Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2015, Article ID 301814, 8 pages
http://dx.doi.org/10.1155/2015/301814

Research Article

On Partial Sum of Tribonacci Numbers

Eunmi Choi and Jiin Jo

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to Eunmi Choi; emc@hnu.kr

Received 30 March 2015; Accepted 28 May 2015

Academic Editor: Nawab Hussain

We study the sum $s_t^{(k,r)} = \sum_{i=0}^{t} T_{ki+r}$ of k step apart Tribonacci numbers for any $1 \le r \le k$. We prove that $s_t^{(k,r)}$ satisfies certain Tribonacci rule $s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + s_{t-3}^{(k,r)} + \lambda$ with integers a_k, b_k, c_k, and λ.

1. Introduction

A Tribonacci sequence $\{T_n\}_{n\ge0}$, which is a generalized Fibonacci sequence $\{F_n\}$, is defined by the Tribonacci rule $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_2 = T_1 = 1$ and $T_0 = 0$. The sequence can be extended to negative subscript n; hence few terms of the sequence are $\ldots, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, \ldots$. Each term in $\{T_n\}$ is called the Tribonacci number.

The sum of Fibonacci numbers is well expressed by $\sum_{i=0}^{n} F_i = F_{n+2} - 1$, and moreover the sum of reciprocal Fibonacci numbers was studied intensively in [1–3]. For the sum $S_n = \sum_{i=0}^{n} T_i$ of Tribonacci numbers, there are some researches including [4–7]. In particular Kilic [6] proved the identity $S_n = (1/2)(T_{n+2} + T_n - 1)$ by means of generating matrix calculations. And Irmak and Alp [5] proved an identity about the k subscripted Tribonacci sum $S_{n,k} = \sum_{i=0}^{n} T_{ki}$ using the three roots of $x^3 - x^2 - x - 1 = 0$.

This paper is devoted to studying the sum of Tribonacci numbers as well as the sum $\sum_{i=0}^{t} T_{ki+r}$ of k step apart Tribonacci numbers for any $1 \le r \le k$. Our method here is to employ the k-step Tribonacci rule $T_n = a_k T_{n-k} + b_k T_{n-2k} + c_k T_{n-3k}$ ($a_k, b_k, c_k \in \mathbb{Z}$) that is a linear combination of k distance Tribonacci numbers in [8]. For this purpose we will display all Tribonacci numbers in rectangle form with k columns, called the k-tribo table:

$$\begin{matrix} T_1 & T_2 & \cdots & T_r & \cdots & T_k \\ T_{k+1} & T_{k+2} & \cdots & T_{k+r} & \cdots & T_{2k} \\ T_{2k+1} & T_{2k+2} & \cdots & T_{2k+r} & \ddots & \end{matrix} \quad (1)$$

Then $\sum_{i=0}^{t} T_{ki+r}$ can be regarded as a partial sum of $t+1$ entries in rth column of the table. We denote it by $s_t^{(k,r)}$ for $1 \le r \le k$.

2. Tribonacci Tables

T_n satisfies a Tribonacci rule $a_1 T_{n-1} + b_1 T_{n-2} + c_1 T_{n-3}$ with $(a_1, b_1, c_1) = (1, 1, 1)$ and a 2-step Tribonacci rule $a_2 T_{n-2} + b_2 T_{n-4} + c_2 T_{n-6}$ with $(a_2, b_2, c_2) = (3, 1, 1)$. Moreover the k-step Tribonacci rules for T_n were proved in [8].

Lemma 1 (see [8]). *Consider $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ with $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$ for any $1 \le k < n$. The $\{a_k\}$ and $\{b_k\}$ satisfy $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ and $b_k = b_{k+1} + b_{k+2} + b_{k+3}$ with $a_1 = 1$, $a_2 = 3$, $a_3 = 7$, and $b_1 = b_2 = 1$, $b_3 = -5$.*

The recurrence $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ implies that $\{a_k\}$ is a Tribonacci type sequence. By extending the subscript k to negative integers, we have $a_0 = 3$, $a_{-1} = -1$, $a_{-2} = -1$, and so forth; thus $\{a_k\} = \{\ldots, -1, -5, 5, -1, -1, 3, 1, 3, 7, 11, \ldots\}$. In particular if $1 \le k \le 10$ then the coefficients (a_k, b_k) are

k	(a_k, b_k)
1	$(1, 1)$
2	$(3, 1)$
3	$(7, -5)$
4	$(11, 5)$
5	$(21, 1)$
6	$(39, -11)$

$$\begin{array}{cc} 7 & (71,15) \\ 8 & (131,-3) \\ 9 & (241,-23) \\ 10 & (443,41) \end{array} \tag{2}$$

For example, T_{31} is expressed by the 10-step Tribonacci rule $T_{31} = 443T_{21} + 41T_{11} + T_1 = 443(121415) + 41(274) + 1 = 53798080$. It shows that T_{31} is a combination of three entries at 1st column of 10-tribo table. Similarly, by taking $k = 11$, we have $a_{11} = 815$ and $b_{11} = -21$, so $T_{31} = 815T_{20} - 21T_9 + T_{-2} = 53798080$ which is a combination of three entries at 9th column of 11-tribo table.

Thus if $n = kt + r$ $(0 \le t,\ 1 \le r \le k)$ then T_n is located at $(t+1)$th row and rth column in the k-tribo table and is a combination of three entries at t, $t-1$, and $(t-2)$th row of rth column. Now for the partial sum $s_t^{(k,r)} = \sum_{i=0}^{t} T_{ki+r}$ of rth column in k-tribo table, let us begin with $k = 3$ and 4.

Theorem 2. *When $k = 3$ or 4, the partial sum $s_t^{(k,r)}$ holds as follows:*

(1)

$$s_t^{(3,r)} = \begin{cases} 7s_{t-1}^{(3,r)} - 5s_{t-2}^{(3,r)} + s_{t-3}^{(3,r)} - 1 & \text{if } r = 1 \\ 7s_{t-1}^{(3,r)} - 5s_{t-2}^{(3,r)} + s_{t-3}^{(3,r)} + 1 & \text{if } r \ne 1 \end{cases} \quad \text{for } t \ge 3$$

$$= \begin{cases} \dfrac{1}{T_3}\left(T_{3(t+1)+r} - 4T_{3t+r} + T_{3(t-1)+r} + 1\right) & \text{if } r = 1 \\ \dfrac{1}{T_3}\left(T_{3(t+1)+r} - 4T_{3t+r} + T_{3(t-1)+r} - 1\right) & \text{if } r \ne 1 \end{cases} \quad \text{for } t \ge 1; \tag{3}$$

(2)

$$s_t^{(4,r)} = \begin{cases} 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} - 4 & \text{if } r = 1 \\ 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} + 4 & \text{if } r \ne 1 \end{cases} \quad \text{for } t \ge 3$$

$$= \begin{cases} \dfrac{1}{T_4^2}\left(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} + 4\right) & \text{if } r = 1 \\ \dfrac{1}{T_4^2}\left(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} - 4\right) & \text{if } r \ne 1 \end{cases} \quad \text{for } t \ge 1. \tag{4}$$

Proof. The 3-tribo table produces a table of $s_t^{(3,r)}$ as follows:

3-tribo table		
1	1	2
4	7	13
24	44	81
149	274	504
927	1705	⋯

t	$s_t^{(3,1)}$	$s_t^{(3,2)}$	$s_t^{(3,3)}$
0	1	1	2
1	5	8	15
2	29	52	96
3	178	326	600
4	1105	2031	3736

(5)

If $t = 4$ then it can be observed that

$$\begin{aligned} s_4^{(3,1)} &= 1105 = 7(178) - 5(29) + 5 - 1 \\ &= 7s_3^{(3,1)} - 5s_2^{(3,1)} + s_1^{(3,1)} - 1, \\ s_4^{(3,2)} &= 2031 = 7(326) - 5(52) + 8 + 1 \\ &= 7s_3^{(3,2)} - 5s_2^{(3,2)} + s_1^{(3,2)} + 1, \\ s_4^{(3,3)} &= 3736 = 7(600) - 5(96) + 15 + 1 \\ &= 7s_3^{(3,3)} - 5s_2^{(3,3)} + s_1^{(3,3)} + 1. \end{aligned} \tag{6}$$

For some $t > 4$, we assume $s_t^{(3,r)} = 7s_{t-1}^{(3,r)} - 5s_{t-2}^{(3,r)} + s_{t-3}^{(3,r)} \pm 1$ with minus sign if $r = 1$, otherwise plus sign. Then the next partial sum $s_{t+1}^{(3,r)}$ satisfies

$$\begin{aligned} s_{t+1}^{(3,r)} &= s_t^{(3,r)} + T_{3(t+1)+r} \\ &= \left(7s_{t-1}^{(3,r)} - 5s_{t-2}^{(3,r)} + s_{t-3}^{(3,r)} \pm 1\right) \\ &\quad + \left(7T_{3t+r} - 5T_{3(t-1)+r} + T_{3(t-2)+r}\right) \\ &= 7s_t^{(3,r)} - 5s_{t-1}^{(3,r)} + s_{t-2}^{(3,r)} \pm 1 \end{aligned} \tag{7}$$

due to Lemma 1. And we also notice that

$$\begin{aligned} T_{10} - 4T_7 + T_4 + 1 &= 149 - 4(24) + 4 + 1 = 2(29) \\ &= 2s_2^{(3,1)} = T_3 s_2^{(3,1)}, \\ T_{11} - 4T_8 + T_5 - 1 &= 274 - 4(44) + 7 - 1 = 2(52) \\ &= 2s_2^{(3,2)} = T_3 s_2^{(3,2)}, \\ T_{12} - 4T_9 + T_6 - 1 &= 504 - 4(81) + 13 - 1 = 2(96) \\ &= 2s_2^{(3,3)} = T_3 s_2^{(3,3)}, \end{aligned} \tag{8}$$

so it shows $T_3 s_2^{(3,r)} = T_{3(3)+r} - 4T_{3(2)+r} + T_{3+r} \pm 1$ with minus sign if $r = 1$, otherwise plus. Hence if we assume

$T_3 s_t^{(3,r)} = T_{3(t+1)+r} - 4T_{3t+r} + T_{3(t-1)+r} \pm 1$ for some t then it follows from Lemma 1 that

$$\begin{aligned} T_3 s_{t+1}^{(3,r)} &= T_3 \left(s_t^{(3,r)} + T_{3(t+1)+r} \right) \\ &= T_{3(t+1)+r} - 4T_{3t+r} + T_{3(t-1)+r} \pm 1 + T_3 T_{3(t+1)+r} \\ &= \left(7T_{3(t+1)+r} - 5T_{3t+r} + T_{3(t-1)+r} \right) - 4T_{3(t+1)+r} \\ &\quad + T_{3t+r} \pm 1 \\ &= T_{3(t+2)+r} - 4T_{3(t+1)+r} + T_{3t+r} \pm 1. \end{aligned} \tag{9}$$

Similarly the 4-tribo table makes the table of $s_t^{(4,r)}$:

4-tribo table

1	1	2	4
7	13	24	44
81	149	274	504
927	1705	3136	5768

t	$s_t^{(4,1)}$	$s_t^{(4,2)}$	$s_t^{(4,3)}$	$s_t^{(4,4)}$
0	1	1	2	4
1	8	14	26	48
2	89	163	300	552
3	1016	1868	3436	6320

(10)

When $t = 3$, experimental observations show that

$$\begin{aligned} s_3^{(4,1)} &= 1016 = (11)\,89 + (5)\,8 + 1 - 4 \\ &= 11 s_2^{(4,1)} + 5 s_1^{(4,1)} + s_0^{(4,1)} - 4, \\ s_3^{(4,2)} &= 1868 = 11\,(163) + 5\,(14) + 5 \\ &= 11 s_2^{(4,2)} + 5 s_1^{(4,2)} + s_0^{(4,2)} + 4, \end{aligned} \tag{11}$$

and so on. Hence if we assume $s_t^{(4,r)} = 11 s_{t-1}^{(4,r)} + 5 s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} \pm 4$ with $-$ sign if $r = 1$, otherwise $+$ sign for some $t > 3$, then $s_{t+1}^{(4,r)}$ is equal to

$$\begin{aligned} s_{t+1}^{(4,r)} &= s_t^{(4,r)} + T_{4(t+1)+r} \\ &= \left(11 s_{t-1}^{(4,r)} + 5 s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} \pm 4 \right) \\ &\quad + \left(11 T_{4t+r} + 5 T_{4(t-1)+r} + T_{4(t-2)+r} \right) \\ &= 11 s_t^{(4,r)} + 5 s_{t-1}^{(4,r)} + s_{t-2}^{(4,r)} \pm 4. \end{aligned} \tag{12}$$

On the other hand, it is easy to see that

$$\begin{aligned} T_{17} + 6T_{13} + T_9 + T_4 &= 10609 + 6\,(927) + 81 + 4 \\ &= 4^2\,(1016) = T_4^2 s_3^{(4,1)}, \\ T_{18} + 6T_{14} + T_{10} - T_4 &= 19513 + 6\,(1705) + 149 - 4 \\ &= 4^2\,(1868) = T_4^2 s_3^{(4,2)}, \\ &\vdots \\ T_{20} + 6T_{16} + T_{12} - T_4 &= 66012 + 6\,(5768) + 504 - 4 \\ &= 4^2\,(6320) = T_4^2 s_3^{(4,4)}. \end{aligned} \tag{13}$$

So by assuming $T_4^2 s_{t-1}^{(4,r)} = T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4$ for some t, we have

$$\begin{aligned} &T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} \pm T_4 \\ &\quad = \left(11T_{4t+r} + 5T_{4(t-1)+r} + T_{4(t-2)+r} \right) + 6T_{4t+r} \\ &\qquad + T_{4(t-1)+r} \pm T_4 \\ &\quad = \left(T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4 \right) + 16T_{4t+r} \\ &\quad = T_4^2 s_{t-1}^{(4,r)} + T_4^2 T_{4t+r} = T_4^2 \left(s_{t-1}^{(4,r)} + T_{4t+r} \right) = T_4^2 s_t^{(4,r)}. \end{aligned} \tag{14}$$

□

We remark that if $r = 4$ in Theorem 2 then we have

$$\begin{aligned} &\frac{1}{T_4^2} \left(T_{4(t+2)} + 6T_{4(t+1)} + T_{4t} - 4 \right) \\ &\quad = s_t^{(4,4)} = \sum_{i=0}^{t} T_{4i+4} = \sum_{i=0}^{t+1} T_{4i}. \end{aligned} \tag{15}$$

Since $T_4 = 4$, our result is equal to Theorem 5 in [6] which was proven by means of generating matrix.

The expression of $s_t^{(4,r)}$ has tails ± 4 according to $r = 1$ or not and equals $T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} \pm 4$ multiplied by $1/T_4^2$. Similarly $s_t^{(3,r)}$ has tails ± 1 according to $r = 1$ or not and equals $T_{3(t+1)+r} - 4T_{3t+r} + T_{3(t-1)+r} \pm 1$ multiplied by $1/T_3$. However, when $k \geq 5$ the expressions of $s_t^{(k,r)}$ quite differ from the case of $k = 3, 4$.

Theorem 3. *Let* $\{\lambda^{(r)}\}_{r=1}^{5} = \{-7, 3, 5, 1, 9\}$ *and* $\{\lambda^{(r)}\}_{r=1}^{6} = \{-12, 6, 4, -2, 8, 10\}$. *Then one has the following:*

(1)

$$\begin{aligned} s_t^{(5,r)} &= 21 s_{t-1}^{(5,r)} + s_{t-2}^{(5,r)} + s_{t-3}^{(5,r)} + \lambda^{(r)} \quad \textit{for } t \geq 3 \\ &= \frac{1}{22} \left(T_{5(t+1)+r} + 2T_{5t+r} + T_{5(t-1)+r} - \lambda^{(r)} \right) \\ &\qquad \textit{for } t \geq 1; \end{aligned} \tag{16}$$

(2)

$$s_t^{(6,r)} = 39s_{t-1}^{(6,r)} - 11s_{t-2}^{(6,r)} + s_{t-3}^{(6,r)} + \lambda^{(r)} \quad \textit{for } t \geq 3$$
$$= \frac{1}{28}\left(T_{6(t+1)+r} - 10T_{6t+r} + T_{6(t-1)+r} - \lambda^{(r)}\right) \quad \textit{for } t \geq 1. \quad (17)$$

Proof. The 5-tribo table yields a table of $s_t^{(5,r)}$ such that

t	$s_t^{(5,1)}$	$s_t^{(5,2)}$	$s_t^{(5,3)}$	$s_t^{(5,4)}$	$s_t^{(5,5)}$
0	1	1	2	4	7
1	14	25	46	85	156
2	288	529	973	1790	3292
3	6056	11138	20486	37680	$\cdots$

(18)

Then by inspecting the table, we find that

$$s_3^{(5,1)} = 21(288) + 14 + 1 - 7 = 21s_2^{(5,1)} + s_1^{(5,1)} + s_0^{(5,1)} - 7,$$
$$s_3^{(5,2)} = 21(529) + 25 + 1 + 3 = 21s_2^{(5,2)} + s_1^{(5,2)} + s_0^{(5,2)} + 3,$$
$$\vdots$$
$$s_3^{(5,5)} = 21(3292) + 156 + 7 + 9 = 21s_2^{(5,5)} + s_1^{(5,5)} + s_0^{(5,5)} + 9, \quad (19)$$

so we have $s_3^{(t,r)} = 21s_2^{(t,r)} + s_1^{(t,r)} + s_{t-3}^{(t,r)} + \lambda^{(r)}$ for $t = 5$ and $1 \leq r \leq t$. Now if we assume the identity is true for some t, then the $t+1$ partial sum follows

$$s_{t+1}^{(5,r)} = s_t^{(5,r)} + T_{5(t+1)+r}$$
$$= \left(21s_{t-1}^{(5,r)} + s_{t-2}^{(5,r)} + s_{t-3}^{(5,r)} + \lambda^{(r)}\right) + \left(21T_{5t+r} + T_{5(t-1)+r} + T_{5(t-2)+r}\right)$$
$$= 21s_t^{(5,r)} + s_{t-1}^{(5,r)} + s_{t-2}^{(5,r)} + \lambda^{(r)} \quad (20)$$

from Lemma 1. Moreover we notice that

$$T_{21} + 2T_{16} + T_{11} + 7 = 121415 + 2(5768) + 274 + 7 = 22(6056) = 22s_3^{(5,1)},$$
$$T_{22} + 2T_{17} + T_{12} - 3 = 223317 + 2(10609) + 504 - 3 = 22(11138) = 22s_3^{(5,2)},$$
$$\vdots$$
$$T_{25} + 2T_{20} + T_{15} - 9 = 1389537 + 2(66012) + 3136 - 9 = 22s_3^{(5,5)}, \quad (21)$$

so we have $T_{5(4)+r} + 2T_{5(3)+r} + T_{5(2)+r} - \lambda^{(r)} = 22s_3^{(5,r)}$. Hence, by assuming $22s_t^{(5,r)} = T_{5(t+1)+r} + 2T_{5t+r} + T_{5(t-1)+r} - \lambda^{(r)}$ for some t, we have

$$22s_{t+1}^{(5,r)} = \left(T_{5(t+1)+r} + 2T_{5t+r} + T_{5(t-1)+r} - \lambda^{(r)}\right) + 22T_{5(t+1)+r}$$
$$= \left(21T_{5(t+1)+r} + T_{5t+r} + T_{5(t-1)+r}\right) + \left(2T_{5(t+1)+r} + T_{5t+r} - \lambda^{(r)}\right)$$
$$= T_{5(t+2)+r} + 2T_{5(t+1)+r} + T_{5t+r} - \lambda^{(r)}. \quad (22)$$

Now, for (2), we construct the table of $s_t^{(6,r)}$ from 6-tribo table:

t	$s_t^{(6,1)}$	$s_t^{(6,2)}$	$s_t^{(6,3)}$	$s_t^{(6,4)}$	$s_t^{(6,5)}$	$s_t^{(6,6)}$
0	1	1	2	4	7	13
1	25	45	83	153	281	517
2	952	1750	3219	5921	10890	20030
3	36842	67762	124634	229238	421634	$\cdots$

(23)

It is not hard to observe

$$s_3^{(6,1)} = 39(952) - 11(25) + 1 - 12 = 39s_2^{(6,1)} - 11s_1^{(6,1)} + s_0^{(6,1)} - 12,$$
$$s_3^{(6,2)} = 39(1750) - 11(45) + 1 + 6 = 39s_2^{(6,2)} - 11s_1^{(6,2)} + s_0^{(6,2)} + 6,$$
$$\vdots$$
$$s_3^{(6,6)} = 39(20030) - 11(517) + 13 + 10 = 39s_2^{(6,6)} - 11s_1^{(6,6)} + s_0^{(6,6)} + 10, \quad (24)$$

so it proves $s_t^{(6,r)} = 39s_{t-1}^{(6,r)} - 11s_{t-2}^{(6,r)} + s_{t-3}^{(6,r)} + \lambda^{(r)}$ when $t = 3$. Thus if we assume the identity holds for some t then Lemma 1 shows that

$$s_{t+1}^{(6,r)} = s_t^{(6,r)} + T_{6(t+1)+r}$$
$$= \left(39s_{t-1}^{(6,r)} - 11s_{t-2}^{(6,r)} + s_{t-3}^{(6,r)} + \lambda^{(r)}\right) + \left(39T_{6t+r} - 11T_{6(t-1)+r} + T_{6(t-2)+r}\right)$$
$$= 39s_t^{(6,r)} - 11s_{t-1}^{(6,r)} + s_{t-2}^{(6,r)} + \lambda^{(r)}. \quad (25)$$

Finally we also notice $28s_2^{(6,r)} = T_{6(3)+r} - 10T_{6(2)+r} + T_{6+r} - \lambda^{(r)}$. Indeed,

$$\begin{aligned} T_{19} - 10T_{13} + T_7 + 12 &= 35890 - 10(927) + 24 + 12 \\ &= 28(952) = 28s_2^{(6,1)}, \\ T_{20} - 10T_{14} + T_{14} - 6 &= 66012 - 10(1705) + 44 - 6 \\ &= 28(1750) = 28s_2^{(6,2)}, \\ &\vdots \\ T_{24} - 10T_{18} + T_{18} - 10 &= 755476 - 10(19513) + 504 \\ &- 10 = 28s_2^{(6,6)}. \end{aligned} \tag{26}$$

So if we assume $28s_{t-1}^{(6,r)} = T_{6t+r} - 10T_{6(t-1)+r} + T_{6(t-2)+r} - \lambda^{(r)}$ then

$$\begin{aligned} 28s_t^{(6,r)} &= \left(T_{6t+r} - 10T_{6(t-1)+r} + T_{6(t-2)+r} - \lambda^{(r)}\right) \\ &\quad + 28T_{6t+r} \\ &= \left(39T_{6t+r} - 11T_{6(t-1)+r} + T_{6(t-2)+r}\right) - 10T_{6t+r} \\ &\quad + T_{6(t-1)+r} - \lambda^{(r)} \\ &= T_{6(t+1)+r} - 10T_{6t+r} + T_{6(t-1)+r} - \lambda^{(r)}. \end{aligned} \tag{27}$$

□

3. The Tails $\lambda^{(k,r)}$ of the Partial Sum $s_t^{(k,r)}$

With the use of coefficients (a_k, b_k) satisfying the k-step Tribonacci rule $T_{kt+r} = a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + T_{k(t-3)+r}$ in Lemma 1, the identities in Theorem 3 can be restated such that

$$\begin{aligned} s_t^{(5,r)} &= a_5 s_{t-1}^{(5,r)} + b_5 s_{t-2}^{(5,r)} + s_{t-3}^{(5,r)} + \lambda^{(r)} \\ &\qquad \text{with } \left\{\lambda^{(r)}\right\}_{r=1}^{5} = \{-7, 3, 5, 1, 9\}, \\ s_t^{(6,r)} &= a_6 s_{t-1}^{(6,r)} + b_6 s_{t-2}^{(6,r)} + s_{t-3}^{(6,r)} + \lambda^{(r)} \\ &\qquad \text{with } \left\{\lambda^{(r)}\right\}_{r=1}^{6} = \{-12, 6, 4, -2, 8, 10\}. \end{aligned} \tag{28}$$

In this sense, we are able to recast Theorem 2 as

$$\begin{aligned} s_t^{(3,r)} &= a_3 s_{t-1}^{(3,r)} + b_3 s_{t-2}^{(3,r)} + s_{t-3}^{(3,r)} + \lambda^{(r)} \\ &\qquad \text{with } \left\{\lambda^{(r)}\right\}_{r=1}^{3} = \{-1, 1, 1\}, \\ s_t^{(4,r)} &= a_4 s_{t-1}^{(4,r)} + b_4 s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} + \lambda^{(r)} \\ &\qquad \text{with } \left\{\lambda^{(r)}\right\}_{r=1}^{4} = \{-4, 4, 4, 4\}. \end{aligned} \tag{29}$$

We will call $\{\lambda^{(r)}\}_{r=1}^{k}$ the tail set of $s_t^{(k,r)} = \sum_{i=0}^{t} T_{ki+r}$.

Theorem 4. *With (a_k, b_k) $(1 \le k \le 10)$ in Lemma 1, one has the following:*

(1) $s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + s_{t-3}^{(k,r)} + \lambda^{(k,r)}$,

(2) $s_t^{(k,r)} = (1/(a_k + b_k))(T_{k(t+1)+r} + (b_k + 1)T_{kt+r} + T_{k(t-1)+r} - \lambda^{(k,r)})$,

where the tails $\{\lambda^{(k,r)}\}$ are defined as follows:

(30)

k	$\left\{\lambda^{(k,r)}\right\}_{r=1}^{k}$
1	$\{1\}$
2	$\{0, 2\}$
3	$\{-1, 1, 1\}$
4	$\{-4, 4, 4, 4\}$
5	$\{-7, 3, 5, 1, 9\}$
6	$\{-12, 6, 4, -2, 8, 10\}$
7	$\{-29, 13, 9, -7, 15, 17, 25\}$
8	$\{-48, 16, 16, -16, 16, 16, 16, 48\}$
9	$\{-87, 35, 21, -31, 25, 15, 9, 49, 73\}$
10	$\{-176, 66, 44, -66, 44, 22, 0, 66, 88, 154\}$

Proof. When $3 \le k \le 6$, (1) is due to Theorems 2 and 3. If $k = 1$ then $s_t^{(1,1)} = \sum_{i=0}^{t} T_{i+1}$ is the sum of all $t+1$ numbers from T_1 to T_{t+1}; hence

$$\begin{aligned} s_1^{(1,1)} + s_2^{(1,1)} + s_3^{(1,1)} &= \left(T_1 + T_2\right) + \left(T_1 + T_2 + T_3\right) \\ &\quad + \left(T_1 + T_2 + T_3 + T_4\right) \\ &= \left(T_1 + T_2\right) + T_3 + T_4 + T_5 + T_1 \\ &\quad - T_3 = s_4^{(1,1)} - 1. \end{aligned} \tag{31}$$

So for any $t > 0$, the identity $s_t^{(1,1)} = s_{t-1}^{(1,1)} + s_{t-2}^{(1,1)} + s_{t-3}^{(1,1)} + \lambda^{(1,1)}$ with $\lambda^{(1,1)} = 1$ can be proved by induction.

If $k = 2$, making use of the table of $s_t^{(2,r)}$, it is easy to see that

$$\begin{aligned} s_3^{(2,1)} &= 34 = 3s_2^{(2,1)} + s_1^{(2,1)} + s_0^{(2,1)} + \lambda^{(2,1)} \\ &\qquad \text{with } \lambda^{(2,1)} = 0, \\ s_3^{(2,2)} &= 62 = 3s_2^{(2,2)} + s_1^{(2,2)} + s_0^{(2,2)} + \lambda^{(2,2)} \\ &\qquad \text{with } \lambda^{(2,2)} = 2. \end{aligned} \tag{32}$$

Thus it can be generalized to $s_t^{(2,r)} = a_2 s_{t-1}^{(2,r)} + b_2 s_{t-2}^{(2,r)} + s_{t-3}^{(2,r)} + \lambda^{(k,r)}$ for any $t > 0$ and $1 \le r \le 2$, since $(a_2, b_2) = (3, 1)$.

Now when $7 \le k \le 10$, (1) can be observed. For instance, the table of $s_t^{(7,r)}$ shows that

$$\begin{aligned} s_3^{(7,1)} &= 71\,(3181) + 15\,(45) + 1 - 29 = 71 s_2^{(7,1)} \\ &\quad + 15 s_1^{(7,1)} + s_0^{(7,1)} - 29, \\ s_3^{(7,2)} &= 71\,(5850) + 15\,(82) + 1 + 13 = 71 s_2^{(7,2)} \\ &\quad + 15 s_1^{(7,2)} + s_0^{(7,2)} + 13, \\ &\ \vdots \\ s_3^{(7,7)} &= 71\,(123144) + 15\,(1729) + 24 + 25 = 71 s_2^{(7,7)} \\ &\quad + 15 s_1^{(7,7)} + s_0^{(7,7)} + 25. \end{aligned} \tag{33}$$

These observations together with mathematical induction imply

$$\begin{aligned} s_t^{(7,r)} &= 71 s_{t-1}^{(7,r)} + 15 s_{t-2}^{(7,r)} + s_{t-3}^{(7,r)} + \lambda^{(7,r)} \\ &= a_7 s_{t-1}^{(7,r)} + b_7 s_{t-2}^{(7,r)} + s_{t-3}^{(7,r)} + \lambda^{(7,r)}. \end{aligned} \tag{34}$$

The rest follows similarly.

Since the coefficients a_k and b_k satisfy $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$ in Lemma 1, $s_t^{(k,r)}$ can be expressed by T_{kt+r}, $T_{k(t-1)+r}$, and $T_{k(t-2)+r}$ with $\lambda^{(k,r)}$ such that

$$\begin{aligned} (a_1 + b_1)\, s_t^{(1,r)} &= 2 s_t^{(1,r)} = T_{(t+1)+r} + 2T_{t+r} + T_{(t-1)+r} \\ &\quad - \lambda^{(1,r)}, \\ (a_2 + b_2)\, s_t^{(2,r)} &= 4 s_t^{(2,r)} = T_{2(t+1)+r} + 2T_{2t+r} + T_{2(t-1)+r} \\ &\quad - \lambda^{(2,r)}, \\ &\ \vdots \\ (a_{10} + b_{10})\, s_t^{(10,r)} &= 484 s_t^{(10,r)} = T_{10(t+1)+r} + 42 T_{10t+r} \\ &\quad + T_{10(t-1)+r} - \lambda^{(10,r)}. \end{aligned} \tag{35}$$

So (2) also follows immediately. □

In particular if $k = 1$ then $r = 1$ and $a_1 = b_1 = \lambda^{(1,1)} = 1$; Theorem 4 shows

$$s_t^{(1,1)} = s_{t-1}^{(1,1)} + s_{t-2}^{(1,1)} + s_{t-3}^{(1,1)} + 1. \tag{36}$$

But since $s_t^{(1,1)} = \sum_{i=0}^{t+1} T_i = S_{t+1}$, we have $S_{t+1} = S_t + S_{t-1} + S_{t-2} + 1$; this is Lemma 1 in [6]. Moreover Theorem 4 implies

$$\begin{aligned} s_t^{(1,1)} &= \frac{1}{2}\left(T_{t+2} + 2T_{t+1} + T_t - 1\right) \\ &= \frac{1}{2}\left(T_{t+3} + T_{t+1} - 1\right). \end{aligned} \tag{37}$$

It means $S_{t+1} = (1/2)(T_{t+3} + T_{t+1} - 1)$, that is, Theorem 2 in [6].

4. Cyclic Rule for Partial Sums

The partial sum $s_t^{(k,r)}$ and its tail $\lambda^{(k,r)}$ were discussed when $1 \le k \le 10$. We investigate them for all $k > 0$ by showing certain cyclic rules.

Theorem 5. *For any $k > 0$, $s_t^{(k,r)}$ satisfies the following cyclic rules:*

(1) $s_t^{(k,r)} + s_t^{(k,r+1)} + s_t^{(k,r+2)} = s_t^{(k,r+3)}$ *for all* $1 \le r \le k-3$;

(2) $s_t^{(k,k-2)} + s_t^{(k,k-1)} + s_t^{(k,k)} = s_{t+1}^{(k,1)} - 1$;

(3) $s_t^{(k,k-1)} + s_t^{(k,k)} + s_{t+1}^{(k,1)} = s_{t+1}^{(k,2)}$, *and* $s_t^{(k,k)} + s_{t+1}^{(k,1)} + s_{t+1}^{(k,2)} = s_{t+1}^{(k,3)}$.

Proof. When $1 \le r \le k-3$, (1) is clear from

$$\begin{aligned} & s_t^{(k,r)} + s_t^{(k,r+1)} + s_t^{(k,r+2)} \\ &\quad = \sum_{i=0}^{t}\left(T_{ki+r} + T_{ki+(r+1)} + T_{ki+(r+2)}\right) = \sum_{i=0}^{t} T_{ki+(r+3)} \\ &\quad = s_t^{(k,r+3)}. \end{aligned} \tag{38}$$

And (2) also follows from

$$\begin{aligned} & s_t^{(k,k-2)} + s_t^{(k,k-1)} + s_t^{(k,k)} \\ &\quad = \sum_{i=0}^{t}\left(T_{ki+(k-2)} + T_{ki+(k-1)} + T_{ki+k}\right) = \sum_{i=0}^{t} T_{ki+(k+1)} \\ &\quad = \sum_{j=1}^{t+1} T_{kj+1} = \sum_{j=0}^{t+1} T_{kj+1} - T_1 = s_{t+1}^{(k,1)} - 1, \end{aligned} \tag{39}$$

since $T_1 = 1$. Moreover

$$\begin{aligned} s_t^{(k,k-1)} + s_t^{(k,k)} + s_{t+1}^{(k,1)} &= \sum_{i=0}^{t} T_{ki+(k-1)} + \sum_{i=0}^{t} T_{ki+k} \\ &\quad + \sum_{i=0}^{t+1} T_{ki+1} \\ &= \sum_{i=1}^{t+1}\left(T_{ki-1} + T_{ki} + T_{ki+1}\right) + T_1 \\ &= \sum_{i=1}^{t+1} T_{ki+2} + T_2 = s_{t+1}^{(k,2)}, \end{aligned} \tag{40}$$

because $T_1 = T_2$. Finally, we also have

$$\begin{aligned} & s_t^{(k,k)} + s_{t+1}^{(k,1)} + s_{t+1}^{(k,2)} \\ &\quad = \sum_{i=1}^{t+1} T_{ki} + \sum_{i=1}^{t+1} T_{ki+1} + T_1 + \sum_{i=1}^{t+1} T_{ki+2} + T_2 \\ &\quad = \sum_{i=1}^{t+1} \left(T_{ki} + T_{ki+1} + T_{ki+2}\right) + T_1 + T_2 \\ &\quad = \sum_{i=1}^{t+1} T_{ki+3} + \left(T_0 + T_1 + T_2\right) = s_{t+1}^{(k,3)}. \end{aligned} \tag{41}$$

□

Next theorem is about the cyclic rule of the tail $\{\lambda^{(k,r)}\}_{r=1}^{k}$ satisfying $s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + s_{t-3}^{(k,r)} + \lambda^{(k,r)}$.

Theorem 6. *For any $k > 0$, $\{\lambda^{(k,r)}\}_{r=1}^{k}$ satisfies the following cyclic rules:*

(1) $\lambda^{(k,r)} + \lambda^{(k,r+1)} + \lambda^{(k,r+2)} = \lambda^{(k,r+3)}$ *for all* $1 \le r \le k-3$;

(2) $\lambda^{(k,k-2)} + \lambda^{(k,k-1)} + \lambda^{(k,k)} = \lambda^{(k,1)} + (a_k + b_k)$;

(3) $\lambda^{(k,k-1)} + \lambda^{(k,k)} + \lambda^{(k,1)} = \lambda^{(k,2)}$, *and* $\lambda^{(k,k)} + \lambda^{(k,1)} + \lambda^{(k,2)} = \lambda^{(k,3)}$.

Proof. When $1 \le k \le 10$, Theorem 4 shows that $\{\lambda^{(k,r)}\}$ satisfies (1). For instance, if $k = 10$, the set $\{\lambda^{(10,r)}\} = \{-176, 66, 44, -66, 44, 22, 0, 66, 88, 154\}$ implies $-176 + 66 + 44 = -66, \ldots, 22 + 0 + 66 = 88$, and $0 + 66 + 88 = 154$.

Suppose $1 \le r \le k-3$. Then Theorem 5 gives rise to

$$\begin{aligned} & \lambda^{(k,r)} + \lambda^{(k,r+1)} + \lambda^{(k,r+2)} \\ &\quad = \left(s_t^{(k,r)} + s_t^{(k,r+1)} + s_t^{(k,r+2)}\right) \\ &\qquad - a_k \left(s_{t-1}^{(k,r)} + s_{t-1}^{(k,r+1)} + s_{t-1}^{(k,r+2)}\right) \\ &\qquad - b_k \left(s_{t-2}^{(k,r)} + s_{t-2}^{(k,r+1)} + s_{t-2}^{(k,r+2)}\right) \\ &\qquad - \left(s_{t-3}^{(k,r)} + s_{t-3}^{(k,r+1)} + s_{t-3}^{(k,r+2)}\right) \\ &\quad = s_t^{(k,r+3)} - a_k s_{t-1}^{(k,r+3)} - b_k s_{t-2}^{(k,r+3)} - s_{t-3}^{(k,r+3)} \\ &\quad = \lambda^{(k,r+3)}. \end{aligned} \tag{42}$$

Now, for (2), (a_k, b_k) in Lemma 1 and $\lambda^{(k,r)}$ in Theorem 4 show

k	$a_k + b_k$	$\lambda^{(k,k-2)} + \lambda^{(k,k-1)} + \lambda^{(k,k)} - (a_k + b_k)$	$\lambda^{(k,1)}$
3	2	$-1 + 1 + 1 - (2)$	-1
4	16	$4 + 4 + 4 - (16)$	-4
5	22	$5 + 1 + 9 - (22)$	-7
		⋮	
10	484	$66 + 88 + 154 - (484)$	-176

(43)

which proves (2) if $1 \le k \le 10$. Now for any $k > 0$, we have

$$\begin{aligned} & \lambda^{(k,k-2)} + \lambda^{(k,k-1)} + \lambda^{(k,k)} \\ &\quad = s_t^{(k,k-2)} + s_t^{(k,k-1)} + s_t^{(k,k)} \\ &\qquad - a_k \left(s_{t-1}^{(k,k-2)} + s_{t-1}^{(k,k-1)} + s_{t-1}^{(k,k)}\right) \\ &\qquad - b_k \left(s_{t-2}^{(k,k-2)} + s_{t-2}^{(k,k-1)} + s_{t-2}^{(k,k)}\right) \\ &\qquad - \left(s_{t-3}^{(k,k-2)} + s_{t-3}^{(k,k-1)} + s_{t-3}^{(k,k)}\right) \\ &\quad = \left(s_{t+1}^{(k,1)} - 1\right) - a_k \left(s_t^{(k,1)} - 1\right) - b_k \left(s_{t-1}^{(k,1)} - 1\right) \\ &\qquad - \left(s_{t-2}^{(k,1)} - 1\right) \\ &\quad = \left(s_{t+1}^{(k,1)} - a_k s_t^{(k,1)} - b_k s_{t-1}^{(k,1)} - s_{t-2}^{(k,1)}\right) - 1 + a_k + b_k \\ &\qquad + 1 = \lambda^{(k,1)} + \left(a_k + b_k\right), \end{aligned} \tag{44}$$

due to Theorem 5. Similarly (3) also follows from Theorem 5 such that

$$\begin{aligned} & \lambda^{(k,k-1)} + \lambda^{(k,k)} + \lambda^{(k,1)} \\ &\quad = s_t^{(k,k-1)} + s_t^{(k,k)} + s_t^{(k,1)} \\ &\qquad - a_k \left(s_{t-1}^{(k,k-1)} + s_{t-1}^{(k,k)} + s_{t-1}^{(k,1)}\right) \\ &\qquad - b_k \left(s_{t-2}^{(k,k-1)} + s_{t-2}^{(k,k)} + s_{t-2}^{(k,1)}\right) \\ &\qquad - \left(s_{t-3}^{(k,k-1)} + s_{t-3}^{(k,k)} + s_{t-3}^{(k,1)}\right) \\ &\quad = s_{t+1}^{(k,2)} - a_k s_t^{(k,2)} - b_k s_{t-1}^{(k,2)} - s_{t-2}^{(k,2)} = \lambda^{(k,2)}. \end{aligned} \tag{45}$$

And the rest follows analogously. □

The identity $\lambda^{(k,r)} = s_t^{(k,r)} - a_k s_{t-1}^{(k,r)} - b_k s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}$ implies that $\{\lambda^{(k,r)}\}$ does not depend on t. Hence without loss of generality if we assume $t = 3$ then

$$\lambda^{(k,r)} = s_3^{(k,r)} - a_k s_2^{(k,r)} - b_k s_1^{(k,r)} - s_0^{(k,r)}. \tag{46}$$

But since $s_0^{(k,r)} = T_r$, $s_1^{(k,r)} = T_r + T_{k+r}$, $s_2^{(k,r)} = T_r + T_{k+r} + T_{2k+r}$, and $s_3^{(k,r)} = T_r + T_{k+r} + T_{2k+r} + T_{3k+r} = 2T_r + (1 + b_k)T_{k+r} + (1 + a_k)T_{2k+r}$, we have

$$\lambda^{(k,r)} = \left(1 - a_k - b_k\right) T_r + \left(1 - a_k\right) T_{k+r} + T_{2k+r}. \tag{47}$$

Therefore we are able to have tails $\{\lambda^{(k,r)}\}$ for all $k > 10$; for instance,

$$\begin{aligned}
\left\{\lambda^{(11,r)}\right\} &= \{-305, 105, 81, -119, 67, 29, \\
&\quad -23, 73, 79, 129, 281\}, \\
\left\{\lambda^{(12,r)}\right\} &= \{-564, 212, 131, -220, 124, 36, \\
&\quad -60, 100, 76, 116, 292, 484\}, \\
\left\{\lambda^{(13,r)}\right\} &= \{-1071, 387, 261, -423, 225, 63, \\
&\quad -135, 153, 81, 99, 333, 513, 945\}, \\
&\vdots
\end{aligned} \tag{48}$$

We remark that a special case $s_{t-1}^{(k,k)} = \sum_{i=0}^{t} T_{ki}$ was proved in [5] by making use of the three roots of $x^3 - x^2 - x - 1 = 0$. Considering the difficulty of finding roots of the cubic polynomial, the identity $s_{t-1}^{(k,k)} = (1/(a_k + b_k))(T_{k(t+1)} + (b_k + 1)T_{kt} + T_{k(t-1)} - \lambda^{(k,k)})$ in Theorem 4 seems a little bit simple and easy.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] K. Kuhapatanakul, "On the sums of reciprocal generalized Fibonacci numbers," *Journal of Integer Sequences*, vol. 16, no. 7, Article ID 13.7.1, 8 pages, 2013.

[2] H. Ohtsuka and S. Nakamura, "On the sum of reciprocal Fibonacci numbers," *The Fibonacci Quarterly*, vol. 46, no. 2, pp. 153–159, 2008.

[3] G. J. Zhang, "The infinite sum of reciprocal of the Fibonacci numbers," *Journal of Mathematical Research and Exposition*, vol. 31, no. 6, pp. 1030–1034, 2011.

[4] J. Feng, "More identities on the Tribonacci numbers," *Ars Combinatoria*, vol. 100, pp. 73–78, 2011.

[5] N. Irmak and M. Alp, "Tribonacci numbers with indices in arithmetic progression and their sums," *Miskolc Mathematical Notes*, vol. 14, no. 1, pp. 125–133, 2013.

[6] E. Kilic, "Tribonacci sequences with certain indices and their sums," *Ars Combinatoria*, vol. 86, pp. 13–22, 2008.

[7] S. Pethe, "Some identities for tribonacci sequences," *Fibonacci Quarterly*, vol. 26, no. 2, pp. 144–151, 1988.

[8] E. Choi and J. Jo, "Identities involving tribonacci numbers," *Journal of the Chungcheong Mathematical Society*, vol. 28, no. 1, pp. 39–51, 2015.