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Research Article

The Problem of Bicenter and Isochronicity for a Class of Quasi Symmetric Planar Systems

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We study a class of quasi symmetric seventh degree systems and obtain the conditions that its two singular points can be two centers at the same step by careful computing and strict proof. In addition, the condition of an isochronous center is also given. In terms of quasi symmetric systems, our work is interesting and obtained conclusions about bicenters are new.

1. Introduction

One of the open problems for planar polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{1}$$

is how to characterize their centers and isochronous centers. Article [1] pointed out that "a center of an analytic system is isochronous if and only if there exists an analytic change of coordinates such that the original system is reduced to a linear system," so an isochronous center is also called a linearizable center. A center is an isochronous center or linearizable center if the period of all periodic solutions is constant.

The main method to investigate centers and isochronous centers problem is the computation of focus values and isochronous constants (see [2–13]), which is a kind of active effective method. The vanishing of all isochronous constants or period constants is a necessary and sufficient condition for the isochronicity. Although theoretically the isochronous center problem can be solved by using the method letting all period constants become zero, in fact only the first few period constants can be given in personal computer. Hence, up to now the sufficient and necessary condition determining an isochronous center can only be found by making some appropriate analytic changes of coordinates which let the original system be reduced to a linear system. This kind of appropriate analytic change is very difficult to be obtained, so only a handful of isochronous systems are investigated. Several classes of known studied isochronous systems are as follows: quadratic isochronous centers (see [14]); isochronous centers of a linear center perturbed by third, fourth, and fifth degree homogeneous polynomials (see [3, 4, 15]); complex polynomial systems (see [1]); reversible systems (see [12, 16]); and isochronous centers of quartic systems with degenerate infinity (see [17]).

For seventh degree system, [18] studied the limit cycles bifurcations. In this paper, we investigate the centers and isochronous centers problem for a class of seventh degree systems with the following form:

$$\begin{aligned} \frac{dx}{dt} &= -(x\delta + y)\left(x^2 + y^2\right)^3 - P_6(x, y), \\ \frac{dy}{dt} &= (x - y\delta)\left(x^2 + y^2\right)^3 + Q_6(x, y), \end{aligned} \tag{2}$$

in which

$$\begin{aligned} P_6(x, y) = {} & (1 + a_1) x^4 y + (3 + 2a_1) x^5 y + 2a_4 x^3 y^2 \\ & + 4a_4 x^4 y^2 - (a_1 - a_3 - 2a_5) x^2 y^3 \\ & + (3 + 2a_5) \times x^3 y^3 + 2x (a_6 + 2a_4 x) y^4 \end{aligned}$$

$$
\begin{aligned}
&- \left(a_3 + 2a_1 x - 2a_5 x\right) y^5 + \frac{1}{2} x^2 (x - y) \\
&\times (x + y)\left(x + 3x^2 + 3y^2\right)\delta, \\
Q_6(x, y) = {}& \frac{1}{2} x^5 (1 + 3x) + a_4 x^4 (1 + 2x) y \\
&- \frac{1}{2}\left(1 + 4a_1 - 2a_5\right) x^3 y^2 - \left(4a_1 - a_5\right) x^4 y^2 \\
&- \left(a_4 - a_6\right) x^2 y^3 - \left(2a_3 + a_5\right) x y^4 \\
&- \frac{1}{2}\left(3 + 8a_1\right) x^2 y^4 - y^5\left(a_6 + 2a_4 x + a_5 y\right) \\
&- x^3 y\left(x + 3x^2 + 3y^2\right)\delta,
\end{aligned} \tag{3}
$$

and δ, a_i $(i \in \{1, 2, 3, 4, 5\})$ are real numbers.

We obtain that the infinity and the elementary singular point $(-1/2, 0)$ of $(2)|_{\delta=0}$ have the same center condition and investigate the isochronous center condition of $(-1/2, 0)$. What is worth pointing out is that the results of bicenters in a polynomial system of degree n are less seen in published papers; our work is new and interesting.

In general, our investigations are shown as follows. Firstly, by making two appropriate transformations (i.e., (32) and (33)) of system (2), system (2) is transformed into system (34); hence, the problem of system (2) center problem and isochronicity is reduced to investigate system (34) center and the isochronous centers problem. Secondly, we prove that system (34) is symmetric about $(-1, 0)$. System (34) has two symmetric elementary singular points (i.e., the origin and $(-2, 0)$), which are from the infinity and the elementary focus $(-1/2, 0)$ of (2) under transformations (32) and (33). Thirdly, through calculating system (34) focal values when $\delta = 0$ and careful analysis, we obtain the condition that the infinity and the elementary focus $(-1/2, 0)$ of $(2)|_{\delta=0}$ become bicenters at the same time. Lastly, we study the above isochronicity problems of $(-1/2, 0)$. During the course of investigating isochronicity of system $(2)|_{\delta=0}$, at first we make use of the method in [19] to compute the first several period constants and find the isochronous centers' necessary condition; next we try to find the sufficient condition. We obtain all sufficient and necessary conditions that the elementary focus $(-1/2, 0)$ of (2) become an isochronous center.

The paper is organized as follows. In Section 2, we introduce preliminary methods to calculate focal values (or Lyapunov constants) and period constants which are necessary for our study in Sections 3 and 4. In Section 3, we make two appropriate transformations which let research on system (2) be reduced to investigate a class of symmetric seventh degree systems in which the first five focal values with more simple expressions are given. Being based on it, we find the condition that the infinity and the elementary singular points of $(2)|_{\delta=0}$ can be bicenters and prove them. In Section 4, by analyzing all center conditions and all obtained expressions of periodic constants, we give all sufficient and necessary conditions that the elementary singular point $(-1/2, 0)$ of $(2)|_{\delta=0}$ becomes an isochronous center and prove them strictly.

2. Preliminary Method to Compute Focal Values and Periodic Constant

In order to continue this study, at first we introduce previous methods to calculate focal values and periodic constants which are necessary for us to verify centers and isochronous centers.

Consider the following real system:

$$
\begin{aligned}
\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x, y), \\
\frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y),
\end{aligned} \tag{4}
$$

where $X_k(x, y)$ and $Y_k(x, y)$ are homogeneous polynomials of degree k about x and y.

By means of transformation

$$
z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{5}
$$

system $(4)|_{\delta=0}$ can be transformed into the following complex system:

$$
\begin{aligned}
\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w), \\
\frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w),
\end{aligned} \tag{6}
$$

where z, w, T are complex variables and

$$
Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta}, \qquad W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}. \tag{7}
$$

Obviously, the coefficients of (6) satisfy conjugate condition; that is,

$$
\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \quad \alpha \geq 0, \ \beta \geq 0, \ \alpha + \beta \geq 2. \tag{8}
$$

System $(4)|_{\delta=0}$ and system (6) are called concomitant systems (for the definition see [11, 12]).

For the complex analytic system (6), making transformation

$$
z = re^{i\theta}, \qquad w = re^{-i\theta}, \qquad T = it, \tag{9}
$$

system (6) can be transformed into

$$
\begin{aligned}
\frac{dr}{dt} &= i\frac{wZ - zW}{2r} = ir\sum_{k=1}^{\infty} \frac{wZ_{k+1} - zW_{k+1}}{2zw} \\
&= \frac{ir}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} \left(a_{\alpha,\beta-1} - b_{\beta,\alpha-1}\right) e^{i(\alpha-\beta)\theta} r^m,
\end{aligned}
$$

$$\frac{d\theta}{dt} = \frac{wZ + zW}{2zw} = 1 + \sum_{k=1}^{\infty} \frac{wZ_{k+1} + zW_{k+1}}{2zw}$$
$$= 1 + \frac{1}{2}\sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} \left(a_{\alpha,\beta-1} + b_{\beta,\alpha-1}\right) e^{i(\alpha-\beta)\theta} r^m. \tag{10}$$

According to the relation between systems (4) and (6), in fact transformation (9) can be regarded as the following real polar coordinate transformation of $(4)|_{\delta=0}$:

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{11}$$

Under transformation (11), from (10) we have

$$\frac{dr}{d\theta} = \frac{(ir/2)\sum_{m=1}^{\infty}\sum_{\alpha+\beta=m+2}\left(a_{\alpha,\beta-1} - b_{\beta,\alpha-1}\right) e^{i(\alpha-\beta)\theta} r^m}{1 + (1/2)\sum_{m=1}^{\infty}\sum_{\alpha+\beta=m+2}\left(a_{\alpha,\beta-1} + b_{\beta,\alpha-1}\right) e^{i(\alpha-\beta)\theta} r^m}. \tag{12}$$

For the complex constant h, $|h| \ll 1$, we write the solution of (12) associated with the initial condition $r|_{\theta=0} = h$ as

$$r = \tilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta) h^k, \tag{13}$$

in which $v_{2k+1}(2\pi)$ $(k = 1, 2, \ldots)$ are called the kth focal value of the origin of (4).

From (13), it is clear that the origin of (4) is a center if and only if all $v_{2k+1}(2\pi) = 0$ $(k = 1, 2, \ldots)$. Hence the computation of focal value plays an important role for settling the center problem. Liu and Li [19] gave some methods to compute focal values. Next we will introduce our method to calculate focal value through the following three lemmas.

Lemma 1 (see [20, 21]). *For system* (6)*, one can derive successively the terms of the following formal series:*

$$M = 1 + \sum_{\alpha+\beta=1}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta}, \tag{14}$$

such that

$$\frac{\partial M}{\partial z} Z - \frac{\partial M}{\partial w} W + \left(\frac{\partial Z}{\partial z} - \frac{\partial W}{\partial w}\right) M = \sum_{m=1}^{\infty} (m+1)\mu_m (zw)^m, \tag{15}$$

where $c_{11} = 1$, $c_{20} = c_{02} = 0$*, for all* $c_{kk} \in R$, $k = 2, 3, \ldots$*, and to any integer* m, μ_m *is determined by the following formulas:*

$$c_{1,1} = 1, \qquad c_{2,0} = c_{0,2} = 0,$$
$$\text{if } (\alpha = \beta = 0,\ \beta \neq 1) \quad \text{or} \quad \alpha < 0, \quad \text{or} \quad \beta < 0, \tag{16}$$
$$\text{then } c_{\alpha,\beta} = 0,$$

or else

$$c_{\alpha,\beta} = \frac{1}{\beta - \alpha} \times \sum_{k+j=3}^{\alpha+\beta+2} \left[(\alpha - k + 1) a_{k,j-1} - (\beta - j + 1) b_{j,k-1}\right] c_{\alpha-k+1,\beta-j+1},$$

$$\mu_m = \sum_{k+j=3}^{2m+2} \left[(m - k + 2) a_{k,j-1} - (m - j + 2) b_{j,k-1}\right] c_{m-k+2,m-j+2}. \tag{17}$$

And μ_k *in Lemma 1 is called kth order singular point value at the origin of system* (6).

Lemma 2 (see [20]). *For system* (4) *and any positive integer* m*, among* $v_{2m}(2\pi)$, $v_k(2\pi)$*, and* $v_k(\pi)$*, there exists the following relation:*

$$v_{2m}(2\pi) = \frac{1}{1 + v_1(\pi)} \times \left[\xi_m^{(0)}\left(v_1(2\pi) - 1\right) + \sum_{k=1}^{m-1} \xi_m^{(k)} v_{2k+1}(2\pi)\right], \tag{18}$$

where $\xi_m^{(k)}$ *are all polynomials of* $v_1(\pi), v_2(\pi), \ldots, v_m(\pi)$ *and* $v_1(2\pi), v_2(2\pi), \ldots, v_m(2\pi)$ *with rational coefficients.*

Obviously, We can imply that $v_{2m}(2\pi) = 0$ when $v_1(2\pi) = 1$, $v_{2k+1}(2\pi) = 0$, $k = 1, 2, \ldots, m-1$.

Lemma 3 (see [20]). *For system* $(4)|_{\delta=0}$, (6)*, and any positive integer* m*, the following assertion holds:*

$$v_{2k+1}(2\pi) = i\pi\left(\mu_m + \sum_{k=1}^{m-1} \xi_m^{(k)} \mu_k\right), \tag{19}$$

where $\xi_m^{(k)}$, $(k = 1, 2, \ldots, m-1)$ *are polynomial functions of coefficients of system* (6).

Obviously, the origin of system (4) is a center if and only if its all focal values vanish, namely, $v_{2k+1} = 0$, $k \in \mathbf{N}$. According to Lemmas 2 and 3, we have the following lemma.

Lemma 4 (see [20]). *For systems* $(4)|_{\delta=0}$ *and* (6)*, the origin is a center if and only if the following relation holds:*

$$\mu_m(2\pi) = 0, \quad m \in \mathbf{N}. \tag{20}$$

Remark 5. In fact, Lemmas 1–4 have given a method to find original center condition of (4).

What is the isochronous center condition of the origin of (4) if the origin of (4) is a center? Next we introduce our method to obtain the isochronous center condition.

We denote that $\tau(\varphi, h) = \int_0^{\varphi} (dt/d\theta) d\theta$. From (10), we have

$$\tau(\varphi, h) = \int_0^{\varphi} \frac{dt}{d\theta} d\theta$$

$$= \int_0^{\varphi} \left[1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} \left(a_{\alpha,\beta-1} + b_{\beta,\alpha-1} \right) \times e^{i(\alpha-\beta)\theta} \widetilde{r}^{m} (\theta, h) \right]^{-1} d\theta. \tag{21}$$

Definition 6. For a sufficiently small complex constant h, the origin of system (6) is called a complex center if $\widetilde{r}(2\pi, h) \equiv h$ of (13), and the origin is a complex isochronous center if

$$\widetilde{r}(2\pi, h) \equiv h, \qquad \tau(2\pi, h) \equiv 2\pi. \tag{22}$$

Lemma 7 (see [19]). *For system* (6)*, one can derive uniquely the formal series*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \qquad \eta = w + \sum_{k+j=2}^{\infty} d_{k,j} w^k z^j, \tag{23}$$

where $c_{k+1,k} = d_{k+1,k} = 0$, $k = 1, 2, \ldots$, p_k *and* q_k *are polynomial in* $a_{\alpha\beta}$, $b_{\alpha\beta}$ *with rational coefficients, such that*

$$\frac{d\xi}{dT} = \xi + \sum_{j=1}^{\infty} p_j \, \xi^{j+1} \eta^j, \qquad \frac{d\eta}{dT} = -\eta - \sum_{j=1}^{\infty} q_j \, \eta^{j+1} \xi^j. \tag{24}$$

Let $\mu_0 = \tau_0 = 0$, $\mu_k = p_k - q_k$, $\tau_k = p_k + q_k$, $k = 1, 2, \ldots$, in which μ_k is kth singular point value of the origin of system (6).

Definition 8. For any positive integer k, one says that $\tau(k) = p_k + q_k$ is the kth complex period constant of origin of system (6).

Lemma 9 (see [19]). *Suppose that the origin of system* (6) *is a complex center (i.e., all* $\mu_m = 0$, $m = 1, 2, \ldots$*) and there exists a positive integer* k*, such that* $\tau_0 = \tau_1 = \cdots = \tau_{k-1} = 0$, $\tau_k \neq 0$*; then*

$$\tau(2\pi, h) = \pi \left[2 - \tau_k h^{2k} + o\left(h^{2k}\right) \right]. \tag{25}$$

It is clear that the origin of system (6) is a complex isochronous center if and only if all $\mu_k = \tau_k = 0$, $k = 1, 2, 3, \ldots$.

For the problem of the computation of τ_k, [14] gives the following two theorems.

Theorem A (see [19]). *For system* (6)*, one can derive uniquely the formal series*

$$f(z, w) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j, \qquad g(z, w) = w + \sum_{k+j=2}^{\infty} d'_{k,j} w^k z^j, \tag{26}$$

where $c'_{k+1,k} = d'_{k+1,k} = 0$, $k = 1, 2, \ldots$, *such that*

$$\frac{df}{dT} = f(z, w) + \sum_{j=1}^{\infty} p'_j \, z^{j+1} w^j, \qquad \frac{dg}{dT} = -g(z, w) - \sum_{j=1}^{\infty} q'_j \, w^{j+1} z^j, \tag{27}$$

and when $k - j - 1 \neq 0$, c'_{kj} *and* $d'_{k,j}$ *are determined by the recursive formulas*

$$c'_{kj} = \frac{1}{j+1-k} \times \sum_{\alpha+\beta=3}^{k+j+1} \left[(k - \alpha + 1) a_{\alpha,\beta-1} - (j - \beta + 1) b_{\beta,\alpha-1} \right] c'_{k-\alpha+1, j-\beta+1},$$
$$d'_{kj} = \frac{1}{j+1-k} \times \sum_{\alpha+\beta=3}^{k+j+1} \left[(k - \alpha + 1) b_{\alpha,\beta-1} - (j - \beta + 1) a_{\beta,\alpha-1} \right] d'_{k-\alpha+1, j-\beta+1}, \tag{28}$$

and for any positive integer j, p'_j, *and* q'_j *are determined by the recursive formulas*

$$p'_j = \sum_{\alpha+\beta=3}^{2j+2} \left[(j - \alpha + 2) a_{\alpha,\beta-1} - (j - \beta + 1) b_{\beta,\alpha-1} \right] c'_{j-\alpha+2, j-\beta+1},$$
$$q'_j = \sum_{\alpha+\beta=3}^{2j+2} \left[(j - \alpha + 2) b_{\alpha,\beta-1} - (j - \beta + 1) a_{\beta,\alpha-1} \right] d'_{j-\alpha+2, j-\beta+1}. \tag{29}$$

In (28) *and* (29)*, we have taken* $c'_{1,0} = d'_{1,0} = 1$, $c'_{0,1} = d'_{0,1} = 0$, *and if* $\alpha < 0$ *or* $\beta < 0$, *we take* $a_{\alpha\beta} = b_{\alpha\beta} = c'_{\alpha\beta} = d'_{\alpha\beta} = 0$.

Theorem B (see [19]). *Let* $p_0 = q_0 = p'_0 = q'_0 = 0$. *If there is a positive integer* m*, such that*

$$p_0 = q_0 = p_1 = q_1 = \cdots = p_{m-1} = q_{m-1} = 0, \tag{30}$$

then

$$p'_0 = q'_0 = p'_1 = q'_1 = \cdots = p'_{m-1} = q'_{m-1} = 0, \qquad p_m = p'_m, \qquad q_m = q'_m, \tag{31}$$

and vice versa.

Actually, Lemma 7 and the above two theorems give an algorithm to compute τ_m. For any positive integer m, in order to compute τ_m, we only need to carry out the addition, subtraction, multiplication, and division to the coefficients of system (6). The algorithm is recursive. It avoids some complicated integrating operations and solving equations. In addition, it can be easily realized by computer algebra systems such as Mathematica.

Notice that the complex period constants are polynomials of the coefficients of system (6). According to the Hilbert basis theorem, there exists $m \in \mathbf{N}$ such that all $\tau_k = 0$ $(k = 1, 2, \ldots)$ if and only if $\tau_1 = \tau_2 = \cdots = \tau_m = 0$. We say that the set $\{\tau_1, \tau_2, \ldots, \tau_m\}$ is a period constant basis of system (6). To determine isochronous center of a system, the key idea is to find a period constant basis.

Remark 10. Lemma 7 and Theorems A and B offer a method to find a necessary condition of isochronicity.

3. The Reduction and Bicenter Condition of System (2)

After introducing the method to calculate the focal values and period constants of system (4), we try to make some appropriate transformations so as to carry out our investigation about system (2).

By means of Bendixson homeomorphous transformation

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{y}{x^2 + y^2}, \tag{32}$$

and time transformation

$$dt = \left(x^2 + y^2\right)^3 d\tau, \tag{33}$$

system (2) can be transformed into the following real system:

$$\begin{aligned} \frac{du}{d\tau} &= \delta u - v + \frac{3\delta}{2} u^2 + 2a_1 uv + \frac{\delta}{2} u^3 + a_1 u^2 v + a_3 v^3, \\ \frac{dv}{d\tau} &= u + \delta v + \frac{3}{2} u^2 + 2a_4 uv + a_5 v^2 + \frac{1}{2} u^3 \\ &\quad + a_4 u^2 v + a_5 u v^2 + a_6 v^3. \end{aligned} \tag{34}$$

After making the above two transformations, the infinity and the elementary focus point $(-1/2, 0)$ of (2), respectively, become the origin and $(-2, 0)$ of system (34). For system (34), we have the following theorem.

Theorem 11. *System* (34) *is a class of z_2-equivariant cubic systems about point* $(-1, 0)$.

Proof. By means of translation transformation

$$u = x - 1, \qquad v = y, \tag{35}$$

system (34) turns into the following system:

$$\begin{aligned} \frac{dx}{d\tau} &= -\frac{\delta}{2} x - (a_1 + 1) y + \frac{\delta}{2} x^3 + a_1 x^2 y + a_3 y^3, \\ \frac{dy}{d\tau} &+ (\delta - a_4) y + \frac{1}{2} x^3 + a_4 x^2 y + a_5 x y^2 + a_6 y^3. \end{aligned} \tag{36}$$

Obviously point $(-x, -y)$ satisfies (36) if (x, y) satisfies (36); then system (36) is a class of z_2-equivariant cubic systems about the origin. Hence system (34) is a class of z_2-equivariant cubic systems about point $(-1, 0)$. Proof ends. □

After making transformations (32) and (33), system (2) becomes a symmetric system about point $(-1, 0)$ (i.e., system (34)), so here we call system (2) a class of quasi symmetric systems. In fact, through investigating the center condition of the origin of system (34), those of the infinity and the elementary focus point $(-1/2, 0)$ of (2) can been forecasted.

In order to investigate the centers problem of (2), we may as well study system (34) or system (36).

Making the transformation

$$z = u + iv, \qquad w = u - iv, \qquad T = i\tau, \quad i = \sqrt{-1}, \tag{37}$$

system $(34)|_\delta = 0$ becomes

$$\begin{aligned} \frac{dz}{dT} &= z + Z_2(z, w) + Z_3(z, w), \\ \frac{dw}{dT} &= -w - W_2(z, w) - W_3(z, w), \end{aligned} \tag{38}$$

in which

$$\begin{aligned} Z_2(z, w) &= \frac{1}{8}(3 - 4a_1 - 4ia_4 - 2a_5) z^2 + \frac{1}{4}(3 + 2a_5) zw \\ &\quad + \frac{1}{8}(3 + 4a_1 + 4ia_4 - 2a_5) w^2, \\ Z_3(z, w) &= \frac{1}{16}(1 - 2a_1 + 2a_3 - 2ia_4 - 2a_5 + 2ia_6) z^3 \\ &\quad + \frac{1}{16}(3 - 2a_1 - 6a_3 - 2ia_4 + 2a_5 - 6ia_6) z^2 w \\ &\quad + \frac{1}{16}(3 + 2a_1 + 6a_3 + 2ia_4 + 2a_5 + 6ia_6) z w^2 \\ &\quad + \frac{1}{16}(1 + 2a_1 - 2a_3 + 2ia_4 - 2a_5 - 2ia_6) w^3, \\ W_2(z, w) &= \frac{1}{8}(3 - 4a_1 + 4ia_4 + 2a_5) w^2 + \frac{1}{4}(3 + 2a_5) zw \\ &\quad + \frac{1}{8}(3 + 4a_1 - 4ia_4 - 2a_5) w^2, \\ W_3(z, w) &= \frac{1}{16}(1 + 2a_1 - 2a_3 - 2ia_4 - 2a_5 + 2ia_6) z^3 \\ &\quad + \frac{1}{16}(3 + 2a_1 + 6a_3 - 2ia_4 + 2a_5 - 6ia_6) z^2 w \\ &\quad + \frac{1}{16}(3 - 2a_1 - 6a_3 + 2ia_4 + 2a_5 + 6ia_6) z w^2 \\ &\quad + \frac{1}{16}(1 - 2a_1 + 2a_3 + 2ia_4 - 2a_5 - 2ia_6) w^3. \end{aligned} \tag{39}$$

System (38) is called the complex concomitant system of system (34). Clearly system (38) belongs to the class of system (6), so we can use the formulas of Lemma 1 and the conclusion of Lemma 4 to compute and simplify the singular points values by using computational software such as Mathematica; we obtain the following theorem.

Theorem 12. *The first 5 singular points' values of the origin for* (38) *are as follows:*

$$\mu_1 = \frac{i}{4}\left(2a_4 + 2a_4a_5 - 3a_6\right);$$
$$\mu_2 = \frac{i}{36}a_4\left[\left(1 + a_5\right)\left(18 + 18a_1 + 8a_4^2\right) - a_3\left(9 + 12a_1 + 12a_5\right)\right]; \quad (40)$$

(1) *if* $a_5 = -1$, *then*

$$\mu_2 = -\frac{i}{12}a_3a_4\left(4a_1 - 1\right);$$
$$\mu_3 = \frac{i}{384}a_3a_4\left(84a_3 - 16a_4^2 - 45\right); \quad (41)$$
$$\mu_4 = \frac{i}{169344}a_4\left(3 + 2a_4^2\right)\left(45 + 16a_4^2\right)^2;$$

(2) *if* $a_5 \neq -1$, *then*

$$\mu_3 = \frac{ia_3a_4}{96\left(1 + a_5\right)}\left(4a_1 - 10a_5 - 11\right)h_0;$$
$$\mu_4 = -\frac{7ia_3a_4}{1500}\left(1 + a_1 - 5a_3\right) \times\left(29a_1^2 - a_1 - 2 + 28a_1^3 + 45a_3 - 105a_1a_3\right); \quad (42)$$
$$\mu_5 = \frac{7i}{36}a_3^4a_4\left(11 - 76a_3 + 140a_3^2\right)$$

in which

$$h_0 = -6 - 10a_1 - 4a_1^2 + 3a_3 + 6a_1a_3 - 10a_5 - 14a_1a_5 - 4a_1^2a_5 + 6a_3a_5 - 4a_5^2 - 4a_1a_5^2. \quad (43)$$

In the above expressions of μ_k, *one lets* $\mu_i = 0$, $i = 1, 2, \ldots, k-1$.

Proof. According to Lemma 1, we have

$$\mu_1 = \frac{i}{4}\left(2a_4 + 2a_4a_5 - 3a_6\right). \quad (44)$$

Let $a_6 = (2/3)a_4(1 + a_5)$; then

$$\mu_1 = 0;$$
$$\mu_2 = \frac{i}{36}a_4\left[\left(1 + a_5\right)\left(18 + 18a_1 + 8a_4^2\right) - a_3\left(9 + 12a_1 + 12a_5\right)\right]. \quad (45)$$

(1) If $a_5 = -1$, then

$$\mu_2 = -\frac{i}{12}a_3a_4\left(4a_1 - 1\right); \quad (46)$$

if $a_3a_4 = 0$, from Lemma 1, we obtain that $\mu_k = 0$, $k = 2, 3, 4, 5$. Hence let $a_1 = 1/4$; then

$$\mu_3 = \frac{i}{384}a_3a_4\left(84a_3 - 16a_4^2 - 45\right). \quad (47)$$

Moreover let $a_3 = (1/84)\ (45 + 16a_4^2)$; then

$$\mu_3 = 0, \qquad \mu_4 = \frac{i}{169344}a_4\left(3 + 2a_4^2\right)\left(45 + 16a_4^2\right)^2; \quad (48)$$

while if $a_4 \neq 0$, then $\mu_4 \neq 0$; at this time only four singular point values can be obtained.

(2) If $a_5 \neq -1$, letting

$$a_4^2 = \frac{3\left(4a_3a_5 - 6a_1a_5 - 6a_5 + 4a_1a_3 + 3a_3 - 6a_1 - 6\right)}{\left(8 + 8a_5\right)}, \quad (49)$$

then

$$\mu_2 = 0; \qquad \mu_3 = \frac{ia_3a_4}{96\left(1 + a_5\right)}\left(4a_1 - 10a_5 - 11\right)h_0. \quad (50)$$

Because $a_3a_4h_0 = 0$ will induce $\mu_3 = \mu_4 = \mu_5 = 0$, let $a_5 = (1/10)\ (4a_1 - 11)$; at this time

$$\mu_4 = -\frac{7ia_3a_4}{1500}\left(1 + a_1 - 5a_3\right) \times\left(29a_1^2 - a_1 - 2 + 28a_1^3 + 45a_3 - 105a_1a_3\right), \quad (51)$$

while $a_3a_4\ (29a_1^2 - a_1 - 2 + 28a_1^3 + 45a_3 - 105a_1a_3) = 0$ will induce $\mu_4 = \mu_5 = 0$, let $a_1 = 5a_3 - 1$; then

$$\mu_5 = \frac{7i}{36}a_3^4a_4\left(11 - 76a_3 + 140a_3^2\right). \quad (52)$$

Proof ends. □

Remark. The equation $11 - 76a_3 + 140a_3^2 = 0$ has not real roots, so only five singular point values exist.

From Theorem 12 and Lemma 4, we have the following theorem.

Theorem 13. *The first 5 focal values of the origin of* (21) *(or the first 5 general focal values of the infinity and the elementary focus* $(-1/2, 0)$ *of* (4)*) are as follows:*

$$v_3 = -\frac{\pi}{4}\left(2a_4 + 2a_4a_5 - 3a_6\right);$$
$$v_5 = -\frac{\pi}{36}a_4\left[\left(1 + a_5\right)\left(18 + 18a_1 + 8a_4^2\right) - a_3\left(9 + 12a_1 + 12a_5\right)\right]. \quad (53)$$

(1) *If* $a_5 = -1$, *then*

$$v_5 = \frac{\pi}{12}a_3a_4\left(4a_1 - 1\right);$$
$$v_7 = -\frac{\pi}{384}a_3a_4\left(84a_3 - 16a_4^2 - 45\right). \quad (54)$$
$$v_9 = -\frac{\pi}{169344}a_4\left(3 + 2a_4^2\right)\left(45 + 16a_4^2\right)^2.$$

(2) *If* $a_5 \neq -1$, *then*

$$
\begin{aligned}
v_7 &= -\frac{\pi a_3 a_4}{96(1+a_5)}(4a_1 - 10a_5 - 11)h_0,\\
v_9 &= \frac{7\pi a_3 a_4}{1500}(1 + a_1 - 5a_3)\\
&\quad\times\left(29a_1^2 - a_1 - 2 + 28a_1^3 + 45a_3 - 105a_1a_3\right),\\
v_{11} &= -\frac{7\pi}{36}a_3^4 a_4\left(11 - 76a_3 + 140a_3^2\right).
\end{aligned}
\tag{55}
$$

in which

$$
\begin{aligned}
h_0 = {}& -6 - 10a_1 - 4a_1^2 + 3a_3 + 6a_1a_3 - 10a_5\\
&- 14a_1a_5 - 4a_1^2a_5 + 6a_3a_5 - 4a_5^2 - 4a_1a_5^2.
\end{aligned}
\tag{56}
$$

In the above expressions of v_{2k+1}, *we have let* $v_{2i-1} = 0$, $i = 2, \dots, k$.

According to Theorem 13, it is easy to obtain the following theorem.

Theorem 14. *The first 5 focal values of the origin of* (34) *(or the first 5 focal values of the infinity and the elementary focus* $(-1/2, 0)$ *of* (2)*) vanish if and only if one of the following conditions holds:*

(H_1) $a_4 = a_6 = 0$;

(H_2) $a_4^2 = -(9/4)(1 + a_1) > 0$, $a_3 = 0$, $a_6 = (2/3)a_4(1 + a_5)$, $a_5 \neq -1$;

(H_3) $a_6 = (2/3)a_4(1 + a_5)$, $a_5 \neq -1$, $a_3 \neq 0$, $a_4^2 = 3\,(4a_3a_5 - 6a_1a_5 - 6a_5 + 4a_1a_3 + 3a_3 - 6a_1 - 6)/(8 + 8a_5) > 0$, $h_0 = 0$.

Theorem 15. *The origin of* (34) *is a center (the infinity and the elementary focus* $(-1/2, 0)$ *of* (2) *are two centers), if and only if one of the conditions* (H_1), (H_2), *and* (H_3) *holds.*

Proof. (1) According to Theorem 14, if the origin of (34) is a center (the infinity and the elementary focus $(-1/2, 0)$ of (2) are two general centers), then one of the conditions (H_1), (H_2), and (H_3) holds. So necessary condition is correct.

(2) Next we prove sufficient condition.

(2.1) If (H_1) holds, then $(36)|_{\delta=0}$ becomes

$$
\begin{aligned}
\frac{dx}{d\tau} &= -(a_1 + 1)y + a_1x^2y + a_3y^3.\\
\frac{dy}{d\tau} &= -\frac{1}{2}x + \frac{1}{2}x^3 + a_5xy^2
\end{aligned}
\tag{57}
$$

Letting

$$
\begin{aligned}
g_1(x, y) &= 2 + a_1 + a_5 - a_1x^2 - a_5x^2 - 2a_3y^2\\
&\quad + 2a_1a_5y^2 - 2a_5^2y^2,\\
g_2(x, y) &= -1 + x^2 + 2a_5y^2,
\end{aligned}
\tag{58}
$$

then system (57) has an integral factor $M_1(x, y) = f_1^{-1}$ and a first integral $F_1(x, y) = f_1 f_2^{(a_1+a_5)}$, in which

$$
f_1 = g_1^2 - \left[2a_3 + (a_1 - a_5)^2\right]g_2^2,
\tag{59}
$$

$$
f_2 = \begin{cases}
\left(\dfrac{g_1 + \sqrt{2a_3 + (a_1 - a_5)^2}\,g_2}{g_1 - \sqrt{2a_3 + (a_1 - a_5)^2}\,g_2}\right)^{1/\sqrt{2a_3 + (a_1 - a_5)^2}}, & \text{if } 2a_3 + (a_1 - a_5)^2 > 0;\\[2ex]
\exp\dfrac{2\left(-1 + x^2 + 2a_5y^2\right)}{2 + (a_1 + a_5)\left(1 - x^2 + a_1y^2 - a_5y^2\right)}, & \text{if } 2a_3 + (a_1 - a_5)^2 = 0;\\[2ex]
\exp\left(\dfrac{2}{\sqrt{-2a_3 - (a_1 - a_5)^2}}\arctan\dfrac{g_2}{g_1}\right), & \text{if } 2a_3 + (a_1 - a_5)^2 < 0.
\end{cases}
\tag{60}
$$

Considering that system (34) can turn into system (36) by making transformations, hence the origin of (34) is a center when (H_1) holds; moreover, the infinity and the elementary focus $(-1/2, 0)$ of (2) are two centers.

(2.2) If (H_2) holds, then $(36)|_{\delta=0}$ becomes

$$
\begin{aligned}
\frac{dx}{d\tau} &= \frac{4}{9}a_4^2y - \frac{1}{9}\left(9 + 4a_4^2\right)x^2y,\\
\frac{dy}{d\tau} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2\\
&\quad + \frac{2}{3}a_4(1 + a_5)y^3,
\end{aligned}
\tag{61}
$$

while system (61) has an integral factor

$$
M_2(x, y) = f_3^{-3} f_4^{(9 - 8a_4^2 + 18a_5)/2}
\tag{62}
$$

and a first integral

$$
\begin{aligned}
F_2 = {}& (3x + 4a_4y) f_3^{-2} f_4^{9(3+2a_5)/2}\\
&-6(1 + a_5)\int\frac{f_4^{9(3+2a_5)/2}dx}{9 + (9 + 4a_4^2)(-1 + x^2)},
\end{aligned}
\tag{63}
$$

in which

$$
\begin{aligned}
f_3 &= 3x + 2a_4y,\\
f_4 &= \begin{cases}
\left[9 + \left(9 + 4a_4^2\right)\left(-1 + x^2\right)\right]^{1/(9+4a_4^2)}, & \text{if } 9 + 4a_4^2 \neq 0;\\
e^{(-1+x^2)/9}, & \text{if } 9 + 4a_4^2 = 0.
\end{cases}
\end{aligned}
\tag{64}
$$

Considering that system (34) can turn into system (36) by making transformations, hence the origin of (34) is a center when (H_2) holds; moreover, the infinity and the elementary focus $(-1/2, 0)$ of (2) are two centers.

(2.3) If (H_3) holds, then $(34)|_{\delta=0}$ has an integral factor $M_3(x,y) = f_5^{2(a_1+a_5)}$ and a first integral $F_3(x,y) = f_5^{(1+2a_1+2a_5)} f_6$, in which

$$\begin{aligned} f_5 &= (a_1+a_5)x - a_4 y, \\ f_6 &= (2+a_1+a_5)[a_4 x + 2(1+a_1)(a_1+a_5)y] \\ &\quad - (1+a_1+a_5) \\ &\quad \times \left[a_4 x^3 + 2(1+a_1)(a_1+a_5)x^2 y \right. \\ &\quad \left. + 2a_4(1+a_5)xy^2 + 2a_3(a_1+a_5)y^3\right]. \end{aligned} \tag{65}$$

Considering that system (34) can turn into system (36) by making transformations, hence the origin of (34) is a center when (H_3) holds; moreover, the infinity and the elementary focus $(-1/2, 0)$ of (2) are two centers. Proof ends. □

4. Isochronous Center Condition of $(-1/2, 0)$ of System (2)

After obtaining the conditions that the infinity and the elementary focus $(-1/2, 0)$ are two centers, next we continue to investigate isochronicity of the elementary focus $(-1/2, 0)$ according to three different cases (i.e., (H_1), (H_2), and (H_3)) in order.

4.1. The Isochronicity of Case (H_1). If condition (H_1) holds, then system $(34)|_{\delta=0}$ becomes

$$\begin{aligned} \frac{du}{d\tau} &= -v + 2a_1 uv + a_1 u^2 v + a_3 v^3, \\ \frac{dv}{d\tau} &= u + \frac{3}{2}u^2 + a_5 v^2 + \frac{1}{2}u^3 + a_5 uv^2. \end{aligned} \tag{66}$$

System (66) complex concomitant system is as follows:

$$\begin{aligned} \frac{dz}{dT} &= z + Z_2(z,w) + Z_3(z,w), \\ \frac{dw}{dT} &= -w - W_2(z,w) - W_3(z,w), \end{aligned} \tag{67}$$

in which

$$\begin{aligned} Z_2(z,w) &= \frac{1}{8}(3-4a_1-2a_5)z^2 + \frac{1}{4}(3+2a_5)zw \\ &\quad + \frac{1}{8}(3+4a_1-2a_5)w^2, \\ Z_3(z,w) &= \frac{1}{16}(1-2a_1+2a_3-2a_5)z^3 \\ &\quad + \frac{1}{16}(3-2a_1-6a_3+2a_5)z^2 w \\ &\quad + \frac{1}{16}(3+2a_1+6a_3+2a_5)zw^2 \\ &\quad + \frac{1}{16}(1+2a_1-2a_3-2a_5)w^3, \\ W_2(z,w) &= \frac{1}{8}(3-4a_1+2a_5)w^2 + \frac{1}{4}(3+2a_5)zw \\ &\quad + \frac{1}{8}(3+4a_1-2a_5)w^2, \\ W_3(z,w) &= \frac{1}{16}(1+2a_1-2a_3-2a_5)z^3 \\ &\quad + \frac{1}{16}(3+2a_1+6a_3+2a_5)z^2 w \\ &\quad + \frac{1}{16}(3-2a_1-6a_3+2a_5)zw^2 \\ &\quad + \frac{1}{16}(1-2a_1+2a_3-2a_5)w^3. \end{aligned} \tag{68}$$

According to formulas (28) and (29) of Theorem A and $\tau_k = p_k + q_k = p'_k + q'_k$, we can compute periodic constants of systems (66) and (67).

Theorem 16. *The first four period constants of the origin of system (66) or (67) are as follows:*

$$\begin{aligned} \tau_1 &= \frac{1}{12}\left(-18 - 4a_1^2 - 9a_3 - 12a_5 + 10a_1 a_5 - 4a_5^2\right), \\ \tau_2 &= \frac{1}{18}\left(-108 - 45a_1 - 12a_1^2 - 117a_5 - 24a_1 a_5 \right. \\ &\quad \left. -2a_1^2 a_5 - 36a_5^2 - 4a_1 a_5^2 - 2a_5^2\right), \\ \tau_3 &= \frac{1}{1080}h_1, \\ \tau_4 &= -\frac{1}{1782606477375000000}(9+a_5)(3+2a_5)^2 h_2. \end{aligned} \tag{69}$$

in which

$$\begin{aligned} h_1 &= \left(9720 + 4050a_1 + 27a_1^2 - 300a_1^3 - 100a_1^4 \right. \\ &\quad + 8802a_5 + 1134a_1 a_5 - 1416a_1^2 a_5 \\ &\quad \left. -200a_1^3 a_5 + 2115a_5^2 + 468a_1 a_5^2 - 100a_1^2 a_5^2\right), \\ h_2 &= 441227872054417689 + 969414967442849922a_5 \\ &\quad + 729276873221792484a_5^2 + 232198402133252079a_5^3 \\ &\quad + 40172927013408528a_5^4 + 3428874437224704a_5^5 \\ &\quad + 49018338325504a_5^6, \end{aligned} \tag{70}$$

In the above expression of τ_m, we have let $\tau_1 = \cdots \tau_{m-1} = 0$, $m = 2, 3, 4$.

Theorem 17. *The first four period constants of the origin of system (66) or (67) are zero, if and only if one of the following two conditions holds:*

(C_1) $a_1 = a_5 = -3/2$, $a_3 = 1/2$;

(C_2) $a_1 = -3$, $a_3 = 0$, $a_5 = -9$.

Proof. By computing the resultant of τ_2 and τ_3 about variable a_1, we have

$$\text{Resultant}\left[\tau_3, \tau_2, a_1\right] = \frac{1}{212576400}\left(9 + a_5\right)\left(3 + 2a_5\right)^2 R, \tag{71}$$

in which

$$R = 4214349 + 14258511a_5 + 18779526a_5^2 + 12158343a_5^3 + 4086747a_5^4 + 727632a_5^5 + 69888a_5^6 + 1024a_5^7. \tag{72}$$

Also the resultant of R and h_2 about a_5 is as follows:

$$\text{Resultant}\left[R, h_2, a_5\right] = -28488259\cdots \neq 0, \tag{73}$$

which shows that $R = 0$ cannot deduce $\tau_2 = \tau_3 = \tau_4 = 0$.

From (71), let $\tau_2 = \tau_3 = \tau_4 = 0$; then Resultant$[\tau_3, \tau_2, a_1] = 0$; hence $a_5 = -3/2$ or $a_5 = -9$.

If $a_5 = -3/2$, then $\tau_4 = 0$; continue to let $\tau_3 = 0$ (i.e., $h_1 = 0$); then $a_1 = -3/2$; next let $\tau_1 = 0$; then $a_3 = 1/2$. Hence condition (C_1) holds. At the same time, if condition (C_1) holds, it is easy to obtain that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$.

If $a_5 = -9$, then $\tau_4 = 0$; continue to let $\tau_3 = 0$ (i.e., $h_1 = 0$); then $a_1 = -3$; next let $\tau_1 = 0$; then $a_3 = 0$. Hence condition (C_2) holds. At the same time, if condition (C_2) holds, it is easy to obtain that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$. □

Clearly, condition (C_1) or (C_2) is necessary for the isochronicity of the origin of system (66) or (67). Moreover, we will prove that condition (C_1) or (C_2) is sufficient condition for the isochronicity of the origin of system (66) or (67).

Theorem 18. *The origin of system* (66) *or* (67) *is an isochronous center if and only if one of (C_1) and (C_2) holds.*

Proof. If the origin of system (66) or (67) is an isochronous center, according to Theorem 17, then one of (C_1) and (C_2) holds. Hence, the necessary condition is proved.

Next we will prove the sufficient condition.

(1) If (C_1) holds, then system (66) becomes

$$\frac{du}{d\tau} = -\frac{1}{2}v\left(2 + 6u + 3u^2 - v^2\right), \quad \frac{dv}{d\tau} = \frac{1}{2}(1 + u)\left(2u + u^2 - 3v^2\right). \tag{74}$$

For (74), make the following analytic polar transformations:

$$u_1 = \frac{2u + 5u^2 + 4u^3 + u^4 + 3v^2 + 4uv^2 + 2u^2v^2 + v^4}{2\left(1 + 2u + u^2 + v^2\right)^2}, \quad v_1 = \frac{(1 + u)\,v}{\left(1 + 2u + u^2 + v^2\right)^2}, \tag{75}$$

which satisfy $du_1/d\tau = u_1$, $dv_1/d\tau = -v_1$, so the origin of system (74) is an isochronous center.

(2) If (C_2) holds, then system (66) becomes

$$\frac{du}{d\tau} = -v - 6uv - 3u^2v, \quad \frac{dv}{d\tau} = u + \frac{3}{2}u^2 - 9v^2 + \frac{1}{2}u^3 - 9uv^2. \tag{76}$$

For (76), make the following analytic polar transformations:

$$u_2 = \frac{u\,(u-1)\,(u-2)}{\sqrt{2}\left(3u - (3/2)u^2 - 1/2\right)^{3/2}}, \quad v_2 = \frac{\sqrt{2}v}{\left(3u - (3/2)u^2 - 1/2\right)^{3/2}}, \tag{77}$$

which satisfy $du_2/d\tau = u_2$, $dv_2/d\tau = -v_2$, so the origin of system (76) is an isochronous center. □

Considering that system (34) is a class of z_2-equivariant cubic systems about point $(-1, 0)$, we have the following Theorem.

Theorem 19. *If one of the conditions (C_1) and (C_2) holds under condition (H_1), then the origin and point $(-2, 0)$ of system* (34) *are two isochronous centers.*

Considering that system (2) can be changed into system (34) under the transformations (32) and (33), ulteriorly, we have the following Theorem.

Theorem 20. *If one of the conditions (C_1) and (C_2) holds under condition (H_1), then the singular point $(-1/2, 0)$ of system* (2) *is an isochronous center.*

4.2. The Isochronicity of Case (H_2). If condition (H_2) holds, then system $(34)|_{\delta=0}$ becomes

$$\begin{aligned}\frac{du}{d\tau} &= -\frac{1}{9}v\left(9 + 18u + 8a_4^2u + 9u^2 + 4a_4^2u^2\right), \\ \frac{dv}{d\tau} &= \frac{1}{6}\left(6u + 9u^2 + 3u^3 + 12a_4uv + 6a_4u^2v + 6a_5v^2 + 6a_5uv^2 + 4a_4v^3 + 4a_4a_5v^3\right).\end{aligned} \tag{78}$$

System (78) complex concomitant system belongs to the form of (38). Hence we can use formulas (28) and (29) of Theorem A and $\tau_k = p_k + q_k = p_k' + q_k'$ to compute periodic constants of system (78).

Theorem 21. *The first three period constants of the origin of system* (78) *are as follows:*

$$\tau_1 = \frac{1}{486}\left(-891 - 306a_4^2 - 32a_4^4 - 891a_5 - 180a_4^2a_5 - 162a_5^2\right),$$

$$\tau_2 = \frac{1}{2916}\left(9+4a_4^2\right) \times \left(729+234a_4^2+32a_4^4+567a_5+108a_4^2a_5\right),$$
$$\tau_3 = -\frac{1}{3346110}\left(9+a_4^2\right)\left(9+4a_4^2\right)\left(999+165a_4^2+16a_4^4\right). \tag{79}$$

In the above expression of τ_m, *we have let* $\tau_1 = \cdots \tau_{m-1} = 0$, $m = 2, 3$.

From Theorem 21, $\tau_3 = 0$ has not real number roots. Hence under condition (H_2), the origin of system (78) cannot become an isochronous center. Accordingly, the singular point $(-1/2, 0)$ of system (2) cannot become an isochronous center under condition (H_2).

4.3. The Isochronicity of Case (H_3). If condition (H_3) holds, then system $(34)|_{\delta=0}$ becomes

$$\frac{du}{d\tau} = -v + 2a_1uv + a_1u^2v + a_3v^3,$$
$$\frac{dv}{d\tau} = u + \frac{3}{2}u^2 + 2a_4uv + a_5v^2 + \frac{1}{2}u^3 + a_4u^2v + a_5uv^2 + a_6v^3, \tag{80}$$

in which the coefficients satisfy condition (H_3).

For convenience, we let $a_4 = b_4r$; then condition (H_3) becomes

$$a_1 = \frac{1}{2}\left(b_4^2 + 2b_4^2r - 2\right),$$
$$a_3 = -\frac{1}{6}b_4\left(3b_4^3 - 12b_4 - 14b_4r + 8b_4^3r - 4b_4r^2 + 4b_4^3r^2\right),$$
$$a_5 = \frac{1}{2}\left(2 - b_4^2 + 2r - 2b_4^2r\right),$$
$$a_6 = \frac{1}{3}\left(4b_4r - b_4^3r + 2b_4r^2 - 2b_4^3r^2\right). \tag{81}$$

System (80) complex concomitant system belongs to the form of (38). Hence we can use formulas (28) and (29) of Theorem A and $\tau_k = p_k + q_k = p_k' + q_k'$ to compute periodic constants of system (80).

Theorem 22. *If condition* (81) *holds (i.e.,* (H_3) *holds), then the first three period constants of the origin of system* (80) *are as follows:*

$$\tau_1 = \frac{1}{6}l_1, \qquad \tau_2 = \frac{1}{96}l_2, \qquad \tau_3 = \frac{1}{414720}l_3, \tag{82}$$

in which l_i, $i = 1, 2, 3, 4$ *are the polynomials of* b_4, r. *These expressions of* l_i *are as follows:*

$$l_1 = -24 + 3b_4^2 - 15r + 18b_4^2r - 3b_4^4r - 2r^2 + 4b_4^2r^2 - 6b_4^4r^2;$$
$$\begin{aligned} l_2 = {} & 336 - 234b_4^2 + 63b_4^4 - 6b_4^6 + 1650r - 1284b_4^2r \\ & + 486b_4^4r - 90b_4^6r + 6b_4^8r + 1565r^2 - 2370b_4^2r^2 \\ & + 1124b_4^4r^2 - 234b_4^6r^2 + 19b_4^8r^2 + 516r^3 \\ & - 1224b_4^2r^3 + 1184b_4^4r^3 - 224b_4^6r^3 + 4b_4^8r^3 \\ & + 52r^4 - 168b_4^2r^4 + 320b_4^4r^4 - 136b_4^6r^4 - 20b_4^8r^4; \end{aligned}$$
$$\begin{aligned} l_3 = {} & -12414600 + 10901385b_4^2 - 4216185b_4^4 \\ & + 963495b_4^6 - 131355b_4^8 + 8100b_4^{10} \\ & - 43826805r + 68754285b_4^2r - 37919070b_4^4r \\ & + 11032200b_4^6r - 1939005b_4^8r + 195615b_4^{10}r \\ & - 8100b_4^{12}r - 80932512r^2 + 169994130b_4^2r^2 \\ & - 121184220b_4^4r^2 + 41520240b_4^6r^2 - 7909800b_4^8r^2 \\ & + 903798b_4^{10}r^2 - 53460b_4^{12}r^2 - 78594084r^3 \\ & + 200907300b_4^2r^3 - 192468840b_4^4r^3 + 78954480b_4^6r^3 \\ & - 15231180b_4^8r^3 + 1650156b_4^{10}r^3 - 138600b_4^{12}r^3 \\ & - 38815064r^4 + 119144360b_4^2r^4 - 150195520b_4^4r^4 \\ & + 82363760b_4^6r^4 - 17141720b_4^8r^4 + 1412136b_4^{10}r^4 \\ & - 196560b_4^{12}r^4 - 9033312r^5 + 33223520b_4^2r^5 \\ & - 55560320b_4^4r^5 + 40281280b_4^6r^5 - 11183680b_4^8r^5 \\ & + 674208b_4^{10}r^5 - 185760b_4^{12}r^5 - 754208r^6 \\ & + 3300480b_4^2r^6 - 7720960b_4^4r^6 + 7271040b_4^6r^6 \\ & - 2734560b_4^8r^6 + 181632b_4^{10}r^6 - 97920b_4^{12}r^6. \end{aligned} \tag{83}$$

Through analyzing the expressions of τ_i, $i = 1, 2, 3$ of Theorem 22, we have the following theorem.

Theorem 23. *Note that* $\tau_1 = \tau_2 = \tau_3 = 0$ *of Theorem 22 have not solutions.*

Proof. By computing carefully in personal computer, we obtain

$$\begin{aligned} R_1 &= \text{Resultant}\left[l_1, l_2, r\right] \\ &= 9216\left(1 + b_4^2\right)^2\left(4 + 5b_4^2 + 8b_4^4 + b_4^6\right) \\ &\quad \times \left(1 - 15b_4^2 - 72b_4^4 + 731b_4^6 + 581b_4^8 - 218b_4^{10} - 2b_4^{12} + 4b_4^{14}\right), \end{aligned} \tag{84}$$

$$\begin{aligned}R_2 = \text{Resultant}\left[l_1, l_3, r\right] = -2654208\left(1 + b_4^2\right)^2 \\ \times \left(-349081952 + 4306069852 b_4^2\right. \\ + 41094839709 b_4^4 - 169659957471 b_4^6 \\ - 996450752748 b_4^8 - 1647467327229 b_4^{10} \\ - 1097111490891 b_4^{12} + 618270753348 b_4^{14} \\ + 1604753703528 b_4^{16} + 406555036384 b_4^{18} \\ - 507767377067 b_4^{20} - 144869567793 b_4^{22} \\ + 23948931114 b_4^{24} + 2359097649 b_4^{26} \\ \left.-735877143 b_4^{28} - 4379940 b_4^{30} + 7310250 b_4^{32}\right),\end{aligned} \tag{85}$$

in which Resultant$[l_i, l_m, r]$ is the resultant of l_i and l_m with respect to r.

Clearly, $R_1 = R_2 = 0$ hold only if $1 + b_4^2 = 0$; namely, $b_4 = \pm i$. If $b_4 = \pm i$, letting $\tau_1 = 0$, then $r = -3/2$. At this time, $a_4 = b_4 r = \pm(3/2)i$, which contradict $a_4 \in \mathbf{R}$. Hence the conclusion of Theorem 23 holds. Proof ends. □

From Theorem 23, equation groups $\tau_1 = \tau_2 = \tau_3 = 0$ have not solutions. Hence under condition (H_3), the origin of system (80) cannot become an isochronous center. Accordingly, singular point $(-1/2, 0)$ of system (2) cannot become an isochronous center under condition (H_3).

Remark 24. Through the above analysis, it is clear that the singular point $(-1/2, 0)$ of system (2) becomes an isochronous center if and only if (H_1) and one of the conditions (C_1) and (C_2) hold.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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