



## Ranking DMUs by ideal points in the presence of fuzzy and ordinal data

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### Abstract

Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision-making units (DMUs). DEA models divided all DMUs in two categories: efficient and inefficient DMUs, and don't able to discriminant between efficient DMUs. On the other hand, the observed values of the input and output data in real-life problems are sometimes imprecise or vague, such as interval data, ordinal data and fuzzy data. This paper develops a new ranking system under the condition of constant returns to scale (CRS) in the presence of imprecise data. In other words, in this paper, we reformulate the conventional ranking method by ideal point as an imprecise data envelopment analysis (DEA) problem, and propose a novel method for ranking the DMUs when the inputs and outputs are fuzzy and/or ordinal or vary in intervals. For this purpose we convert all data into interval data. In order to convert each fuzzy number into interval data we use the nearest weighted interval approximation of fuzzy numbers by applying the weighting function and also we convert each ordinal data into interval one. By this manner we could convert all data into interval data. The numerical example illustrates the process of ranking all the DMUs in the presence of fuzzy, ordinal and interval data.

*Key words:* Data envelopment analysis (DEA), Decision making unit (DMU), Ideal points, Interval data, Nearest weighted interval approximation, fuzzy data, ordinal data.

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## 1 Introduction

In evaluating the relative efficiency of each Decision making units (DMUs) by DEA, we obtain scores between zero and one. Therefore, DEA models discriminate DMUs into two categories: efficient DMUs and inefficient DMUs. In this way, usually more than one unit may be efficient in the DEA models and their scores are 1. Although efficiency score can be a criterion for ranking inefficient DMUs, this criterion cannot rank efficient DMUs. Therefore, the researchers proposed some methods to difference these efficient units. This concept has named ranking efficient units in the DEA. Therefore selecting the best ranking method or the way of combining different ranking methods for ranking DMUs is an important point in ranking DMUs in DEA. Several authors have proposed methods for ranking the best performers. For a review of ranking methods, readers are referred to Adler et al. [1]. In some cases, the models proposed by Andersen and Petersen [2] and Mehrabian et al. [34] can be infeasible. In addition to this difficulty, the Andersen and Petersen model may be unstable because of extreme sensitivity to small variations in the data when some DMUs have relatively small values for some of their inputs. Jahanshahloo et al. [23] present a method for ranking extreme efficient decision making units in data envelopment analysis models with constant and variable returns to scale. In their method, they exploit the leave-one-out idea and  $l_1$ -norm also, Jahanshahloo et al. [24] proposed a ranking system for extreme efficient DMUs based upon the omission of efficient DMUs from reference set of the inefficient DMUs. Li et al. [31] developed a super-efficiency model to overcome some deficiencies in the earlier models. Izadikhah [19] proposed a method for ranking decision making units with interval data by introducing two efficient and inefficient frontiers. Wang et al. [42] proposed a methodology for ranking decision making units. That methodology ranks DMUs by imposing an appropriate minimum weight restriction on all inputs and outputs, which is decided by a decision maker (DM) or an assessor in terms of the solutions to a series of linear programming (LP) models that are specially constructed to determine a maximin weight for each DEA efficient unit. Liu and Peng [33] proposed a methodology to determine one common set of weights for the performance indices of only DEA efficient DMUs. Then, these DMUs are ranked according to the efficiency score weighted by the common set of weights. For the decision maker, this ranking is based on the optimization of the group's efficiency. Jahanshahloo et al. [21] proposed two ranking methods. In the first method, an ideal line was defined and determined a common set of weights for efficient DMUs then a new efficiency score obtained and ranked them with it. In the second method, a special line was defined then compared all efficient DMUs with it and ranked them. Wang et al. [41] proposed a new methodology based on regression analysis to seek a common set of weights that are easy to estimate and can produce a full ranking for DMUs. Chen and Deng [5] proposed a new

method for ranking units. Their method develop a new ranking system under the condition of variable returns to scale (VRS) based on a measure of cross-dependence efficiency, where the evaluation for an efficient DMU is dependent of the efficiency changes of all inefficient units due to its absence in the reference set, while the appraisal of inefficient DMUs depends on the influence of the exclusion of each efficient unit from the reference set. Recently, Rezai Balf et al. [35] proposed a method for ranking extreme efficient decision making units (DMUs). Their method uses  $L_\infty$  or (Tchebycheff) Norm, and it seems to have some superiority over other existing methods, because this method is able to remove the existing difficulties in some methods, such as Andersen and Petersen (AP) that it is sometimes infeasible. Hosseinzadeh Lotfi et al. [18] proposed a methodology for ranking decision making units by using a goal programming model. Hosseinzadeh Lotfi et al. [17] proposed a method for ranking DMUs. They consider some CCR efficient DMUs, and then rank them by using some ranking methods, each of which is important and significant. Afterwards, by using TOPSIS method, they suggested the ranks of efficient DMUs. Jahanshahloo et al. [22] proposed some different methods and compared them. The original DEA models [3] assumed that inputs and outputs are measured by exact values on a ratio scale this assumption may not be valid i.e. some or all of inputs and outputs may be imprecise. "Imprecise data" implies that some data are known only to the exact that the true values lie within prescribed bounds while other data are known only in terms of ordinal relations. Ordinal data is one of the imprecise data. Cooper et al [9] discuss how to deal with bounded data and weak ordinal data and provide a unified IDEA model when weight restrictions are also present. Cook et al. [8] mixtures of exact and ordinal data. In addition to it, in this context, one can read Cook et al. [6,7], and Zhu [46]. The imprecise data representation with interval, ordinal, and ratio interval data was initially proposed by Cooper et al. [9,11] to study the uncertainty in DEA. Soon after, many researchers adopted the concept and proposed different DEA models with interval data in the DEA literature [12,15,26,39,40]. Due to the existence of uncertainty, DEA sometimes faces the situation of imprecise data, especially when a set of DMUs contain missing data, ordinal data, interval data, or fuzzy data. Therefore, how to evaluate the efficiency of a set of DMUs in interval environments is a problem worth studying. Cooper et al. [9,11] were the first to study how to deal with imprecise data such as bounded data. Kim et al. [28] also used an analogous scale transformation and variable alternation method, but they did not take the interval data situation into account. Lee et al. [29] extended the idea of IDEA (imprecise DEA) to the additive model. Despotis and Smirlis [12] also studied the problem of IDEA, but developed an alternative approach for dealing with imprecise data in DEA. Entani et al. [15] proposed a DEA model with interval efficiencies measured from both the optimistic and the pessimistic viewpoints. In these studies, the great majority of input and output variables focus on crisp data, and they cannot easily measure linguistic terms. Kao

and Liu [?] argue that we cannot gather crisp data because respondents cannot easily decide on the values by means of intuition. Sengupta [38] is considered the pioneer in solving these kinds of issues; he proposed a fuzzy objective function and constraints using results from Zimmermann [43,44]. Though fuzzy concepts have been applied to many fields and easy to adopt, as such, some research works attempted to solve the problems with the limitation of conventional data envelopment analysis (DEA) by introducing the fuzzy concepts (Cooper, Park, and Yu, [9]; Despotis and Smirlis [12]; Guo and Tanaka, [16]; Jahanshahloo et al. [25]. Cooper et al. [9] addressed the problem of imprecise data in DEA in its general form. Furthermore, Despotis and Smirlis [12] calculated upper and lower bounds for the radial efficiency scores of DMUs with interval data. Although DEA offers many advantages relative to many other statistical approaches, some limitations have to be considered. One important problem involves its sensitivity to data, so we should have accurate measurement of inputs and outputs in order to successfully apply DEA. However, in many situations, such as in a manufacturing system, a production process or a service system, inputs and outputs are volatile and complex so that it is difficult to measure them in an accurate way. Instead the data can be given as a fuzzy variable. Many fuzzy approaches have been introduced in the DEA literature [9,11,15,27]. Recently, Guo and Tanaka [16] and Lertworasirikul et al. [30] applied possibility measure proposed by Zadeh [45] to the fuzzy DEA model. Although possibility measure has been widely used, it has no self-duality property which is absolutely needed in both theory and practice. In order to define a self-dual measure, Liu and Liu [32] presented credibility measure in 2002. This paper will extend the CCR model to a fuzzy DEA model based on credibility measure, and then give a fuzzy ranking method to rank all the DMUs with fuzzy inputs and outputs. Recently, Jahanshahloo et al. [21] proposed a method for ranking DMUs with interval data. For ranking with crisp data, they first calculate the ideal point of each DMU and a special DMU, then they measure the distance of the ideal point from the special DMU; a DMU has a better rank if it has a shorter distance. Therefore they could rank non-extreme DMUs, as well. And also they generalized their method for interval data. Rostamy-Malkhalifeh and Aghayi [36] present a new method for ranking Units based on their overall profit efficiency. This ranking is also used for DMUs with interval data. The proposed method is used by AP model and  $l_1$  norm with interval data. They proposed two models for finding rank of DMUs by evaluation through AP model and also we introduce two models for comparing efficient DMUs in optimistic and pessimistic prices by  $l_1$ -norm. Therefore, in this paper we develop a methodology for ranking DMUs in the presence of fuzzy and ordinal data by using ideal points. For doing this we convert each fuzzy and ordinal data into interval one.

Rest of the paper is organized as follows: In section two, we review some basic concept about basic definitions and notions about classic DEA models, Imprecise data and Ranking DMUs by ideal points with interval data.

In section three, we will focus on the proposed method. In section four, numerical examples are demonstrated. And finally, the conclusion is discussed in Section five.

## 2 Preliminaries

In this section we present some required concepts. First, we assume that there are DMUs, where each  $DMU_j$  ( $j=1, \dots, n$ ), uses  $m$  different inputs,  $x_{ij}$  ( $i=1, \dots, m$ ), to produce  $s$  different outputs,  $y_{rj}$  ( $r=1, \dots, s$ ). We assume that the data set are positive and deterministic.

### 2.1 Classic DEA models

In this section we review some classic DEA models.

#### 2.1.1 Input-oriented CCR model

One the basic model used to evaluated DMUs is the input-oriented CCR model introduced by Charnes et al. [4], the CCR efficiency is obtained by calculating following model:

$$\begin{aligned}
 & \min \theta_o \\
 & \text{s.t.} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_o x_{io}, \quad i = 1, \dots, m, \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \dots, s, \\
 & \lambda \geq 0
 \end{aligned} \tag{2.1}$$

Model (2.1) known as envelopment form of CCR model. Dual form of model (2.1), which known as multiplier form of CCR is expressed as follows:

$$\begin{aligned}
& \text{Max} \sum_{r=1}^s u_r y_{ro} \\
& \text{s.t.} \\
& \sum_{i=1}^m v_i x_{io} = 1 \\
& \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij}, \quad j = 1, \dots, n, \\
& v_i, u_r \geq 0 \quad i = 1, \dots, m, r = 1, \dots, s
\end{aligned} \tag{2.2}$$

**Definition 2.1**  $DMU_o$  is CCR-efficient if:

1.  $\theta_o^* = 0$
2. All slack variables are zero in alternative optimal solution.

### 2.1.2 Ranking model

Super-efficiency model introduced by Andersen and Petersen for ranking efficient units is defined as follows:

$$\begin{aligned}
& \min \theta_o - \epsilon \left( \sum_{i=1}^m s_i^- + \sum_{i=1}^m s_r^+ \right) \\
& \text{s.t.} \\
& \sum_{j=1, j \neq o}^n \lambda_j x_{ij} + s_i^- = \theta_o x_{io}, \quad i = 1, \dots, m, \\
& \sum_{j=1, j \neq o}^n \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\
& \lambda_j, s_i^-, s_r^+ \geq 0, \quad j = 1, \dots, n, i = 1, \dots, m, r = 1, \dots, s
\end{aligned} \tag{2.3}$$

Efficient DMUs have super-efficiency score greater than or equal to 1, while inefficient score DMUs have super-efficiency score less than 1.

AP model, in some cases, breaks down with zero data and may be unstable because of extreme sensitivity to small variation in the data when some DMUs have relatively small values for some of its inputs.

### 2.2 Imprecise data

In this section we discuss about imprecise data, and how to convert them into interval data.

### 2.2.1 Ordinal data and converting them into interval data

In this section, we consider the transformation of ordinal preference information about the output and input  $y_{rj}$  and  $x_{ij}$  ( $j = 1, \dots, n$ ).

For weak ordinal preference information  $y_{r1} \geq \dots \geq y_{rn}$  and  $x_{i1} \geq \dots \geq x_{in}$ , we have the following ordinal relationships after scale transformation:

$$1 \geq \hat{y}_{r1} \geq \dots \geq \hat{y}_{rn} \geq \sigma_r \text{ and } 1 \geq \hat{x}_{i1} \geq \dots \geq \hat{x}_{in} \geq \epsilon_i$$

Where  $\sigma_r$  is a small positive number reflecting the ratio of the possible minimum of  $\langle x_{ij} | j = 1, \dots, n \rangle$  to its possible maximum. It can be approximately estimated by the decision maker. It is referred as the ratio parameter for convenience. The resultant permissible interval for each  $\hat{x}_{ij}, \hat{y}_{rj}$  is given by:

$$\begin{aligned} \hat{y}_{rj} &\in [\sigma_r, 1], (j = 1, \dots, n) \\ \hat{x}_{ij} &\in [\epsilon_i, 1], (j = 1, \dots, n) \end{aligned}$$

For strong ordinal preference information:

$$1 \geq \hat{y}_{rj}, \hat{y}_{rj} \geq \chi_r \hat{y}_{rj+1}, (j = 1, \dots, n-1) \text{ and } \hat{y}_{rj} \geq \sigma_r \quad (2.4)$$

$$1 \geq \hat{x}_{ij}, \hat{x}_{ij} \geq \eta_r \hat{x}_{ij+1}, (j = 1, \dots, n-1) \text{ and } \hat{x}_{ij} \geq \epsilon_i \quad (2.5)$$

Where  $\epsilon_i$  and  $\sigma_r$  are preference intensity parameters satisfying  $\chi_r, \eta_i > 1$ , provided by the decision maker and  $\epsilon_i, \sigma_r$  are the ratio parameters also provided by decision maker. The resultant permissible interval for each  $\hat{y}_{rj}$  and  $\hat{x}_{ij}$  can be derived as follows:

$$\hat{y}_{rj} \in [\sigma_r \chi_r^{n-j}, \chi_r^{1-j}], j = 1, \dots, n \text{ with } \sigma_r \leq \chi_r^{1-n} \quad (2.6)$$

$$\hat{x}_{ij} \in [\epsilon_i \eta_i^{n-j}, \eta_i^{1-j}], j = 1, \dots, n \text{ with } \epsilon_i \leq \eta_i^{1-n} \quad (2.7)$$

So all the ordinal preference information converted into the interval data.

**Remark 2.1** We will mention each input and output that is definitive transform into interval data as follow:

$$\begin{aligned} x_{ij} &\in [x_{ij}^L, x_{ij}^U] \text{ where } x_{ij}^L = x_{ij}^U = x_{ij} \\ y_{rj} &\in [y_{rj}^L, y_{rj}^U] \text{ where } y_{rj}^L = y_{rj}^U = y_{rj} \end{aligned} \quad (2.8)$$

### 2.2.2 Fuzzy numbers

In this subsection we review some definitions and notions about fuzzy numbers and possibility space. And after that we review the nearest

weighted interval approximations.

### 2.2.3 Basic definitions and notions about fuzzy numbers and possibility space

Let  $R$  be the set of all real numbers. We assume that a fuzzy number  $A$  for all  $x \in R$  can be expressed as follows:

$$A(x) = \begin{cases} A_L(x), & x \in [a, b]; \\ 1, & x \in [b, c]; \\ A_R(x), & x \in [c, d]; \\ 0, & \text{Otherwise.} \end{cases} \quad (I)$$

Where  $a, b, c$  and  $d$  are real numbers such that  $a < b \leq c < d$ ,  $A_L$  is real-valued function that is increasing and right continuous and  $A_R$  is a real-valued function that is decreasing and left continuous. Notice that (I) is an L-R fuzzy number with strictly monotone shape function as proposed by Dubois and Prade in 1981, and also described in Dubois and Prade [13,14]. Each fuzzy number  $A$  described by (I) has the following  $\alpha$ -level sets ( $\alpha$ -cut).

**Definition 2.2** We denote an  $\alpha$ -cut of fuzzy number  $a$  by  $A_\alpha$  which is defined as:

$$A_\alpha = \{x | A(x) \geq \alpha\} \quad (2.9)$$

an  $\alpha$ -cut of  $a$  can be stated as  $A_\alpha = [A_L^{-1}(\alpha), A_U^{-1}(\alpha)] = [\underline{a}(\alpha), \bar{a}(\alpha)]$  for all  $\alpha \in [0, 1]$ .

We denote the family of fuzzy numbers by  $\xi$ .

**Example 2.1** Let  $A \in \xi$  be a fuzzy number with the following membership function

$$A = \begin{cases} A = 1 - \left\{ \frac{(x-5)^2}{4} \right\}, & 3 \leq x \leq 7; \\ 0, & \text{Otherwise.} \end{cases}$$

then the  $\alpha$ -cut of  $A$  is as follow:

$$A_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)] = [5 - 2\sqrt{1 - \alpha}, 5 + 2\sqrt{1 - \alpha}], \alpha \in [0, 1] \quad (2.10)$$



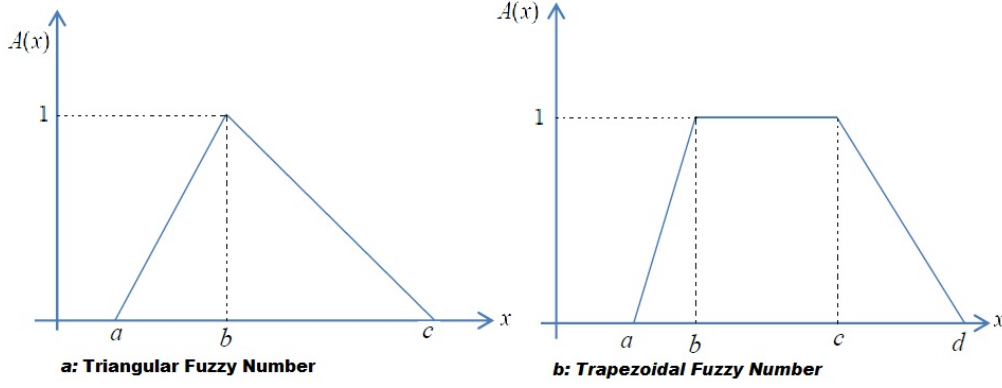


Fig. 1. The membership functions of Fuzzy numbers

**Definition 2.3** A fuzzy number  $A = (a, b, c, d)$  is called a trapezoidal fuzzy number if its membership function  $A(x)$  has the following form:

$$A(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a, b]; \\ 1, & x \in [b, c]; \\ \frac{d-x}{d-c}, & x \in [c, d]; \\ 0, & \text{Otherwise.} \end{cases} \quad (II)$$

if  $b = c$ , then  $A = (a, b, c)$  is a triangular fuzzy number (see Fig. 1).

#### 2.2.4 The nearest weighted interval approximations

In this subsection, we recall the concept of the nearest weighted interval approximation to a fuzzy number. The information of this section are taken from [37].

**Definition 2.4** A weighting function is a function as  $f = (\underline{f}, \bar{f} : ([0, 1], [0, 1])) \rightarrow (R, R)$  such that the function  $\underline{f}, \bar{f}$  are non-negative, monotone increasing and satisfies the following normalization condition(see Saeidifar (2011)):

$$\int_0^1 \underline{f}(\alpha) d\alpha = \int_0^1 \bar{f}(\alpha) d\alpha$$

Note that if  $g = (g, \bar{g}) : ([0, 1], [0, 1]) \rightarrow (R, R)$  is a function that is non-negative and monotone increasing, then we can consider

$$\underline{f}(\alpha) = \frac{g(\alpha)}{\int_0^1 g(\alpha) d\alpha}, \bar{f}(\alpha) = \frac{\bar{g}(\alpha)}{\int_0^1 \bar{g}(\alpha) d\alpha}$$

**Remark 2.2** The function  $f(\alpha)$  can be understood as the weight of our

interval approximation, the property of monotone increasing of function  $f(\alpha)$  means that the higher the cut level is, the more important its weight is in determining the interval approximation of fuzzy numbers. In applications, the function  $f(\alpha)$  can be chosen according to the actual situation.

**Definition 2.5** Let  $A \in \xi$  be a fuzzy number with  $A_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)]$  and  $f(\alpha) = (\underline{f}(\alpha), \bar{f}(\alpha))$  be a weighted function. Then the nearest  $f = (\underline{f}, \bar{f})$ -weighted interval approximation of  $A$  is defined as, Then the interval:

$$NWIA_f(A) = [C_L^f, C_U^f] = \left[ \int_0^1 \underline{f}(\alpha) d\alpha, \int_0^1 \bar{f}(\alpha) d\alpha \right] \quad (2.11)$$

Where,  $C_L^f$  is the nearest lower weighted point approximation ( $NLWPA_{\underline{f}}(A)$ ) and Where,  $C_U^f$  is the nearest upper weighted point approximation ( $NLWPA_{\bar{f}}(A)$ ) of fuzzy number  $A$ .

**Theorem 2.1** Let  $A \in \xi$  be a fuzzy number with  $A_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)]$  and  $f(\alpha) = (\underline{f}(\alpha), \bar{f}(\alpha))$  be a weighted function. Then, the interval

$$NWIA_f(A) = [NWIA_{\underline{f}}(A), NWIA_{\bar{f}}(A)]$$

Is the nearest weighted interval approximation to fuzzy number  $A$ .

Obviously, the weighted interval approximation defined by Eq.(2.11) synthetically reflects the information on every membership degree. Its advantage is that different  $\alpha$ - cut set plays different roles.

**Theorem 2.2** Let  $A, B \in \xi$ , let  $f(\alpha) = (\underline{f}(\alpha), \bar{f}(\alpha))$  be a weighting function, and let  $\lambda \in R$  then we have:

$$\begin{aligned} NWIA_{\underline{f}}(A + B) &= NWIA_{\underline{f}}(A) + NWIA_{\underline{f}}(B) \\ NWIA_{\underline{f}}(\lambda A) &= \lambda NWIA_{\underline{f}}(A) \end{aligned} \quad (2.12)$$

**Corollary 2.1** Let  $A = (a, b, c, d)$  be a trapezoidal fuzzy number, and  $f(\alpha) = (\underline{f}(\alpha), \bar{f}(\alpha))$  be a weighting function. Then  
1. for  $f(\alpha) = (1, 1)$ :

$$NWIA_{\underline{f}}(A) = \left[ \frac{a+b}{2}, \frac{c+d}{2} \right] \quad (2.13)$$

2. for  $f(\alpha) = (2\alpha, 2\alpha)$ :

$$NWIA_{\underline{f}}(A) = \left[ \frac{a+2b}{2}, \frac{2c+d}{2} \right] \quad (2.14)$$

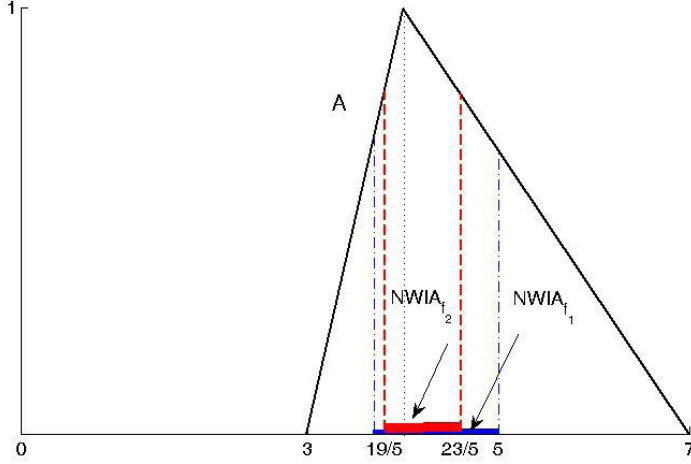


Fig. 2. Fuzzy number (Ex. 2)

3. for  $f(\alpha) = (n\alpha^{n-1}, n\alpha^{n-1})$ ,  $n \in N$  (natural number):

$$NWIA_{\underline{f}}(A) = \left[ \frac{a+nb}{2}, \frac{nc+d}{2} \right] \quad (2.15)$$

**Example 2.2** Let a  $A = (3, 4, 7)$  be a triangular fuzzy number and also  $f_1(\alpha) = (2\alpha, 2\alpha)$  and  $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$  be two weighting functions. Then the nearest weighted interval to  $A$  is as follows (see Fig. 2):

$$NWIA_{f_1} = \left[ \frac{11}{3}, 5 \right], NWIA_{f_2} = \left[ \frac{19}{5}, \frac{23}{5} \right] \quad (2.16)$$

**Example 2.3** Let a  $A = (3, 7, 8, 13)$  be a trapezoidal fuzzy number and also  $f_1(\alpha) = (2\alpha, 2\alpha)$  and  $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$  be two weighting functions. Then the nearest weighted interval to  $A$  is as follows (see Fig. 3):

$$NWIA_{f_1} = \left[ \frac{17}{3}, \frac{29}{3} \right] NWIA_{f_2} = \left[ \frac{31}{5}, 9 \right] \quad (2.17)$$

**Example 2.4** Let  $A$  be a fuzzy number with the following membership function

$$\tilde{A} = \begin{cases} A = 1 - \left\{ \frac{(x-5)^2}{4} \right\}, & 3 \leq x \leq 7; \\ 0, & \text{Otherwise.} \end{cases}$$

And let  $f_1(\alpha) = (2\alpha, 2\alpha)$  and  $f_2(\alpha) = (4\alpha^3, 4\alpha^3)$  be two weighting functions. Then the nearest weighted interval to  $A$  is as follows (see Fig. 4):

$$NWIA_{f_1} = \left[ \frac{59}{15}, \frac{91}{15} \right] NWIA_{f_2} = \left[ \frac{1319}{315}, \frac{1831}{315} \right] \quad (2.18)$$

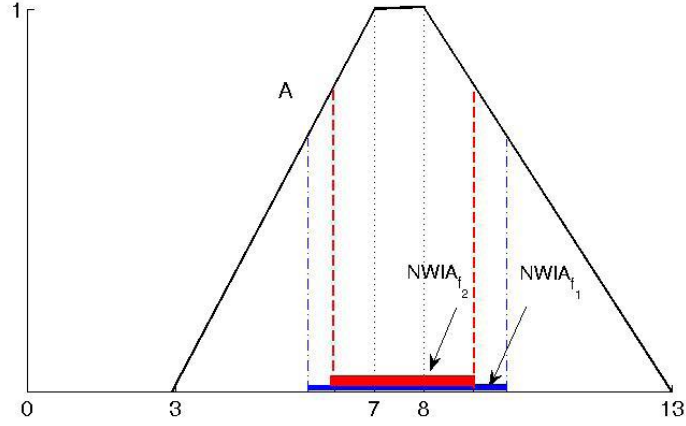


Fig. 3. Fuzzy number (Ex. 3)

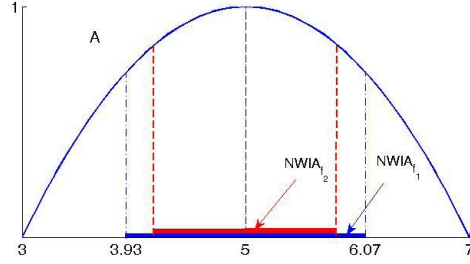


Fig. 4. Fuzzy number (Ex. 4)

### 2.3 Ranking DMUs by ideal points with interval data

In this section, unlike the original DEA model, we assume further that the levels of inputs and outputs are not known exactly, the true input and output data known to lie within bounded intervals, i.e.  $x_{ij} \in [x_{ij}^L, x_{ij}^U]$  and  $y_{rj} \in [y_{rj}^L, y_{rj}^U]$  with upper and lower bounds of the intervals given as constants and assumed strictly positive i.e.  $x_{ij}^L > 0$  and  $y_{rj}^L > 0$ .

In this case, the efficiency can be an interval. The upper limit of interval efficiency is obtained from the optimistic viewpoint and the lower limit is obtained from the pessimistic viewpoint. The following model provides such an upper bound for  $DMU_o$ :

$$\theta_o^U = \max_{U,V} = \frac{\frac{U^T[Y_o^L, Y_o^U]}{V^T[Y_o^L, Y_o^U]}}{\max_j \frac{U^T[Y_j^L, Y_j^U]}{V^T[Y_j^L, Y_j^U]}} U \geq 0, V \geq 0 \quad (2.19)$$

Where vector-variables  $U$  and  $V$  are weights for outputs and inputs, respectively, which to be estimated. We denote by  $\theta_o^U$  the efficiency score attained by  $DMU_o$  in (2.19). The model below provides a lower bound

of the efficiency score for  $DMU_o$ :

$$\theta_o^L = \min_{U,V} = \frac{\frac{U^T[Y_o^L, Y_o^U]}{V^T[Y_o^L, Y_o^U]}}{\max_j \frac{U^T[Y_j^L, Y_j^U]}{V^T[Y_j^L, Y_j^U]}} U \geq 0, V \geq 0 \quad (2.20)$$

Since, above mentioned models are non-linear; we obtain the upper ( $H_o^U$ ) and lower ( $L_o^U$ ) bound of  $\theta_o^U$ , and the upper ( $H_o^L$ ) and lower ( $L_o^L$ ) bound of  $\theta_o^L$ .

$$\begin{aligned} H_o^U &= \max_{U,V} \frac{U^T Y_o^U}{V^T X_o^L} \\ \text{s.t.} \quad & \max(\max_{j \neq o} \frac{U^T Y_j^L}{V^T X_j^U}, \frac{U^T Y_o^U}{V^T X_o^L}) = 1 \\ & U \geq 0, V \geq 0 \end{aligned} \quad (2.21)$$

$$\begin{aligned} H_o^L &= \min_{U,V} \frac{U^T Y_o^U}{V^T X_o^L} \\ \text{s.t.} \quad & \max(\max_{j \neq o} \frac{U^T Y_j^L}{V^T X_j^U}, \frac{U^T Y_o^U}{V^T X_o^L}) = 1 \\ & U \geq 0, V \geq 0 \end{aligned} \quad (2.22)$$

$$\begin{aligned} L_o^U &= \max_{U,V} \frac{U^T Y_o^L}{V^T X_o^U} \\ \text{s.t.} \quad & \max(\max_{j \neq o} \frac{U^T Y_j^U}{V^T X_j^L}, \frac{U^T Y_o^L}{V^T X_o^U}) = 1 \\ & U \geq 0, V \geq 0 \end{aligned} \quad (2.23)$$

$$\begin{aligned} L_o^L &= \min_{U,V} \frac{U^T Y_o^L}{V^T X_o^U} \\ \text{s.t.} \quad & \max(\max_{j \neq o} \frac{U^T Y_j^U}{V^T X_j^L}, \frac{U^T Y_o^L}{V^T X_o^U}) = 1 \\ & U \geq 0, V \geq 0 \end{aligned} \quad (2.24)$$

These models easily can be simplified according to Jahanshahloo et al. [21] and [?]. According to Jahanshahloo et al. [21], the ideal points with interval data for can be defined as follows:

$$\bar{x}_{io}^L = \min_r \left\{ \frac{Y_{ro}^L}{\max_j \frac{y_{rj}^U}{x_{ij}^L}} \right\}, \quad i = 1, \dots, m, \quad (2.25)$$

$$\bar{x}_{io}^U = \min_r \left\{ \frac{Y_{ro}^U}{\max_j \frac{y_{rj}^L}{x_{ij}^U}} \right\}, \quad i = 1, \dots, m, \quad (2.26)$$

$$\bar{y}_{ro}^L = \max_j \left\{ \left( \max_j \frac{y_{rj}^L}{x_{ij}^L} x_{io}^L \right) \right\}, \quad r = 1, \dots, s, \quad (2.27)$$

$$\bar{y}_{ro}^U = \max_j \left\{ \left( \max_j \frac{y_{rj}^U}{x_{ij}^U} x_{io}^U \right) \right\}, \quad r = 1, \dots, s, \quad (2.28)$$

On the basis of the above efficiency score intervals,  $DMU_s$  can be classified in three subsets as follow:

$$E^{++} = \{j \in a, \dots, n | L_o^U = 1\}$$

$$E^+ = \{j \in a, \dots, n | L_o^U < 1, H_o^U = 1\}$$

$$E^- = \{j \in a, \dots, n | H_o^U < 1\}$$

Also, we calculate the special  $DMU$  by the following relations:

$$\begin{aligned} \tilde{x}_i &= \max \left( \max_{j \in E^{++}} \bar{x}_{ij}^U, \max_j x_{ij}^U \right), \\ \tilde{y}_r &= \max \left( \max_{j \in E^{++}} \bar{y}_{rj}^L, \max_j y_{rj}^L \right). \end{aligned} \quad (2.29)$$

Then, we calculate the distance of the ideal point of  $DMU_o$  in its best situation  $(\bar{X}_o^L, \bar{y}_o^U)$  from the special  $DMU(\Gamma_o^g)$  and the distance of the ideal point of in its worst situation from  $(\bar{X}_o^U, \bar{y}_o^L)$  the special  $DMU(\Gamma_o^b)$  by the following models:

$$\begin{aligned} \Gamma_o^g &= \min \left( \frac{1}{m} \sum_{i=1}^m \theta_i \right) / \left( \frac{1}{s} \sum_{r=1}^s \phi_r \right) \\ s.t. \quad \tilde{x}_{io} &= \theta_i \bar{x}_{io}^L, & i &= 1, \dots, m, \\ \tilde{y}_{ro} &= \phi_r \bar{y}_{ro}^U, & r &= 1, \dots, s \end{aligned} \quad (2.30)$$

$$\begin{aligned} \Gamma_o^b &= \min \left( \frac{1}{m} \sum_{i=1}^m \theta_i \right) / \left( \frac{1}{s} \sum_{r=1}^s \phi_r \right) \\ s.t. \quad \tilde{x}_{io} &= \theta_i \bar{x}_{io}^U, & i &= 1, \dots, m, \\ \tilde{y}_{ro} &= \phi_r \bar{y}_{ro}^L, & r &= 1, \dots, s \end{aligned} \quad (2.31)$$

Hence, we define a ranking criterion for  $DMU_s$  as follows:

$$\Psi_o = \Gamma_o^g + \Gamma_o^b, \quad (2.32)$$

By attention to Jahanshahloo et al. [21], we know that the rank of  $DMU_o$  is better than other's if it has a lower  $\Psi_o$ .

Table 1  
The data set.

	I1	I 2	I3	I4	O1	O2	O3	O4
1	252	6	(3,5.5,8)	134	[48,53]	1	[3119,3122]	64
2	253	7	$\tilde{x}_{23}$	134	[44,50]	3	[3120,3126]	51
3	248	2	$\tilde{x}_{33}$	127	[55,56]	14	[3141,3141]	78
4	253	11	(3,7.5,10)	134	[47,52]	13	[3110,3132]	55
5	251	14	$\tilde{x}_{53}$	134	[46,50]	2	[3130,3139]	53
6	250	12	(5,6,7,8.5)	131	[48,51]	10	[3115,3121]	52
7	252	5	$\tilde{x}_{73}$	132	[47,53]	5	[3124,3125]	69
8	250	3	$\tilde{x}_{83}$	131	[42,51]	12	[3129,3138]	62
9	248	1	$\tilde{x}_{93}$	127	[55,56]	15	[3140,3141]	79
10	254	15	$\tilde{x}_{10,3}$	132	[49,55]	6	[3120,3139]	60
11	252	8	(6,8.5,11.5)	134	[47,53]	7	[3127,3138]	64
12	253	9	(2,5,6,7)	132	[48,54]	9	[3124,3137]	59
13	250	10	(6.8,7.8,9.8)	130	[48,55]	11	[3119,3134]	67
14	251	4	(6.7,9.7,10.2,12.3)	131	[43,51]	4	[3121,3136]	61
15	250	13	(5.7,7.4,9.7)	130	[42,50]	8	[3128,3135]	60

### 3 An algorithm for ranking $DMU_s$ in the presence of fuzzy and ordinal data

We assume that, there are  $n$  homogeneous  $DMU_s$ , and each  $DMU_j$  uses  $m$  inputs ( $i = 1, \dots, m$ ) to produce  $s$  outputs ( $r = 1, \dots, s$ ). We also assume that inputs and outputs aren't necessarily deterministic and they may be as definitive, fuzzy, ordinal or interval data.

We consider five steps achieve for ranking these  $DMU_s$  and total results:  
Step 1: Firstly, by using the formula (2.6),(2.7),(2.8)and (2.11) we can convert all of the data into interval data. Therefore each input  $x_{ij}$  as from  $[x_{ij}^L, x_{ij}^U]$  and output  $y_{rj}$  as from  $[y_{rj}^L, y_{rj}^U]$ . Step 2: By using the formula(2.25)-(2.28), we achieve the Ideal point for interval data.

Step 3: By using the formula (2.29), special point can be obtained.

Step 4:By using the formula (2.30) and (2.32), we calculate the value of  $\Gamma_o^b$  and  $\Gamma_o^g$

Step 5:For each  $DMU_o$  we calculate the value of as follows:

$$\Psi_o = \Gamma_o^b + \Gamma_o^g$$

Table 2  
The interval form of Input data of the Table 1.

	I1L	I2U	I2L	I2U	I3L	I3U	I4L	I4U
1	252	252	0.112	0.387	4.875	6.125	134	134
2	253	253	0.125	0.43	1.375	3.447	134	134
3	248	248	0.074	0.254	0.75	1.724	127	127
4	253	253	0.191	0.656	6.375	8.125	134	134
5	251	251	0.262	0.9	4.086	5.914	134	134
6	250	250	0.212	0.729	5.75	7.375	131	131
7	252	252	0.101	0.348	1.857	2.143	132	132
8	250	250	0.082	0.282	2.212	4.143	131	131
9	248	248	0.066	0.228	0.857	2.143	127	127
10	254	254	0.291	1	2.644	3.356	132	132
11	252	252	0.139	0.478	7.875	9.25	134	134
12	253	253	0.154	0.531	4.25	6.25	132	132
13	250	250	0.172	0.59	7.55	8.3	130	130
14	251	251	0.091	0.313	8.95	10.725	131	131
15	250	250	0.236	0.81	6.8	7.975	130	130

Step 6: We can rank the  $DMU_s$  by using the value of  $\Psi_o$ . The best-rank  $DMU$  is the one with the minimum value of  $\Psi_o$ .

#### 4 Application

In this section, we show the ability of the provided approach using a numerical example. We apply the proposed method for evaluating 15 units, which each unit uses four inputs to produce four outputs. The inputs 1 and 4 are completely known, the input 2 is of the form of ordinal data and input 3 is of the form of fuzzy data. Also, the outputs 1 and 3 are of the form of interval data, the output 2 is of the form of ordinal data and output 4 is completely known. The data set for this example are shown in Table 1. In Table 1, we can see that the data of input 3 are fuzzy data in general form. The membership functions of general fuzzy numbers of input 3 in Table 1 are as follows:



$$\tilde{x}_{23} = \begin{cases} 0, & x \leq 1; \\ \frac{x-1}{0.5}, & 1 \leq x \leq 1.5; \\ 1, & 1.5 \leq x \leq 2; \\ e^{-\frac{(x-2)^2}{8}}, & 2 \leq x. \end{cases} \quad \tilde{x}_{33} = \begin{cases} x, & 0 \leq x \leq 1; \\ e^{-\frac{(x-1)^2}{2}}, & 1 \leq x; \\ 0, & \text{Otherwise.} \end{cases}$$

$$\tilde{x}_{53} = \begin{cases} \frac{1-(x-5)^2}{4}, & x \in [3, 7]; \\ 0, & \text{Otherwise.} \end{cases} \quad \tilde{x}_{73} = \begin{cases} (x-1)^2, & x \in [1, 2]; \\ (x-3)^2, & x \in [2, 3]; \\ 0, & \text{Otherwise.} \end{cases}$$

$$\tilde{x}_{83} = \begin{cases} \frac{(x-1)^2}{2}, & x \in [1, 3]; \\ 1, & x \in [3, 4]; \\ (5-x)^2, & x \in [4, 5]; \\ 0, & \text{Otherwise.} \end{cases} \quad \tilde{x}_{93} = \begin{cases} x^2, & x \in [0, 1]; \\ 1, & x \in [1, 2]; \\ (3-x)^2, & x \in [2, 3]; \\ 0, & \text{Otherwise.} \end{cases}$$

$$\tilde{x}_{10,3} = \begin{cases} \frac{2}{1+(x-3)^2} - 1, & x \in [2, 4]; \\ 0, & \text{Otherwise.} \end{cases}$$

In order to convert the fuzzy data into interval ones, we use a weighting function  $f(\alpha, \alpha) = 3\alpha^2$  in procedure mentioned in section 2.2.2. Also by a procedure mentioned in section 2.2.1, we convert the ordinal data into interval data. Therefore the interval forms of Inputs data of Table 1 are obtained as Table 2, and the interval forms of Outputs data of Table 1 are obtained as Table 3.

In order to rank these 15 units, we need to calculate  $L_j^U j = (1, \dots, n)$ , (see Table 4). From Table 4, it can be seen that only units 3 and 9 are in  $E^{++}$  and the other units are in  $E^+$ . For ranking the units in  $E^+$  we use the fact that if  $DMU_j(j = 1, \dots, n)$  has a higher  $L_j^U$ , then it has a better rank. In Table 4 we set the rank of  $DMU_s$  3 and 9 in 1<sup>th</sup> position, and the rank of other  $DMU_s$  are calculated with respect to the values of  $L_j^U$  (For more details see the third column of Table 4).

So far we've done the first step of the proposed algorithm. Now we need to calculate the ideal points for  $DMU_s$  in  $E^{++}$ . Ideal point of  $DMU_s$  can be obtained by formula (2.25)-(2.28). Table 5 shows the input data of the ideal points for  $DMU_s$  3 and 9.

Table 6 shows the output data of the ideal points for  $DMU_s$  3 and 9. For ranking these two units in  $E^{++}$  first we determine their ideal points and the special DMU. The upper and lower bounds of the data of their ideal points and the inputs and outputs of the special DMU are shown

Table 3  
The interval form of Output data of the Table 1.

	O1L	O1U	O2L	O2U	O3L	O3U	O4L	O4U
1	48	53	0.066	0.38	3119	3122	64	64
2	44	50	0.076	0.436	3120	3126	51	51
3	55	56	0.163	0.933	3141	3141	78	78
4	47	52	0.152	0.871	3110	3132	55	55
5	46	50	0.071	0.407	3130	3139	53	53
6	48	51	0.124	0.808	3115	3121	52	52
7	47	53	0.087	0.501	3124	3125	69	69
8	42	51	0.142	0.813	3129	3138	62	62
9	55	56	0.175	1	3140	3141	79	79
10	49	55	0.094	0.537	3120	3139	60	60
11	47	53	0.1	0.575	3127	3138	64	64
12	48	54	0.115	0.661	3124	3137	59	59
13	48	55	0.132	0.758	3119	3134	67	67
14	43	51	0.081	0.468	3121	3136	61	61
15	42	50	0.108	0.616	3128	3135	60	60

in Tables 5-7, respectively. In Table 3, "L" and "U" indices indicate the lower and upper bounds of intervals, respectively. We denote the special *DMU* by  $(\tilde{x}, \tilde{y})$ . Inputs and outputs of special *DMU* can be obtained by formula (2.29) as Table 7.

This is complete the step 3. Now for each  $DMU \in E^{++}$  we calculate the value of  $\Gamma_o^b$  and  $\Gamma_o^b$ . This is done by using the formula (2.30) and (2.32) and are shown in Table 8.

This is complete the step 3. Now for each  $DMU \in E^{++}$  we calculate the value of  $\Gamma_o^b$  and  $\Gamma_o^b$ . This is done by using the formula (30) and (31) and are shown in Table 8.

Now, for each  $DMU \in E^{++}$  we calculate the value of  $\Psi_o$  as fourth column of Table 8. The ranking order of these DMUs are shown in the final column of Table 8. We can see that  $DMU_9$  is the best *DMU* because it has the minimum value of  $\Psi_o$ .

Table 4  
Values of  $L_o^U$ .

$DMU_s$	$L_o^U$	Rank
1	0.9772	11
2	0.9737	13
3	1.0000	1
4	0.9706	14
5	0.9846	6
6	0.9838	7
7	0.9788	10
8	0.9882	4
9	1.0000	1
10	0.9699	15
11	0.9797	9
12	0.9749	12
13	0.9851	5
14	0.9818	8
15	0.9897	3

Table 5  
Input data of Ideal Points.

	I1L	I1U	I2L	I2U	I3L	I3U	I4L	I4U
3	40.424	244.8608	0.010758	0.225114	0.131029	1.724	20.701	125.3924
9	43.4	248	0.01155	0.228	0.140675	1.724	22.225	127

Table 6  
Output data of Ideal Points.

	O1L	O1U	O2L	O2U	O3L	O3U	O4L	O4U
3	55	215.5152	0.175	3.848485	3141	12088.09	79	304.0303
9	55	193.4545	0.175	3.454545	3141	10850.73	79	272.9091

Table 7  
Inputs and outputs of special DMU.

special $DMU$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$
$(\tilde{x}, \tilde{y})$	254	1	10.724	134	42	0.066	3110	51

Table 8  
The final results and ranking.

$DMU_s$	$\Gamma_o^b$	$\Gamma_o^g$	$\Psi_o$	Ranking
3	4.60	294.42	299.02	2
9	4.57	246.16	250.73	1

## 5 Conclusion

Due to its widely used practical background, data envelopment analysis (DEA) has become a pop area of research, but lack of discrimination power is a drawback of DEA that has aroused considerable research interest in the DEA literature. Therefore, ranking of  $DMU_s$  in DEA is an important phase for efficiency evaluation of  $DMU_s$ .

In the conventional DEA, all the data assume the form of specific numerical values. However, the observed values of the input and output data in real-life problems are sometimes imprecise or vague, such as interval data, ordinal data and fuzzy data. The imprecise or vague data in the DEA models have been examined in the literature in different ways. For this purpose, different methods with different properties to achieve full ranking in the presence of imprecise data have been proposed. Recently, Jahanshahloo et al. [20] proposed a new method that can rank all  $DMU_s$  with interval data completely. Although they proposed a ranking method by ideal points in the presence of interval data, it is not perfect for ranking  $DMU_s$  with fuzzy and/or ordinal data. Therefore, a new approach to ranking  $DMU_s$  in the presence of fuzzy, ordinal and interval data is suggested in the current study. The introduced ranking system can similarly be performed in the assessment of  $DMU_s$  with exact data.

In order to convert each fuzzy number into interval data we used the nearest weighted interval approximation of fuzzy numbers by applying the weighting function  $f_{\alpha,\alpha} = 3\alpha^2$  (see section 2.2.2) and by a method discussed in section 2.2.1 we converted each ordinal data into interval one. By this manner we could convert all data into interval data. Then for ranking  $DMU_s$  we used the method proposed in [20]. This method is able to rank all  $DMU_s$  with any kind of fuzzy and/or ordinal data. It has been shown that the proposed ranking methodology can successfully distinguish between all  $DMU_s$  with fuzzy and/or ordinal data and therefore makes a new contribution to DEA ranking. The numerical example illustrated the process of ranking all the  $DMU_s$  in the presence of fuzzy, ordinal and interval data.

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