



Tripled and coincidence fixed point theorems for contractive mappings satisfying Φ -maps in partially ordered metric spaces

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Abstract

In this paper, we study some tripled fixed and coincidence point theorems for two mappings $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ satisfying a nonlinear contraction based on ϕ -maps. Our results extend and improve many existing results in the literature. Also, we introduce an example to support the validity of our results.

1 Introduction and Preliminaries

The notion of a coupled fixed point for a mapping $F: X \times X \rightarrow X$ was initiated by Gnana Bhaskar and Lakshmikantham [1] and they proved some interesting coupled fixed point theorems, while Ćirić and Lakshmikantham [2] introduced the concept of the coupled fixed point for two mappings $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ and established many existing theorems. Vasile Berinde and Marin Borcut [3, 4] initiated the concept of a tripled fixed point of the mapping $F: X \times X \times X \rightarrow X$ and the concept of a tripled coincidence point of the two mappings $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ and extended the results of Gnana Bhaskar and Lakshmikantham [1] and Ćirić and Lakshmikantham [2] to the interesting tripled fixed and coincidence point theorems. For some

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coupled fixed point results in different metric spaces we refer the reader to [5]-[25].

Consistent with Berinde and Borcut [3, 4], we give the following definitions and preliminaries.

Definition 1.1 ([3]). Let X be a nonempty set and $F: X \times X \times X \rightarrow X$ be a given mapping. An element $(x, y, z) \in X \times X \times X$ is called a *tripled fixed point* of F if

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Let (X, d) be a metric space. The mapping

$$\bar{d}: X \times X \times X \rightarrow \mathbb{R}, \quad \bar{d}((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w),$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d .

Definition 1.2 ([3]). Let (X, \leq) be a partially ordered set and $F: X \times X \times X \rightarrow X$ be a mapping. We say that F has the *mixed monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and z and is monotone non-increasing in y ; that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 & \quad \text{implies} \quad F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 & \quad \text{implies} \quad F(x, y_1, z) \geq F(x, y_2, z), \end{aligned}$$

and

$$z_1, z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies} \quad F(x, y, z_1) \leq F(x, y, z_2).$$

The main results of [3] are:

Theorem 1.1 ([3]). Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a continuous mapping such that F has the mixed monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w) \quad (1)$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Theorem 1.2 ([3]). Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w)$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. Assume that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Also, Borcut and Berinde [4] introduced the concept of a tripled coincidence point of mappings $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$.

Definition 1.3 ([4]). Let X be a nonempty set. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings. An element $(x, y, z) \in X \times X \times X$ is called a *triple coincidence point* of F and g if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz.$$

Definition 1.4 ([4]). Let (X, \leq) be a partially ordered set and $F: X \times X \times X \rightarrow X$ be a mapping. We say that F has the *mixed g -monotone property* if $F(x, y, z)$ is monotone non-decreasing in x and z and is monotone non-increasing in y ; that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \leq gx_2 \quad \text{implies} \quad & F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad gy_1 \leq gy_2 \quad \text{implies} \quad & F(x, y_1, z) \geq F(x, y_2, z), \end{aligned}$$

and

$$z_1, z_2 \in X, \quad gz_1 \leq gz_2 \quad \text{implies} \quad F(x, y, z_1) \leq F(x, y, z_2).$$

The main results of [4] is:

Theorem 1.3 ([4]). Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ such that F has the mixed g -monotone property. Assume that there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(gx, gu) + kd(gy, gv) + ld(gz, gw) \tag{2}$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \geq gw$. Suppose that $F(X \times X \times X \subseteq gX)$, g is continuous and commute with F and also suppose either

- (a) F is continuous or
- (b) X has the following property:

- (i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g a tripled coincidence point.

By following Matkowski [26], we let Φ be the set of all non-decreasing functions $\phi: [0, +\infty) \rightarrow [0, +\infty)$ such that $\lim_{n \rightarrow +\infty} \phi^n(t) = 0$, for all $t > 0$, where ϕ^n denotes the n -th iterate of ϕ . Then its an easy matter to show that:

- (1) $\phi(t) < t$ for all $t > 0$,
- (2) $\phi(0) = 0$.

If $\phi \in \Phi$, then we call ϕ to be a ϕ -map.

In this paper, we utilize the notion of ϕ -map to prove a number of tripled fixed and coincidence point results for two mapping $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ in ordered metric spaces. Our results generalize Theorems 1.1 and 1.3. Also, we support our results by introducing a nontrivial example satisfying our main results and doesn't satisfy the conditions 1 and 2 of Theorems 1.1 and 1.3 respectively.

2 Main Results

Our first result is the following.

Theorem 2.1. *Let (X, \leq) be a partially ordered set and (X, d) be a metric space. Let $F: X \times X \times X \rightarrow X$, $g: X \rightarrow X$ be two mappings. Suppose the following*

- 1. gX is a complete subspace of X ,
- 2. $F(X \times X \times X) \subseteq gX$,
- 3. F has the mixed g -monotone property,
- 4. F is continuous and
- 5. g is continuous and commute with F .

Assume that there exists $\phi \in \Phi$ such that

$$d(F(x, y, z), F(u, v, w)) \leq \phi \left(\max \left\{ d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(u, v, w), gu), d(F(w, v, u), gw) \right\} \right), \tag{3}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

Proof. Suppose there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$. Define $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Then $gx_0 \leq gx_1$, $gy_0 \geq gy_1$ and $gz_0 \leq gz_1$. Again, define $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Since F has the mixed g -monotone property, we have $gx_0 \leq gx_1 \leq gx_2$, $gy_2 \leq gy_1 \leq gy_0$ and $gz_0 \leq gz_1 \leq gz_2$. Continuing this process, we can construct three sequences (gx_n) , (gy_n) and (gz_n) in X such that

$$gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}) \leq gx_{n+1} = F(x_n, y_n, z_n),$$

$$gy_{n+1} = F(y_n, x_n, y_n) \leq gy_n = F(y_{n-1}, x_{n-1}, y_{n-1}),$$

and

$$gz_n = F(z_{n-1}, y_{n-1}, x_{n-1}) \leq gz_{n+1} = F(z_n, y_n, x_n).$$

If, for some integer n , we have $(gx_{n+1}, gy_{n+1}, gz_{n+1}) = (gx_n, gy_n, gz_n)$, then $F(x_n, y_n, z_n) = gx_n$, $F(y_n, x_n, y_n) = gy_n$, and $F(z_n, y_n, x_n) = gz_n$; that is, (x_n, y_n, z_n) is a tripled coincidence point of F . Thus we shall assume that $(gx_{n+1}, gy_{n+1}, gz_{n+1}) \neq (gx_n, gy_n, gz_n)$ for all $n \in \mathbb{N}$; that is, we assume that either $gx_{n+1} \neq gx_n$ or $gy_{n+1} \neq gy_n$ or $gz_{n+1} \neq gz_n$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} & d(gx_{n+1}, gx_n) \\ &= d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leq \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\ &\quad \left. \left. d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n), \right. \right. \\ &\quad \left. \left. d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1}), d(F(z_{n-1}, y_{n-1}, x_{n-1}), gz_{n-1}) \right\} \right) \\ &= \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\ &\quad \left. \left. d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right) \\ &\leq \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\ &\quad \left. \left. d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right), \end{aligned} \tag{4}$$

$$\begin{aligned}
& d(gy_n, gy_{n+1}) \\
= & d(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\
\leq & \phi \left(\max \left\{ d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(F(y_{n-1}, x_{n-1}, y_{n-1}), gy_{n-1}), \right. \right. \\
& \left. \left. d(F(y_n, x_n, y_n), gy_n) \right\} \right) \\
= & \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(y_{n+1}, y_n)\}) \\
\leq & \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), d(gx_{n+1}, gx_n), \right. \right. \\
& \left. \left. d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right), \tag{5}
\end{aligned}$$

and

$$\begin{aligned}
& d(gz_{n+1}, gz_n) \\
= & d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
\leq & \phi \left(\max \left\{ d(gz_n, gz_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1}), \right. \right. \\
& d(F(z_n, y_n, x_n), gz_n), d(F(x_n, y_n, z_n), gx_n), \\
& \left. \left. d(F(z_{n-1}, y_{n-1}, x_{n-1}), gz_{n-1}), d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1}) \right\} \right) \\
= & \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\
& \left. \left. d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right) \\
\leq & \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\
& \left. \left. d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right), \tag{6}
\end{aligned}$$

From (4), (5), and (6), it follows that

$$\begin{aligned}
& \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\
\leq & \phi \left(\max \left\{ d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \right. \right. \\
& \left. \left. d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right). \tag{7}
\end{aligned}$$

If

$$\begin{aligned}
& \max \{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1}), \\
& \quad d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n)\} \\
= & \max \{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\},
\end{aligned}$$

then from (7) we have

$$\begin{aligned} & \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\ & \leq \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\}) \\ & < \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\}, \end{aligned}$$

a contradiction. Thus (7) becomes

$$\begin{aligned} & \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\ & \leq \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\}). \end{aligned} \tag{8}$$

By repeating (8) n -times, we get that

$$\begin{aligned} & \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} \\ & \leq \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gz_n, gz_{n-1})\}) \\ & \leq \phi^2(\max\{d(gx_{n-1}, gx_{n-2}), d(gy_{n-1}, gy_{n-2}), d(gz_{n-1}, gz_{n-2})\}) \\ & \vdots \\ & \leq \phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}). \end{aligned} \tag{9}$$

Now, we shall show that (gx_n) , (gy_n) , and (gz_n) are Cauchy sequence in X . Let $\epsilon > 0$. Since

$$\lim_{n \rightarrow +\infty} \phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}) = 0$$

and $\epsilon > \phi(\epsilon)$, there exist $n_0 \in \mathbb{N}$ such that

$$\phi^n(\max\{d(gx_1, gx_0), d(gy_1, gy_0), d(gz_1, gz_0)\}) < \epsilon - \phi(\epsilon),$$

for all $n \geq n_0$.

This implies that,

$$\max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1}), d(gz_{n+1}, gz_n)\} < \epsilon - \phi(\epsilon), \tag{10}$$

for all $n \geq n_0$.

For $m, n \in \mathbb{N}$, we will prove by induction on m that

$$\max\{d(gx_n, gx_m), d(gy_n, gy_m), d(gz_n, gz_m)\} < \epsilon, \tag{11}$$

for all $m \geq n \geq n_0$.

For $m = n + 1$, (11) is true by using (10) and noting that $\epsilon - \phi(\epsilon) < \epsilon$. Now suppose that (11) holds for $m = k$; that is,

$$\max\{d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k)\} < \epsilon. \tag{12}$$

For $m = k + 1$, we have

$$\begin{aligned}
& d(gx_n, gx_{k+1}) \\
\leq & d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{k+1}) \\
\leq & \epsilon - \phi(\epsilon) + d(F(x_k, y_k, z_k), F(x_n, y_n, z_n)) \\
\leq & \epsilon - \phi(\epsilon) + \phi\left(\max\left\{d(gx_k, gx_n), d(gy_k, gy_n), d(gz_k, gz_n), \right. \right. \\
& d(F(x_k, y_k, z_k), gx_k), d(F(z_k, y_k, x_k), gz_k), \\
& \left. \left. d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n)\right\}\right) \\
= & \epsilon - \phi(\epsilon) + \phi\left(\max\left\{d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), \right. \right. \\
& d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), d(gz_{k+1}, gz_k)\left.\right\}\right) \\
\leq & \epsilon - \phi(\epsilon) + \phi\left(\max\left\{d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), \right. \right. \\
& d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), \\
& \left. \left. d(gy_{k+1}, gy_k), d(gz_{k+1}, gz_k)\right\}\right). \tag{13}
\end{aligned}$$

Using (10) and (12), we have

$$\begin{aligned}
& \max\left\{d(gx_n, gx_k), d(gy_n, gy_k), d(gz_n, gz_k), d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), \right. \\
& \left. d(gz_{n+1}, gz_n), d(gx_{k+1}, gx_k), d(gy_{k+1}, gy_k), d(gz_{k+1}, gz_k)\right\} \\
\leq & \max\{\epsilon, \epsilon - \phi(\epsilon)\} = \epsilon. \tag{14}
\end{aligned}$$

From (13), (14) and the properties of ϕ , we obtain

$$d(gx_n, gx_{n+1}) < \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon.$$

Similarly, we show that

$$d(gy_n, gy_{k+1}) < \epsilon,$$

and

$$d(gz_n, gz_{k+1}) < \epsilon.$$

Hence, we have

$$\max\{d(gx_n, gx_{k+1}), d(gy_n, gy_{k+1}), d(gz_n, gz_{k+1})\} < \epsilon.$$

Thus (11) holds for all $m \geq n \geq n_0$. Hence (gx_n) , (gy_n) and (gz_n) are Cauchy sequences in gX .

Since gX is complete, there exist $p, q, r \in gX$ such that (gx_n) , (gy_n) and (gz_n) converge to p , q , and r respectively. Choose $x, y, z \in X$ such that $p = gx, q = gy$ and $r = gz$. Now, we show that (p, q, r) is a coincidence point of F . Since F, g are commute and F is continuous, we have

$$ggx_{n+1} = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n) \rightarrow F(p, q, r).$$

Also, since g is continuous and $gx_n \rightarrow p$, we have $ggx_n \rightarrow gp$. By uniqueness of limit, we get $F(p, q, r) = gp$. Similarly, we show that $gq = F(q, p, q)$ and $gr = F(r, q, p)$. So (p, q, r) is a tripled coincidence point of F and g . \square

By taking $\phi(t) = kt, k \in [0, 1)$ in Theorem 2.1, we have the following:

Corollary 2.1. *Let (X, \leq) be a partially ordered set and (X, d) be a metric space. Let $F: X \times X \times X \rightarrow X, g: X \rightarrow X$ be two mappings. Suppose the following:*

1. gX is a complete subspace of X ,
2. $F(X \times X \times X) \subseteq gX$,
3. F has the mixed g -monotone property,
4. F is continuous and
5. g is continuous and commute with F .

Assume that there exists $k \in [0, 1)$ such that

$$d(F(x, y, z), F(u, v, w)) \leq k \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(u, v, w), gu), d(F(w, v, u), gw)\},$$

for all $x, y, z, u, v, w \in X$ with $x \geq u, y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0), gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

As a consequence of Corollary 2.1, we have the following:

Corollary 2.2. *Let (X, \leq) be a partially ordered set and (X, d) be a metric space. Let $F: X \times X \times X \rightarrow X, g: X \rightarrow X$ be two mappings. Suppose the following*

1. gX is a complete subspace of X ,
2. $F(X \times X \times X) \subseteq gX$,
3. F has the mixed g -monotone property,
4. F is continuous and
5. g is continuous and commute with F .

Assume that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ with $\sum_{i=1}^7 a_i < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq a_1d(gx, gu) + a_2d(gy, gv) + a_3d(gz, gw) + a_4d(F(x, y, z), gx) + a_5d(F(z, y, x), gz) + a_6d(F(u, v, w), gu) + a_7d(F(w, v, u), gw),$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

By taking $g = i_X$ (the identity mapping on X) in Theorem 2.1, Corollary 2.1 and Corollary 2.2, we have the following:

Corollary 2.3. *Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \phi \left(\max \left\{ d(x, u), d(y, v), d(z, w), d(F(x, y, z), x), d(F(z, y, x), z), d(F(u, v, w), u), d(F(w, v, u), w) \right\} \right),$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Corollary 2.4. *Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq k \max \left\{ d(x, u), d(y, v), d(z, w), d(F(x, y, z), x), d(F(z, y, x), z), d(F(u, v, w), u), d(F(w, v, u), w) \right\},$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Corollary 2.5. *Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property. Assume that there exists $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ in $[0, 1)$ with $\sum_{i=1}^7 a_i < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w) + a_4 d(F(x, y, z), x) + a_5 d(F(z, y, x), z) + a_6 d(F(u, v, w), u) + a_7 d(F(w, v, u), w),$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

By adding an additional hypotheses, the continuity of F and g in Theorem 2.1 can be dropped.

Theorem 2.2. Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings. Suppose that there exists $\phi \in \Phi$ such that

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \leq \phi(\max \{d(gx, gu), d(gy, gv), d(gz, gw), \\ & d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(u, v, w), gu), d(F(w, v, u), gw)\}), \end{aligned} \quad (15)$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \leq gw$. Suppose the following:

1. gX is a complete subspace of X ,
2. $F(X \times X \times X) \subseteq gX$,
3. F has the mixed g -monotone property,
4. F and g are commute.

Also, assume that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

Proof. By following the same process in Theorem 2.1, we construct three Cauchy sequences (gx_n) , (gy_n) and (gz_n) in gX with

$$gx_1 \leq gx_2 \leq \dots \leq gx_n \leq \dots,$$

$$gy_1 \geq gy_2 \geq \dots \geq gy_n \geq \dots,$$

and

$$gz_1 \leq gz_2 \leq \dots \leq gz_n \leq \dots$$

such that $gx_n \rightarrow p = gx \in gX$, $gy_n \rightarrow q = gy \in gX$, and $gz_n \rightarrow r = gz \in gX$, where $x, y, z \in X$. By the hypotheses on X , we have $gx_n \leq gx$, $gy_n \geq gy$ and $gz_n \leq gz$ for all $n \in \mathbb{N}$. From (15), we have

$$\begin{aligned} & d(F(x, y, z), x_{n+1}) \\ &= d(F(x, y, z), F(x_n, y_n, z_n)) \\ &\leq \phi\left(\max \left\{d(x, x_n), d(y, y_n), d(z, z_n), d(F(x, y, z), gx), d(F(z, y, x), gz), \right. \right. \\ & \left. \left. d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n)\right\}\right), \end{aligned}$$

$$\begin{aligned}
&= \phi \left(\max \left\{ d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(F(x, y, z), gx), \right. \right. \\
&\quad \left. \left. d(F(z, y, x), gz), d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right), \\
&\leq \phi \left(\max \left\{ d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(F(x, y, z), gx), \right. \right. \\
&\quad \left. \left. d(F(z, y, x), gz), d(F(y, x, y), gy), d(gx_{n+1}, gx_n), \right. \right. \\
&\quad \left. \left. d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right), \tag{16}
\end{aligned}$$

$$\begin{aligned}
&d(y_{n+1}, F(y, x, y)) \\
&= d(F(y_n, x_n, y_n), F(y, x, y)) \\
&\leq \phi \left(\max \left\{ d(y_n, y), d(x_n, x), d(F(y_n, x_n, y_n), gy_n), d(F(y, x, y), gy) \right\} \right) \\
&= \phi \left(\max \left\{ d(gy_n, gy), d(gx_n, gx), d(gy_{n+1}, gy_n), d(F(y, x, y), gy) \right\} \right) \\
&\leq \phi \left(\max \left\{ d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), \right. \right. \\
&\quad \left. \left. d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(y, x, y), gy), \right. \right. \\
&\quad \left. \left. d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right) \tag{17}
\end{aligned}$$

and

$$\begin{aligned}
&d(F(z, y, x), z_{n+1}) \\
&= d(F(z, y, x), F(z_n, y_n, x_n)) \\
&\leq \phi \left(\max \left\{ d(z, z_n), d(y, y_n), d(x, x_n), d(F(z, y, x), gz), d(F(x, y, z), gx), \right. \right. \\
&\quad \left. \left. d(F(z_n, y_n, x_n), gz_n), d(F(x_n, y_n, z_n), gx_n) \right\} \right), \\
&\leq \phi \left(\max \left\{ d(gz, gz_n), d(gy, gy_n), d(gx, gx_n), d(F(z, y, x), gz), \right. \right. \\
&\quad \left. \left. d(F(x, y, z), gx), d(gz_{n+1}, gz_n), d(gx_{n+1}, gx_n) \right\} \right), \\
&\leq \phi \left(\max \left\{ d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(F(x, y, z), gx), \right. \right. \\
&\quad \left. \left. d(F(z, y, x), gz), d(F(y, x, y), gy), d(gx_{n+1}, gx_n), \right. \right. \\
&\quad \left. \left. d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n) \right\} \right). \tag{18}
\end{aligned}$$

From (16)-(18), we have

$$\begin{aligned} & \max\{d(F(x, y, z), x_{n+1}), d(y_{n+1}, F(y, x, y)), d(F(z, y, x), z_{n+1})\} \\ \leq & \phi\left(\max\left\{d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(F(x, y, z), gx), \right. \right. \\ & d(F(z, y, x), gz), d(F(y, x, y), gy), d(gx_{n+1}, gx_n), \\ & \left. \left. d(gy_{n+1}, gy_n), d(gz_{n+1}, gz_n)\right\}\right). \end{aligned} \quad (19)$$

Our claim is:

$$\max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\} = 0.$$

To prove our claim, suppose that

$$\max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\} \neq 0.$$

Let

$$\epsilon = \max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\}.$$

Since $\epsilon > 0$, $d(gx, gx_n) \rightarrow 0$, $d(gy, gy_n) \rightarrow 0$, $d(gz, gz_n) \rightarrow 0$, $d(gx_n, gx_{n+1}) \rightarrow 0$, $d(gy_n, gy_{n+1}) \rightarrow 0$, and $d(gz_n, gz_{n+1}) \rightarrow 0$, we choose $n_0 \in \mathbb{N}$ such that

$$d(x, x_n) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

$$d(y, y_n) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

$$d(z, z_n) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

$$d(x_n, x_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

$$d(y_n, y_{n+1}) < e \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

and

$$d(z_n, z_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Thus (19) becomes

$$\begin{aligned} & \max\{d(F(x, y, z), x_{n+1}), d(y_{n+1}, F(y, x, y)), d(F(z, y, x), z_{n+1})\} \\ \leq & \phi\left(\max\left\{\frac{\epsilon}{2}, d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(y, x, y), gy)\right\}\right) \\ = & \phi\left(\max\left\{d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(y, x, y), gy)\right\}\right) \end{aligned} \quad (20)$$

for all $n \geq n_0$.

Letting $n \rightarrow +\infty$ in (20) it follows that

$$\begin{aligned} & \max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\} \\ \leq & \phi\left(\max\left\{d(F(x, y, z), gx), d(F(z, y, x), gz), d(F(y, x, y), gy)\right\}\right) \\ < & \max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\}, \end{aligned}$$

a contradiction. Therefore

$$\max\{d(F(x, y, z), gx), d(gy, F(y, x, y)), d(F(z, y, x), gz)\} = 0$$

and hence $d(F(x, y, z), gx) = 0$, $d(gy, F(y, x, y)) = 0$ and $d(F(z, y, x), gz) = 0$. Thus $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$; that is (x, y, z) is a tripled fixed point of F and g . \square

By taking $\phi(t) = kt$, $k \in [0, 1)$ in Theorem 2.2, we have the following:

Corollary 2.6. *Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings. Suppose that there exists $k \in [0, 1)$ such that*

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ \leq & k \max\{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ & d(F(z, y, x), gz), d(F(u, v, w), gu), d(F(w, v, u), gw)\}, \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \leq gw$. Suppose the following:

1. gX is a complete subspace of X ,
2. $F(X \times X \times X) \subseteq gX$,
3. F has the mixed g -monotone property,
4. F and g are commute.

Also, assume that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

As a consequence result of Corollary 2.6, we have

Corollary 2.7. *Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings.*

Suppose that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ with $\sum_{i=1}^7 a_i < 1$ such that

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ & \leq a_1 d(gx, gu) + a_2 d(gy, gv) + a_3 d(gz, gw) + a_4 d(F(x, y, z), gx) + \\ & \quad a_5 d(F(z, y, x), gz) + a_6 d(F(u, v, w), gu) + a_7 d(F(w, v, u), gw), \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$, and $gz \leq gw$. Suppose the following:

1. gX is a complete subspace of X ,
2. $F(X \times X \times X) \subseteq gX$,
3. F has the mixed g -monotone property,
4. F and g are commute.

Also, assume that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point.

By taking $g = i_X$ (the identity mapping on X) in Theorem 2.2, Corollary 2.6 and Corollary 2.7, we have the following:

Corollary 2.8. Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \leq \phi \left(\max \left\{ d(x, u), d(y, v), d(z, w), d(F(x, y, z), x), \right. \right. \\ \left. \left. d(F(z, y, x), z), d(F(u, v, w), u), d(F(w, v, u), w) \right\} \right) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. Assume also that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Corollary 2.9. Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \leq k \max \left\{ d(x, u), d(y, v), d(z, w), d(F(x, y, z), x), \right. \\ \left. d(F(z, y, x), z), d(F(u, v, w), u), d(F(w, v, u), w) \right\} \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \leq w$. Assume also that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Corollary 2.10 ([3]). Let (X, \leq) be a partially ordered set and (X, d) be a complete metric space. Let $F: X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property. Assume that there exist $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in [0, 1)$ with $\sum_{i=1}^7 a_i < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq a_1d(x, u) + a_2d(y, v) + a_3d(z, w) + a_4d(F(x, y, z), x) + a_5d(F(z, y, x), z) + a_6d(F(u, v, w), u) + a_7d(F(w, v, u), w),$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$, and $z \geq w$. Assume that X has the following properties:

- i. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,
- ii. if a non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all $n \in \mathbb{N}$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then F has a tripled fixed point.

Now, we prove a uniqueness theorem for a tripled fixed point.

Theorem 2.3. In addition to the hypotheses of Theorem 2.1 (respectfully Theorem 2.2) suppose that

$$[(gx_0 \leq gy_0) \wedge (gz_0 \leq gy_0)] \vee [(gy_0 \leq gx_0) \wedge (gy_0 \leq gz_0)].$$

Then $gx = gy = gz$.

Proof. Suppose to the contrary, that is $gx \neq gy$ or $gy \neq gz$ or $gx \neq gz$. Let

$$\epsilon = \max\{d(gx, gy), d(gx, gz), d(gy, gz)\}.$$

Since $\epsilon > 0$, $d(gx_n, gx_{n+1}) \rightarrow 0$, $d(gy_n, gy_{n+1}) \rightarrow 0$ and $d(gz_n, gz_{n+1}) \rightarrow 0$ there exist $n_0 > 0$ such that

$$d(gx_n, gx_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0,$$

$$d(gy_n, gy_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0$$

and

$$d(gz_n, gz_{n+1}) < \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Without loss of generality, we may assume that $gx_0 \leq gy_0$ and $gz_0 \leq gy_0$. By the mixed monotone property of F , we have $gx_n \leq gy_n$ and $gz_n \leq gy_n$ for all $n \in \mathbb{N}$. Thus by (3), we have

$$\begin{aligned}
& d(gy_{n+1}, gx_{n+1}) \\
&= d(F(y_n, x_n, y_n), F(x_n, y_n, z_n)) \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), d(F(y_n, x_n, y_n), gy_n), \right. \right. \\
&\quad \left. \left. d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n) \right\} \right) \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n, \right. \right. \\
&\quad \left. \left. d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right) \tag{21}
\end{aligned}$$

and

$$\begin{aligned}
& d(gy_{n+1}, gz_{n+1}) \\
&= d(F(y_n, x_n, y_n), F(z_n, y_n, x_n)) \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), d(F(y_n, x_n, y_n), gy_n), \right. \right. \\
&\quad \left. \left. d(F(x_n, y_n, z_n), gx_n), d(F(z_n, y_n, x_n), gz_n) \right\} \right) \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n, \right. \right. \\
&\quad \left. \left. d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right) \tag{22}
\end{aligned}$$

By (21) and (22), we have

$$\begin{aligned}
& \max \{ d(gy_{n+1}, gx_{n+1}), d(gy_{n+1}, gz_{n+1}) \} \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), d(gy_{n+1}), gy_n, \right. \right. \\
&\quad \left. \left. d(gx_{n+1}, gx_n), d(gz_{n+1}, gz_n) \right\} \right).
\end{aligned}$$

For $n \geq n_0$, we have

$$\begin{aligned}
& \max \{ d(gy_{n+1}, gx_{n+1}), d(gy_{n+1}, gz_{n+1}) \} \\
&\leq \phi \left(\max \left\{ d(gy_n, gx_n), d(gy_n, gz_n), \frac{\epsilon}{2} \right\} \right) \\
&\leq \phi^2 \left(\max \left\{ d(gy_{n-1}, gx_{n-1}), d(gy_{n-1}, gz_{n-1}), \frac{\epsilon}{2} \right\} \right)
\end{aligned}$$

$$\begin{aligned} &\leq \phi^3 \left(\max \left\{ d(gy_{n-2}, gx_{n-2}), d(gy_{n-2}, gz_{n-2}), \frac{\epsilon}{2} \right\} \right) \\ &\quad \vdots \\ &\leq \phi^{n-n_0} \left(\max \left\{ d(gy_{n_0}, gx_{n_0}), d(gy_{n_0}, gz_{n_0}), \frac{\epsilon}{2} \right\} \right). \end{aligned} \tag{23}$$

On letting $n \rightarrow +\infty$ in (23) and using the property of ϕ and the fact that d is continuous on its variables, we get that $\max\{d(gy, gx), d(gy, gz)\} = 0$. Hence $gy = gz = gx$, a contradiction. \square

Corollary 2.11. *In addition to the hypotheses of Corollary 2.1 (respectfully Corollary 2.6) suppose that*

$$[(gx_0 \leq gy_0) \wedge (gz_0 \leq gy_0)] \vee [(gy_0 \leq gx_0) \wedge (gy_0 \leq gz_0)].$$

Then $gx = gy = gz$.

Now, we introduce an example to support the useability of our results:

Example 2.1. Let $X = [0, 1]$ with usual ordered. Define $d: X \times X \rightarrow X$ by $d(x, y) = |x - y|$. Define $g: X \rightarrow X$ and $F: X \times X \times X \rightarrow X$ by $gx = \frac{3}{4}x$, and

$$F(x, y, z) = \begin{cases} 0, & y \geq \min\{x, z\}; \\ \frac{1}{3}(\min\{x, z\} - y), & y < \min\{x, z\}. \end{cases}$$

Then:

1. gX is a complete subspace of X .
2. $F(X \times X \times X) \subseteq gX$.
3. F and g are commute.
4. F has the mixed g -monotone property.
5. For $x, y, z, u, v, w \in X$ we have

$$\begin{aligned} &d(F(x, y, z), F(u, v, w)) \\ &\leq \frac{8}{9} \max \{ d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ &\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw) \} \end{aligned}$$

holds, for all $gx \geq gu, gy \leq gv$ and $gz \geq gw$.

6. There are no $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w)$$

holds for all $x \geq u, y \leq v$ and $z \geq w$.

7. There are no $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(gx, gu) + kd(gy, gv) + ld(gz, gw)$$

holds for all $gx \geq gu, gy \leq gv$ and $gz \geq gw$.

Proof. The proof of (1), (2) and (3) are clear. To prove (4), let $x, y, z \in X$. To show that $F(x, y, z)$ is monotone g -non-decreasing in x , let $x_1, x_2 \in X$ with $gx_1 \leq gx_2$. Then $\frac{3}{4}x_1 \leq \frac{3}{4}x_2$ and hence $x_1 \leq x_2$. If $y \geq \min\{x_1, z\}$, then $F(x_1, y, z) = 0 \leq F(x_2, y, z)$. If $y < \min\{x_1, z\}$, then

$$F(x_1, y, z) = \frac{1}{3}(\min\{x_1, z\} - y) \leq \frac{1}{3}(\min\{x_2, z\} - y) = F(x_2, y, z).$$

Therefore, $F(x, y, z)$ is monotone g -non-decreasing in x . Similarly, we may show that $F(x, y, z)$ is monotone g -non-decreasing in z and monotone g -non-increasing in y . To prove (5), given $x, y, z, u, v, w \in X$ with $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. Then $\frac{3}{4}x \geq \frac{3}{4}u$, $\frac{3}{4}y \leq \frac{3}{4}v$ and $\frac{3}{4}z \geq \frac{3}{4}w$. Hence $x \geq u$, $y \leq v$ and $z \geq w$. So, we have the following cases:

Case 1: $y \geq \min\{x, z\}$ and $v \geq \min\{u, w\}$. Here, we have

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) = 0 \\ & \leq \frac{1}{2} \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ & \quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\}. \end{aligned}$$

Case 2: $y \geq \min\{x, z\}$ and $v < \min\{u, w\}$. This case is impossible since

$$y \leq v < \min\{u, w\} \leq \min\{x, z\}.$$

Case 3: $y < \min\{x, z\}$ and $v \geq \min\{u, w\}$.

Suppose $w \leq v$, then $w - y \leq v - y$ and hence

$$\begin{aligned} & \min\{x, z\} - y \\ & \leq z - y = z - w + w - y \\ & \leq z - w + v - y \\ & = \frac{4}{3} \left[\frac{3}{4}(z - w) + \frac{3}{4}(v - y) \right] \\ & = \frac{4}{3} \left[(gz - gw) + (gv - gy) \right] \\ & = \frac{4}{3} [d(gz, gw) + d(gv, gy)] \\ & \leq \frac{8}{3} \max \{d(gy, gv), d(gz, gw)\} \\ & \leq \frac{8}{3} \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ & \quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& d(F(x, y, z), F(u, v, w)) \\
&= d\left(\frac{1}{3}(\min\{x, z\} - y), 0\right) \\
&= \frac{1}{3}(\min\{x, z\} - y) \\
&\leq \frac{8}{9} \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\
&\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\}.
\end{aligned}$$

Suppose $u \leq v$, then $u - y \leq v - y$ and hence

$$\begin{aligned}
& \min\{x, z\} - y \\
&\leq x - y \\
&= x - u + u - y \\
&\leq x - u + v - y \\
&= \frac{4}{3}\left[\frac{3}{4}(x - u) + \frac{3}{4}(u - y)\right] \\
&= \frac{4}{3}\left[(gx - gu) + (gu - gy)\right] \\
&= \frac{4}{3}\left[d(gx, gu) + d(gv, gy)\right] \\
&\leq \frac{8}{3} \max\{d(gx, gu), d(gy, gv)\} \\
&\leq \frac{8}{3} \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\
&\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& d(F(x, y, z), F(u, v, w)) \\
&= d\left(\frac{1}{3}(\min\{x, z\} - y), 0\right) \\
&= \frac{1}{3}(\min\{x, z\} - y) \\
&\leq \frac{8}{9} \max \{d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\
&\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw)\}.
\end{aligned}$$

Case 4: $y < \min\{x, z\}$ and $v < \min\{u, w\}$.

Since $x \geq u$ and $z \geq w$, then $\min\{x, z\} \geq \min\{u, w\}$. Thus

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ &= d\left(\frac{1}{3}(\min\{x, z\} - y), \frac{1}{3}(\min\{u, w\} - v)\right) \\ &= \frac{1}{3} \left| (\min\{x, z\} - \min\{u, w\}) + (v - y) \right| \\ &= \frac{1}{3} \left[(\min\{x, z\} - \min\{u, w\}) + (v - y) \right] \end{aligned}$$

If $\min\{u, w\} = u$, then $\min\{x, z\} - \min\{u, w\} \leq x - u$, and hence

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ &\leq \frac{1}{3} \left[(x - u) + (v - y) \right] \\ &= \left(\frac{1}{3}\right) \left(\frac{4}{3}\right) \left[\frac{3}{4}(x - u) + \frac{3}{4}(v - y)\right] \\ &= \frac{4}{9} \left[(gx - gu) + (gv - gy) \right] \\ &= \frac{4}{9} \left[d(gx, gu) + d(gy, gv) \right] \\ &\leq \frac{8}{9} \max \{ d(gx, gu), d(gy, gv) \} \\ &\leq \frac{8}{9} \max \{ d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ &\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw) \}. \end{aligned}$$

If $\min\{u, w\} = w$, then $\min\{x, z\} - \min\{u, w\} \leq z - w$, and hence

$$\begin{aligned} & d(F(x, y, z), F(u, v, w)) \\ &\leq \frac{1}{3} \left[(z - w) + (v - y) \right] \\ &= \left(\frac{1}{3}\right) \left(\frac{4}{3}\right) \left[\frac{3}{4}(z - w) + \frac{3}{4}(v - y)\right] \\ &= \frac{4}{9} \left[(gz - gw) + (gv - gy) \right] \\ &= \frac{4}{9} \left[d(gz, gw) + d(gy, gv) \right] \\ &\leq \frac{8}{9} \max \{ d(gy, gv), d(gz, gw) \} \\ &\leq \frac{8}{9} \max \{ d(gx, gu), d(gy, gv), d(gz, gw), d(F(x, y, z), gx), \\ &\quad d(F(z, y, x), gx), d(F(u, v, w), gu), d(F(w, v, u), gw) \}. \end{aligned}$$

To prove (6), suppose there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w)$$

holds for all $x \geq u$, $y \leq v$ and $z \geq w$. Then

$$d(F(1, 0, 1), F(0, 0, 1)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq j, \quad (24)$$

$$d(F(1, 0, 1), F(1, 1, 1)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq k, \quad (25)$$

and

$$d(F(1, 0, 1), F(1, 0, 0)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq l. \quad (26)$$

From (24), (25), and (26), we have $j + k + l \geq 1$, a contradiction.

To prove (7), suppose there exist $j, k, l \in [0, 1)$ with $j + k + l < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq jd(gx, gu) + kd(gy, gv) + ld(gz, gw)$$

holds for all $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. Then

$$d(F(1, 0, 1), F(0, 0, 1)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq \frac{3}{4}j, \quad (27)$$

$$d(F(1, 0, 1), F(1, 1, 1)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq \frac{3}{4}k, \quad (28)$$

and

$$d(F(1, 0, 1), F(1, 0, 0)) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3} \leq \frac{3}{4}l. \quad (29)$$

From (27), (28), and (29), we have $j + k + l \geq \frac{12}{9}$, a contradiction.

Thus by Theorems 2.1 and 2.3, F and g have a tripled coincidence point. Here, $(0, 0, 0)$ is the tripled coincidence point of F and g . \square

Remarks:

1. Example 2.1 does not satisfy condition 1 of Theorem 1.1 (Theorem 7 of [3]).
2. Example 2.1 does not satisfy condition 2 of Theorem 1.3 (Theorem 4 of [4]).
3. Theorem 1.3 (Theorem 4 of [4]) is a special case of Corollaries 2.2 and 2.7.
4. Theorem 1.1 (Theorem 7 of [3]) is a special case of Corollary 2.5.
5. Theorem 1.2 (Theorem 8 of [3]) is a special case of Corollary 2.10.

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