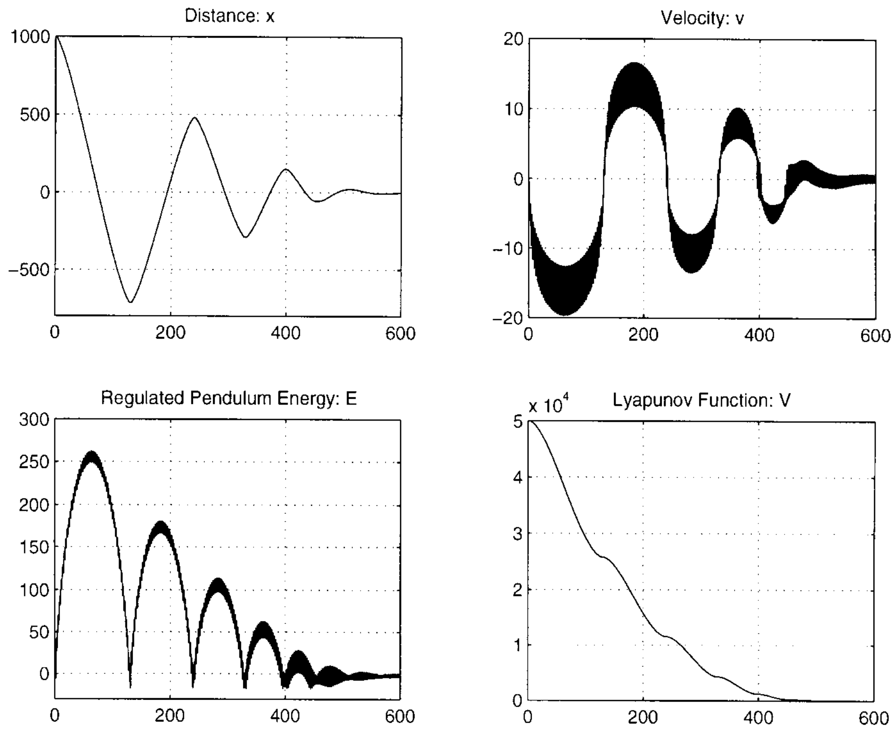

Figure 5. Convergence to the desired set with large initial conditions.

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APPENDIX

Proofs of Propositions 6 and 7. Consider the linear approximation of the closed-loop system in the equilibrium points (16) and (17). It has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_3 \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_3 \end{bmatrix} = \left.\begin{bmatrix} \dfrac{\partial F_1}{\partial \dot{x}} & \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial \dot{\theta}} & \dfrac{\partial F_1}{\partial \theta} \\ 1 & 0 & 0 & 0 \\ \dfrac{\partial F_2}{\partial \dot{x}} & \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial \dot{\theta}} & \dfrac{\partial F_2}{\partial \theta} \\ 0 & 0 & 1 & 0 \end{bmatrix}\right|_{(0,\pi,0,0)\text{ or }(0,0,0,0)} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_3 \end{bmatrix} \tag{A1}$$

where

$$F_1(q, \dot{q}) = \frac{1}{1 + \sin^2\theta}[\sin\theta(\dot{\theta}^2 - g\cos\theta) + f]$$

$$F_2(q, \dot{q}) = \frac{1}{1 + \sin^2\theta}[-\dot{\theta}^2\sin\theta\cos\theta + 2g\sin\theta - f\cos\theta].$$

Straightforward calculations show that for these equilibriums

$$\frac{\partial F_1}{\partial \dot{\theta}} = 0, \qquad \frac{\partial F_2}{\partial \dot{\theta}} = 0$$

$$\frac{\partial F_1}{\partial \theta} = \frac{k_v g}{k_v + k_E g(\cos\theta - 1)} - g, \qquad \frac{\partial F_2}{\partial \theta} = \cos\theta\left(2g - \frac{k_v g}{k_v + k_E g(\cos\theta - 1)}\right)$$

$$\frac{\partial F_1}{\partial \dot{x}} = -\frac{\dot{\phi}(0)}{k_v + k_E g(\cos\theta - 1)}, \qquad \frac{\partial F_2}{\partial \dot{x}} = \cos\theta\frac{\dot{\phi}(0)}{k_v + k_E g(\cos\theta - 1)}$$

$$\frac{\partial F_1}{\partial x} = -\frac{k_x}{k_v + k_E g(\cos\theta - 1)}, \qquad \frac{\partial F_2}{\partial x} = \cos\theta\frac{k_x}{k_v + k_E g(\cos\theta - 1)}$$

where θ is either 0 or π. Due to zeros on the first line of the last formulas the characteristic polynomial of the matrix A, see (A1), is

$$\det(\lambda I_4 - A) = \lambda^4 + \lambda^3\left(-\frac{\partial F_1}{\partial \dot{x}}\right) + \lambda^2\left(-\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial \theta}\right) + \lambda\left(\frac{\partial F_1}{\partial \dot{x}}\frac{\partial F_2}{\partial \theta} - \frac{\partial F_2}{\partial \dot{x}}\frac{\partial F_1}{\partial \theta}\right) + \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial \theta} - \frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial \theta} \tag{A2}$$

Consider first the downward equilibrium (16). For this point polynomial (A2) takes the form

$$p(\lambda) = \lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 \tag{A3}$$

where

$$\alpha_1 = \frac{\dot{\phi}(0)}{k_v - 2g\,k_E}, \qquad \alpha_2 = \frac{k_x + g\,k_v - 4g^2\,k_E}{k_v - 2g\,k_E}$$

$$\alpha_3 = \frac{g\,\dot{\phi}(0)}{k_v - 2g\,k_E}, \qquad \alpha_4 = \frac{g\,k_x}{k_v - 2gk_E}$$

Due to the assumptions one can easily verify that for any controller $f \in \mathscr{F}_\rho$

$$\alpha_1 > 0, \quad \alpha_3 > 0, \quad \alpha_4 > 0$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the roots of polynomial (A3). If all λ_i, $i = 1, \dots, 4$, are complex then $\lambda_1 = \overline{\lambda}_2$, $\lambda_3 = \overline{\lambda}_4$ and

$$2\,\mathrm{Re}\,\lambda_1 + 2\,\mathrm{Re}\,\lambda_3 = -\alpha_1 < 0$$

Therefore at least two of λ_i, $i = 1, \ldots, 4$, have negative real parts. Suppose that two of λ_i, $i = 1, \ldots, 4$, for example λ_1 and λ_2, are real constants. Then $\lambda_3 = \bar{\lambda}_4$ and

$$\lambda_1 + \lambda_2 + 2\,\mathrm{Re}\,\lambda_3 = -\alpha_1 < 0$$

If $\mathrm{Re}\,\lambda_3 \geqslant 0$, otherwise one has two complex roots λ_3, λ_4 with negative real parts, then at least one of λ_1, λ_2 is negative. But due to the inequality

$$\lambda_1 \lambda_2 |\lambda_3|^2 = \alpha_4 > 0 \tag{A4}$$

another real root should be also negative. Suppose that all λ_i, $i = 1, \ldots, 4$ are real. By the inequality

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -\alpha_1 < 0$$

at least one of them is negative. Due to the inequality

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = \alpha_4 > 0 \tag{A5}$$

the polynomial (A3) should have at least two negative roots. Thus, it is shown that the linear approximation of the closed-loop system with any $f \in \mathscr{F}_\rho$ in the downward equilibrium (16) always has at least two eigenvalues with negative real part. This implies that the stable manifold of this equilibrium is at least two dimensional.

Suppose now that the parameters of the controller $f \in \mathscr{F}$ in addition satisfy inequality (29), which means that the value of the constant α_2 is negative, $\alpha_2 < 0$. We are going to check in the previous manner that under this condition polynomial (A3) always has two roots with positive real part.

Indeed, if all roots, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, are complex, in particular, $\lambda_1 = \bar{\lambda}_2$, $\lambda_3 = \bar{\lambda}_4$. It follows from the inequality

$$|\lambda_1|^2 + |\lambda_3|^2 + 4\,\mathrm{Re}\,\lambda_1\,\mathrm{Re}\,\lambda_3 = \alpha_2 < 0$$

that $\mathrm{Re}\,\lambda_1$ and $\mathrm{Re}\,\lambda_2$ have different signs, i.e. two roots of (A3) have positive real parts.

Suppose that polynomial (A3) has two real roots, for example λ_1, λ_2. By inequality (A4) they have the same sign, i.e. $\lambda_1\lambda_2 > 0$. Then due to the relation

$$|\lambda_3|^2 + \lambda_1\lambda_2 + (\lambda_1 + \lambda_2) 2\,\mathrm{Re}\,\lambda_3 = \alpha_2 < 0$$

the constants $\lambda_1 + \lambda_2$ and $\mathrm{Re}\,\lambda_3$ have different signs, i.e. two roots of (A3) have positive real parts.

Suppose that all roots are real. Due to (A5) the number of negative roots is even. If one supposes that all roots are negative then it contradicts the inequality

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \alpha_2 < 0$$

Proposition (7) is proved. □

Consider the upright equilibrium (17). For this point polynomial (A2) takes the form (A3) with the following coefficients:

$$\alpha_1 = \frac{\dot{\phi}(0)}{k_v}, \quad \alpha_2 = \frac{k_x}{k_v} - g, \quad \alpha_3 = -\frac{g\dot{\phi}(0)}{k_v}, \quad \alpha_4 = -\frac{g\,k_x}{k_v}$$

Due to assumptions one has

$$\alpha_1 > 0, \quad \alpha_3 < 0, \quad \alpha_4 < 0$$

The value of α_4 is negative. Hence polynomial (A3) has at least one positive λ_1 and one negative λ_2 roots. Suppose that another two roots λ_3, λ_4 are complex, $\lambda_3 = \bar{\lambda}_4$. Then one has

$$(\lambda_1 + \lambda_2) + 2\,\mathrm{Re}\,\lambda_3 = -\alpha_1 < 0 \tag{A6}$$

$$(\lambda_1 + \lambda_2)\cdot|\lambda_3|^2 + \lambda_1\lambda_2\,2\,\mathrm{Re}\,\lambda_3 = -\alpha_3 > 0 \tag{A7}$$

If one assumes that $\mathrm{Re}\,\lambda_3 \geqslant 0$ then by (A6) the value of $\lambda_1 + \lambda_2$ is negative. Moreover, the value of $\lambda_1\lambda_2$ is also negative. Thus the left-hand side of (A7) should be non-positive, but it is positive. Therefore, $\mathrm{Re}\,\lambda_3 < 0$.

Suppose that all roots are real. Then $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_1\lambda_2\lambda_3\lambda_4 = \alpha_4 < 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -\alpha_1 < 0$ and

$$(\lambda_1 + \lambda_2)\lambda_3\lambda_4 + (\lambda_3 + \lambda_4)\,\lambda_1\lambda_2 = -\alpha_3 > 0 \tag{A8}$$

Obviously $\lambda_3\lambda_4 > 0$. The assumption $(\lambda_3 + \lambda_4) > 0$ implies that $(\lambda_1 + \lambda_2) < 0$ and that

$$(\lambda_1 + \lambda_2)\,\lambda_3\lambda_4 + (\lambda_3 + \lambda_4)\,\lambda_1\lambda_2 < 0.$$

This contradicts (A8). Proposition 6 is proved. □

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