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Partial Compensation of Large Scale Discrete Systems

Nicholas Baine, Terry Kolakowski, Julie Lee, and Pradeep Misra

Abstract— **This paper addresses the problem of partial state feedback compensation for large scale discrete systems. The eigenvalues of the closed-loop matrix should lie within a designated region of the z-domain to satisfy both stability and damping requirements. The system is to be compensated in such a way that only the eigenvalues that lie outside the desired region are affected. This is achieved through the use of the fast matrix sector function to decompose the system without solving for the eigenvalues. The decomposed system is then controlled using LQR design techniques.**

I. INTRODUCTION

The control of large scale systems, such as large space structures[1] and networks[2], continues to provide challenging computational problems. For systems on the order of a hundred states or more, the conventional algorithms, pole placement and linear quadratic control, are computationally impractical. With that, this paper expands on earlier work by Misra et al [3], [4], who worked on partial compensation of high order continuous systems, and shows the development of a partial compensation technique in the discrete domain.

We first assume that the system is represented by its state equations:

$$S : \dot{x}(k) = Ax(k) + Bu(k) \tag{1}$$

where, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, n is assumed to be large. Often it is computationally impractical to design a controller to reassign all eigenvalues of the system through pole placement or design an LQR controller.

In this paper, we introduce systematic approach to design a stabilizing controller that also achieves desired damping (ζ) and degree of stability (distance from the unit circle in the z-domain of some desired value α). The objective of this method is to design a state feedback that affects only the eigenvalues of the system that do not satisfy the desired damping and stability margin specifications. To do this, the system is block triangularly decomposed as shown below:

$$S : \dot{x}(k) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x(k) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \tag{2}$$

The subsystem $S_b : (A_{22}, B_2)$ contains the eigenvalues that lie either outside the stability margin (α) or do not have the desired damping (ζ).

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N. Baine is a Ph.D. student in the Department of Electrical Engineering, Wright State University, Dayton, OH 45435, USA baine.3@wright.edu

T. Kolakowski is a Ph.D. student in the Department of Electrical Engineering, Wright State University, Dayton, OH 45435, USA kolakowski.2@wright.edu

J. Lee is a Ph.D. student in the Department of Electrical Engineering, Wright State University, Dayton, OH 45435, USA lee.67@wright.edu

P. Misra is faculty in the Department of Electrical Engineering, Wright State University, Dayton, OH 45435, USA pmisra@cs.wright.edu

It should be mentioned that in [5], Saad proposes two methods to decompose the system into subsystems based on dominant and non-dominant eigenvalues. The first of these methods utilizes subspace iteration with Chebyshev acceleration [6], the second method applies Arnoldi process to compute k largest eigenvalues of the state matrix. Both of these methods are well suited for computation of eigenvalues of large sparse matrices; hence, they accomplish the decomposition into dominant and non-dominant eigenvalue sub-matrices efficiently. The downside of this process is that k is not known *a priori*, and it cannot address damping or degree of stability issues addressed in this paper.

In the work of Hench *et al* [7], the authors used a periodic Riccati equation approach to accomplish the placement of eigenvalues in a prescribed region bounded by ζ-curves. There are differences in this work and the work done by [7]. The scale of the systems considered in this paper is much larger, making it necessary to separate the system into two subsystems. One made of acceptable eigenvalues and another containing all the eigenvalues that need to be relocated. Unlike the present work, the approach in [7] uses the solution of periodic ARE (Algebraic Riccati Equation) which is computationally more expensive.

In [3], [4], the authors partitioned a large scale *analog* system into subsystems using matrix sector functions and then used LQR design to move the undesirable eigenvalues to a designated region of acceptable stability and damping. This paper modifies the method in [4] for use in the *discrete* domain. This paper makes use of the more efficient matrix sector algorithm proposed by Shieh *et al* [8], [9] instead of Halley's iteration formula for solving non-linear equations [10], which was used in [4].

The layout of this paper is as follows: In Section 2, some relevant information regarding matrix sector functions and pole placement on a disk are reviewed. Section 3 describes an efficient algorithm to isolate S_b from S along with the algorithms to design a feedback controller that achieves desired damping and stability margin specifications. This algorithm is then illustrated with an example in Section 4.

II. Background

A. Matrix Sector Function

A generalized matrix sector function is to be used to separate the eigenvalues of the system relative to their being within a circular region in the λ-plane. If $\hat{C}$ is a circle of radius ρ with a center located at the origin of the λ-plane, as seen in Fig 1, the generalized bilinear transform, g(A), is defined as:

$$g(A) = (A - \rho I_m)(A + \rho I_m)^{-1} \tag{3}$$

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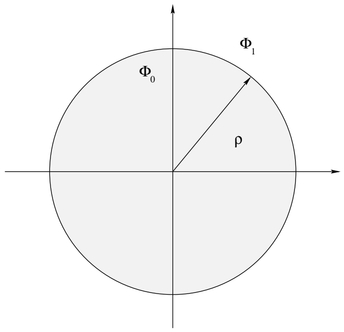

Fig. 1. Circular Matrix Sector Mapping

where $A \in \mathbb{C}^{nxm}$ with $\sigma(A) = \lambda_i : i = 1, 2, \cdots, m,\ g(\lambda_i) \neq 0$ and $det(A + \rho I_m) \neq 0$.

The fast matrix sector algorithm [8] is summarized in the equations below:

Definitions

$$H(i) \triangleq [H_1^T(i), H_2^T(i)]^T \tag{4}$$

$$\bar{H}(i) \triangleq [\bar{H}_1^T(i), \bar{H}_2^T(i)]^T \tag{5}$$

$$\mathbf{H}(i) \triangleq \begin{bmatrix} H_1(i) & H_2(i) \\ H_2(i) & H_1(i) \end{bmatrix} \in \mathbb{C}^{nmxnm} \tag{6}$$

Initialization

$$H(0) \triangleq [I_m, (gA)^T]^T \in \mathbb{C}^{nmxm}, n = 2 \tag{7}$$

Iteration

$$\bar{H}(i+1) = \mathbf{H}^{k-1}(i) H(i) \tag{8}$$

$$H(i+1) = \bar{H}(i+1)\bar{H}_1^{-1}(i+1) \tag{9}$$

Termination

$$\left\| H_p(i) - H_p(i-1) \right\| < \epsilon \tag{10}$$

The recursive fast matrix algorithm should be terminated when an acceptable error tolerance ϵ is reached. Upon convergence, the desired generalized matrix sector function is

$$S_2^{(0)} g(A) = \frac{1}{2}(I_m + S_2 g(a)) \tag{11}$$

where,

$$S_2 g(A) = \lim_{i \to \infty} H_2(i) \tag{12}$$

and the trace of $S_2^{(0)} g(A)$ is equal to the number of eigenvalues of A that lie on the interior of the circle $\hat{C}$. For more information regarding this method refer to [8], [9], [11], [12] and [13].

B. Pole Placement on a Disk

As done in [14] the eigenvalues of a matrix $(A - \gamma I)/r$ lie in a unit circle about the origin if and only if there exists a positive definite matrix P satisfying

$$\frac{(A - \gamma I)^T)}{r} P \frac{A - \gamma I)}{r} - P = -\frac{Q}{r^2} \tag{13}$$

where $Q(= C^T C)$ is a positive semi-definite matrix and (A,C) is an observable pair.

When under state feedback compensation ($u = Fx$), the eigenvalues of $(A + BF)$ will lie within a disk of radius r and center γ if and only if there exists a positive definite matrix P satisfying

$$\begin{gathered} -\gamma(A + BF)^T P - \gamma P(A + BF) + \\ (A + BF)^T P(A + BF) + (\gamma^2 - r^2)P = -Q \end{gathered} \tag{14}$$

where Q is a positive semi-definite matrix. In which case, the feedback matrix F is given by

$$F = -(r^2 R + B^T P B)^{-1} B^T P(A - \gamma I)3 \tag{15}$$

where R is any arbitrary positive definite matrix and P is the symmetric positive definite solution to the Ricatti equation below:

$$\begin{gathered} -Q = \frac{(A - \gamma I)^T}{r} P \frac{(A - \gamma I)}{r} - \\ P \frac{(A - \gamma I)^T}{r} P B (r^2 R + B^T P B)^{-1} B^T P \frac{(A - \gamma I)}{r} \end{gathered} \tag{16}$$

and $Q = C^T C$.

III. Main Results

A. Selection of Region

Before decomposing the system, a region needs to be selected which has the desired damping and stability margin. It is in our best interest to keep this area as large as possible to maintain robustness and keep gains manageable. The following is the development of an exact relationship between the radius and center of a circle of maximum area and the prescribed damping and stability.

1) Vertically Constrained Region: For our circular region, the limiting distance will be either in the vertical or horizontal direction. If it is limited in the vertical direction it will be due to the ζ-curve (damping factor), as shown in Fig. 2. If this is the case, the relationships for the center and radius of the circle are developed below.

Given $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n^2 s + \omega_n^2}$, a second order transfer function in the s-plane has eigenvalues located at $s_{1,2} = -\zeta\omega_n \pm j\omega_n$. The equivalent eigenvalues in the z-plane are located at

$$z = e^{sT}|_{s_{1,2}} = e^{-\zeta\omega_n T} \angle(\pm\omega_n T \sqrt{1 - \zeta^2}) = r\angle \pm \Theta$$

which gives leads to the following three equations.

$$e^{-\zeta\omega_n T} = r \tag{17}$$

$$\zeta\omega_n T = \ln r \tag{18}$$

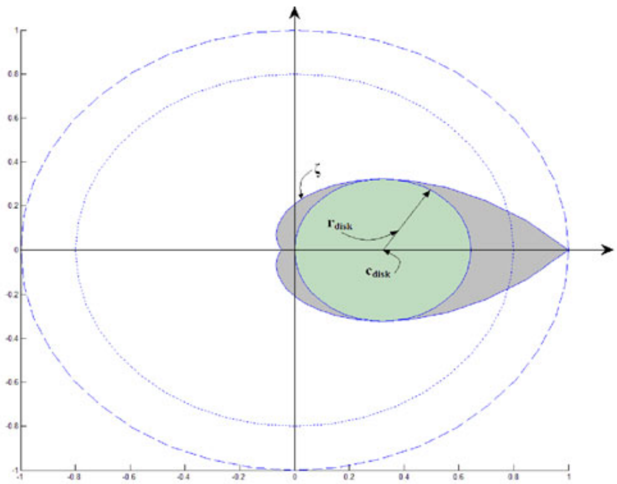

Fig. 2. Vertically Constrained Region

$$\omega_n T \sqrt{1-\zeta^2} = \Theta \tag{19}$$

Taking the ratio of (18) and (19) yields

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{-lnr}{\Theta} \tag{20}$$

Then solve for r to obtain,

$$e^{\left(\frac{-\zeta\Theta}{\sqrt{1-\zeta^2}}\right)} = r \tag{21}$$

Converting (19) and (21) to rectangular form yields:

$$Re = e^{\left(\frac{-\zeta\Theta}{\sqrt{1-\zeta^2}}\right)} cos(\Theta) \tag{22}$$

$$Im = e^{\left(\frac{-\zeta\Theta}{\sqrt{1-\zeta^2}}\right)} sin(\Theta) \tag{23}$$

Solving for the value of θ such that the vertical clearance of the constant ζ-curve is maximized in terms of ζ yields:

$$Im'(\Theta) = \frac{-\zeta}{\sqrt{1-\zeta^2}} e^{\left(\frac{-\zeta\Theta}{\sqrt{1-\zeta^2}}\right)} sin(\Theta) + e^{\left(\frac{-\zeta\Theta}{\sqrt{1-\zeta^2}}\right)} cos(\Theta) = 0 \tag{24}$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} sin(\Theta) = cos(\Theta) \tag{25}$$

$$\Theta = tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{26}$$

Plugging θ into the parametric equations for the curve, (22) and (23), to find the points on the zeta curve that could bound the maximum area circle in the z-plane satisfying our requirements. This gives us:

$$Re = e^{\left(\frac{-\zeta}{\sqrt{1-\zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)} cos\left(tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right) \tag{27}$$

$$Im = e^{\left(\frac{-\zeta}{\sqrt{1-\zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)} sin\left(tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right) \tag{28}$$

Simplifying further yields

$$Re = \zeta e^{\left(\frac{-\zeta}{\sqrt{1-zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)} \tag{29}$$

$$Im = \sqrt{1-\zeta^2} e^{\left(\frac{-\zeta}{\sqrt{1-\zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)} \tag{30}$$

which restated in terms of the circular region are

$$r_{disk} = Im = \sqrt{1-\zeta^2} e^{\left(\frac{-\zeta}{\sqrt{1-\zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)} \tag{31}$$

$$c_{disk} = \left(Re = \zeta e^{\left(\frac{-\zeta}{\sqrt{1-zeta^2}} tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)}, 0\right) \tag{32}$$

2) Horizontally Constrained Region: The above shows the relationships if the dominant constraint is the damping factor. If the stability factor α has an effect such as Fig. 3, the relationships are as follow:

$$x_1 = -e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \tag{33}$$

$$x_2 = \alpha \tag{34}$$

where x_1 is our left most boundary due to the ζ-curve and x_2 is the right most boundary due to the α-curve. This gives us the following relationships:

$$r_{disk} = \frac{x_2 - x_1}{2} = \frac{\alpha + e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{2} \tag{35}$$

$$c_{disk} = \left(\frac{\alpha - e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{2}, 0\right) \tag{36}$$

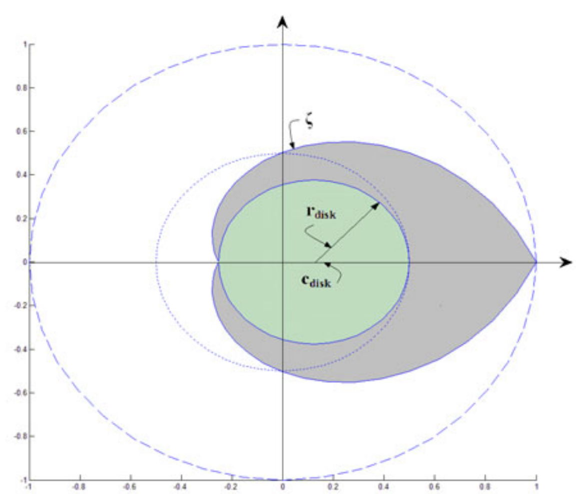

Fig. 3. Horizontally Constrained Region

B. Block Triangular Decomposition of the System

In this section, a method is developed to efficiently perform triangular decomposition on the controllable system pair (A,B) such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{37}$$

where the eigenvalues of A_{11} are in the desirable region and the eigenvalues of A_{22} need to be relocated to the desired region. One option to accomplish this is through Schur decomposition and then rearranging the eigenvalues along the diagonal [15]. Unfortunately, Schur decomposition is computationally expensive and will incur inaccuracies when used on large scale systems making the results unreliable. Below, a matrix sector function approach will be used to decompose the system into the form in (37).

As mentioned earlier, the eigenvalues of the system are to be moved to a circular region that satisfies both a stability and damping specification. After determining the location of that circular region, center and radius, the system can be separated into subsystems. To do this, the matrix sector algorithm is applied to $A' = A - c_{disk} I_n$, which divides the complex plane into two sectors.

After convergence, $S_2(A)$ has eigenvalues at ± 1 with distinct eigenvectors. The next step is to perform a QR decomposition on $(S_2(A) - I_m)$

$$Q^T (S_2(A) - I_n) = R, \qquad R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \tag{38}$$

where $R_{11} \in \mathbb{R}^{n_1 x n_1}$ and $n_2 = n - n_1$. The matrix Q can now be used to transform the system (A, B) to the block triangular form:

$$(A, B) := (Q^T A Q, Q^T B) := \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) \tag{39}$$

$A_{11} \in \mathbb{R}^{n_1 x n_1}$, $A_{22} \in \mathbb{R}^{n_2 x n_2}$, where A_{11} contains the eigenvalues of the system that lie within the circular region and A_{22} contains the eigenvalues of the system that lie outside the circular region and may not meet the stability and damping specifications, where

$$n_2 << n. \tag{40}$$

C. LQR Design

The last part is to design a controller for the subsystem (A_{22}, B_2) that does not meet specifications. This is accomplished by solving the aforementioned Ricatti equation using (A_{22}, B_2) as seen in (41).

$$-Q = \frac{(A_{22} - \gamma I)^T}{r} P \frac{(A_{22} - \gamma I)}{r} - \\ P \frac{(A_{22} - \gamma I)^T}{r} P B_2 (r^2 R + B_2^T P B_2)^{-1} B_2^T P \frac{(A_{22} - \gamma I)}{r} \tag{41}$$

IV. Illustrative Example

In this section, an example is presented demonstrating the effectiveness of this technique. For this example, a 100th order system was created with 5 outlying eigenvalues and 9 marginal eigenvalues. This system was generated using a random number generator and shifting the eigenvalues toward the center of the desired region ($\zeta = 0.707$ and $\alpha = 0.8$). Nine of the points naturally fell outside of the circle. Then five more points were added using a diagonal matrix, D_5 , that were further from the circle for demonstration purposes as seen in (42), where Ψ is a randomly generated matrix with eigenvalues centered around the center of our desired region. The system is then transformed using an arbitrary orthogonal matrix M ($MM^T = I$).

$$A = M \begin{bmatrix} \Psi & 0 \\ 0 & D_5 \end{bmatrix} M^T, D_5 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix} \tag{42}$$

For this example, the input matrix, B, was a randomly generated matrix of size $n \times 10$.

In Figure 4, the eigenvalues of the uncompensated system are indicated by 'o'. The system was partitioned into a "good" 86th order system and a "bad" 14th order system. The eigenvalues of the "bad" subsystem were reassigned using LQR design, while leaving the "good" subsystem unaffected. The eigenvalues of the final system are designated by 'x' in Figure 4. Without this technique, reassigning all the eigenvalues to the desired region would have required a 100th order LQR problem rather than a relatively small 14th order problem.

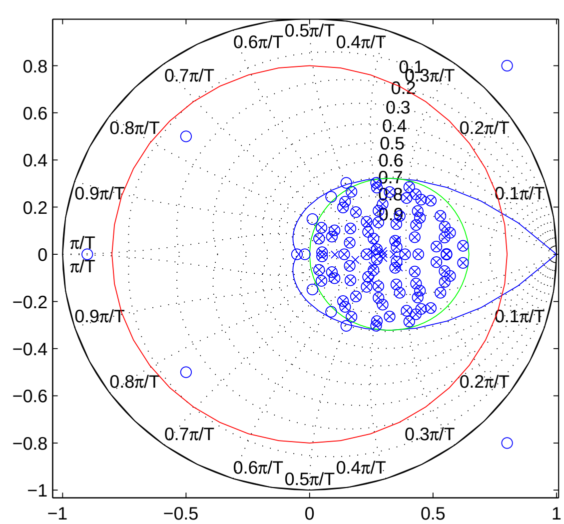

Fig. 4. Compensated and Uncompensated Eigenvalues.

V. Conclusion

This paper presented a procedure for partial state feedback compensation of a large scale discrete system. The method shown here partitions the system into two subsystems through the use of block triangular decomposition. A controller is then designed to compensate the "bad" subsystem, moving eigenvalues to a desired region through the use of LQR design techniques. This dramatically reduces the size of the problem by only controlling eigenvalues out side of desired specifications.

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