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Cr-Submanifolds of a Nearly Hyperbolic Kenmotsu Manifold Admitting a Quater Symmetric Metric Connection

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ABSTRACT
We consider a nearly hyperbolic Kenmotsu manifold with a quarter symmetric metric connection and study CR-submanifolds of a nearly hyperbolic Kenmotsu manifold with quarter symmetric metric connection. We also study parallel distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric metric connection and find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric metric connection.
***Keywords*:** CR-submanifolds, nearly hyperbolic Kenmotsu manifold, quarter symmetric metric connection, integrability conditions, parallel distribution.

I. Introduction

The study of CR-submanifolds of a Kaehler manifold was initiated by A. Bejancu in ([1], [2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [4] and M. Kobayashi in [14]. CR-submanifolds of Kenmotsu manifold was studied by A. Bejancu and N. Papaghuic in [3]. Later, several geometers (see, [6], [9] [10], [11] [12] [16] [17]) enriched the study of CR- submanifolds of almost contact manifolds.The almost hyperbolic (f, g, η, ξ) -structure was defined and studied by Upadhyay and Dube in [13]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [7]. On the other hand, S. Golab introduced the idea of semi-symmetric and quarter symmetric connections in [5]. CR-submanifolds of LP-Sasakian manifold with quarter symmetric metric connection were studied by the first author and S.K. Lovejoy Das in [8]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by the first author, M.D. Siddiqi and S. Rizvi in [15]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Kenmotsu manifold with a quarter symmetric metric connection.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic Kenmotsu manifold with a quarter symmetric metric connection. In section 3, some properties of CR- Submanifold of nearly hyperbolic Kenmotsu manifold with a quarter symmetric metric connection are investigated. In section 4,We study some result on parallel distribution on ξ-horizontal and ξ-vertical CR- Submanifold of nearly hyperbolic Kenmotsu manifold with a quarter symmetric metric connections.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure$(\emptyset, \xi, \eta . g)$, where a tensor $\emptyset$ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following

(2.1)$\emptyset^2 X = X + \eta(X)\boldsymbol{\xi}, \qquad g(X, \xi) = \eta(X),$

(2.2)$\eta(\xi) = -1, \qquad \emptyset(\xi) = 0, \qquad \eta o \emptyset = 0,$

(2.3)$g(\emptyset X, \emptyset Y) = -g(X, Y) - \eta(X)\eta(Y)$

For any X, Y tangent to $\bar{M}$ [17]. In this case

(2.4)$g(\emptyset X, Y) = -g(\emptyset Y, X)$.

An almost hyperbolic contact metric structure $(\emptyset, \xi, \eta, g)$ on $\bar{M}$ is called hyperbolic Kenmotsu manifold [7] if and only if

(2.5)$(\nabla_X \emptyset)Y = g(\emptyset X, Y)\xi - \eta(Y)\emptyset X$

for all X, Y tangent to $\bar{M}$.

Further, an almost hyperbolic contact metric manifold $\bar{M}$ on $(\emptyset, \xi, \eta, g)$ is called nearly-hyperbolic Kenmotsu [7] if

$$(2.6)(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X.$$

Now, Let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let $TM \text{ and } T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M

respectively and ∇^* be the induced Levi-Civita connection on N, then the Gauss and Weingarten formulas are given respectively by
(2.7)$\nabla_X Y = \nabla_X^* Y + h(X,Y)$,
(2.8)$\nabla_X N = -A_N X + \nabla_X^\perp N$
for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is a connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N as
(2.9)$g(A_N X, Y) = g(h(X,Y), N)$
for any $x \in M$ and $X \in T_x M$.We write
(2.10)$X = PX + QX$,
where$PX \in D$ and $QX \in D^\perp$.
Similarly, for N normal to M, we have
(2.11)$\emptyset N = BN + CN$,
where, BN ($resp.CN$) is the tangential component($resp.normal\ component$) of $\emptyset N$.
Now, we define a quarter symmetric metric connection by
(2.12) $\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi$
Such that,
$(\overline{\nabla}_X g)(Y,Z) = -\eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y)$
Using (2.13) and (2.6), We have
(2.13) $(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = -\eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2g(X,Y)\xi$

An almost hyperbolic contact manifold $\overline{M}$ satisfying (2.13) is called nearly hyperbolic Kenmotsu manifold with quarter symmetric metric connection.
For a nearly hyperbolic Kenmotsu manifold with quarter symmetric metric connection, we have
(2.14)$\overline{\nabla}_X \boldsymbol{\xi} = -\boldsymbol{\alpha}(\emptyset X) + \beta(X - \eta(X)\xi) + \emptyset X$.
Gauss and Weingarten formula for quarter symmetric non-metric connection are given respectively by
(2.15)$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$,
(2.16) $\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N - g(\phi X, N)\xi$.

Definition 1. An m-dimensional submanifold M of$\overline{M}$ is called a CR-Submanifold of almost nearly hyperbolic contact manifold $\overline{M}$, if there exist a differentiable distribution $D: x \longrightarrow D_x$ on M satisfying the following conditions:

i. D is invariant, that is $\emptyset D_x \subset D_x$ $for\ each\ x \in M$.
ii. The complementary orthogonal distribution $D^\perp$of D is anti-invariant, that is $\emptyset D_x^\perp \subset T_x^\perp M$. If dim $D_x^\perp = 0$ ($resp.$, dim $D_x = 0$), then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D ($resp., D^\perp$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^\perp)$ is called $\xi - horizontal\ (resp., vertical)\ if\ \xi_X \in D_X (resp., \xi_X \in$ $D X\perp$.

III. Some Basic Lemmas

Lemma -3.1 If M be a CR-submanifold of nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection. Then

(3.1) $-\eta(X)(PY) - \eta(Y)(PX) - \eta(X)(\phi PY) - \eta(Y)(\phi PX) + 2g(X,Y)(P\xi) + (\phi P)\nabla_X Y + (\phi P)\nabla_Y X = (P)\nabla_X \phi PY + (P)\nabla_Y \phi PX - (P)A_{\phi QY} X - (P)A_{\phi QX} Y - g(\phi X, \phi QY)(P\xi) - g(\phi Y, \phi QX)(P\xi)$

(3.2) $-\eta(X)(QY) - \eta(Y)(QX) + 2g(X,Y)(Q\xi) + 2Bh(Y,X) = (Q)\nabla_X \phi PY + (Q)\nabla_Y \phi PX - (Q)A_{\phi QY} X - (Q)A_{\phi QX} Y - g(\phi X, \phi QY)(Q\xi) - g(\phi Y, \phi QX)(Q\xi)$

(3.3)$- \eta(X)(\phi QY) - \eta(Y)(\phi QX) + (\phi Q)\nabla_X Y + (\phi Q)\nabla_Y X + 2Ch(Y,X) = h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX$

Proof: From (2.10) , we have

$$\phi Y = \phi PY + \phi QY$$

Differentiating covariantly and using (2.15) and (2.16), we get

$$(\overline{\nabla}_X \phi)Y + \phi \nabla_X Y + \phi h(X,Y) = \nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY} X + \nabla_X^\perp \phi QY - g(\phi X, \phi QY)\xi$$

Interchanging X and Y

$$(\overline{\nabla}_Y \phi)X + \phi \nabla_Y X + \phi h(Y,X) = \nabla_Y \phi PX + h(Y, \phi PX) - A_{\phi QX} Y + \nabla_Y^\perp \phi QX - g(\phi Y, \phi QX)\xi$$

Adding above two equations, we obtain

$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X + \phi\nabla_X Y + \phi\nabla_Y X + 2\phi h(Y,X) = \nabla_X\phi PY + \nabla_Y\phi PX + h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY}X - A_{\phi QX}Y + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX - g(\phi X,\phi QY)\xi - g(\phi Y,\phi QX)\xi$$

Using eqution (2.13) in above equation, we get

(3.4) $- \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\, g(X,Y)\xi + \phi\nabla_X Y + \phi\nabla_Y X + 2\phi h(Y,X) = \nabla_X\phi PY + \nabla_Y\phi PX + h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY}X - A_{\phi QX}Y + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi - g(\phi X,\phi QY)\xi - g(\phi Y,\phi QX)\xi$

Comparing tangential, Vertical and Normal components from both sides of (3.4), we get the desired results Hence the Lemma is Proved.

Lemma -3.2 If M be a CR-submanifold of nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection. Then

(3.5) $2(\overline{\nabla}_X\emptyset)Y = - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\, g(X,Y)\xi + \nabla_X\emptyset Y + h(X,\emptyset Y) - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$

(3.6) $2(\overline{\nabla}_Y\emptyset)X = - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\, g(X,Y)\xi - \nabla_X\emptyset Y - h(X,\emptyset Y) + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y]$

For any X ,Y $\in D$.

Proof : From Gauss Formula (2.15), We get

(3.7) $\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = \nabla_X\emptyset Y + h(X,\emptyset Y) - \nabla_Y\emptyset X - h(Y,\emptyset X)$

Also we have

(3.8) $\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = (\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X + \emptyset[X,Y]$

From (3.7) and (3.8)

(3.9) $(\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X = \nabla_X\emptyset Y + h(X,\emptyset Y) - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$

Adding (2.13) and (3.9)

$$2(\overline{\nabla}_X\emptyset)Y = - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\, g(X,Y)\xi + \nabla_X\emptyset Y + h(X,\emptyset Y) - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$$

Subtracting (3.9) from (2.13) ,

$$2(\overline{\nabla}_Y\emptyset)X = - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\, g(X,Y)\xi - \nabla_X\emptyset Y - h(X,\emptyset Y) + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y]$$

Hence Lemma is proved.

Corollary 3.3. If M be a ξ −vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection. Then

$2(\overline{\nabla}_X\emptyset)Y = \nabla_X\emptyset Y - \nabla_Y\emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) + 2\, g(X,Y)\xi - \emptyset[X,Y]$

and

$2(\overline{\nabla}_Y\emptyset)X = \nabla_Y\emptyset X - \nabla_X\emptyset Y + h(Y,\emptyset X) - h(X,\emptyset Y) + 2\, g(X,Y)\xi + \emptyset[X,Y]$

For any X,Y ∈ D.

Lemma 3.4 If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection . Then

(3.10) $2(\overline{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \emptyset[Y,Z] - \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z - \eta(Z)\phi Y + 2\, g(Y,Z)\xi$

(3.11) $2(\overline{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \emptyset[Y,Z] - \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z - \eta(Z)\phi Y + 2\, g(Y,Z)\xi$

For any Y,Z $\in D^{\perp}$ **.**

Proof : Let, Y,Z $\in D^{\perp}$. From Weingarten formula (2.16), we get

(3.12) $\overline{\nabla}_Y\emptyset Z - \overline{\nabla}_Z\emptyset Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y$

Also we have, from equation (3.8)

(3.13) $\overline{\nabla}_Y\emptyset Z - \overline{\nabla}_Z\emptyset Y = (\overline{\nabla}_Y\emptyset)Z - (\overline{\nabla}_Z\emptyset)Y + \emptyset[Y,Z]$

From (3.12) and (3.13)

(3.14) $(\overline{\nabla}_Y\emptyset)Z - (\overline{\nabla}_Z\emptyset)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \emptyset[Y,Z]$

For nearly Hyperbolic Kenmotsu manifold, we have

(3.15) $(\overline{\nabla}_Y\phi)Z + (\overline{\nabla}_Z\phi)Y = - \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z - \eta(Z)\phi Y + 2\, g(Y,Z)\xi$

Adding (3.14) and (3.15),

$2(\overline{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \emptyset[Y,Z] - \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z - \eta(Z)\phi Y + 2\,g(Y,Z)\xi$

Subtracting (3.14) from (3.15),

$2(\overline{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \emptyset[Y,Z] - \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z - \eta(Z)\phi Y + 2\,g(Y,Z)\xi$

For Y,Z ∈ $D^{\perp}$

Hence Lemma is proved.

Corollary 3.5 If M be a $\xi -$ horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection . Then

$$\boldsymbol{2(\overline{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \emptyset[Y,Z] + 2\,g(Y,Z)\xi}$$ **,and**

$$\boldsymbol{2(\overline{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \emptyset[Y,Z] + 2\,g(Y,Z)\xi}$$

For any Y,Z ∈ $D^{\perp}$.

Lemma 3.6 If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection . Then

(3.16) $$\boldsymbol{2(\overline{\nabla}_X\emptyset)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\,g(X,Y)\xi}$$

And

(3.17) $$\boldsymbol{2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + g(\phi X,\phi Y)\xi + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y] - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\,g(X,Y)\xi}$$

For any X ∈ D , Y ∈ $D^{\perp}$

Proof : Let X ∈ D ,Y ∈ $D^{\perp}$

From Gauss and Weingarten formulae, we have

(3.18) $\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X)$

Also we have

(3.19) $\overline{\nabla}_X\emptyset Y - \overline{\nabla}_Y\emptyset X = (\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X + \emptyset[X,Y]$

From (3.18) and (3.19)

(3.20) $(\overline{\nabla}_X\emptyset)Y - (\overline{\nabla}_Y\emptyset)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$

Also For nearly Hyperbolic Kenmotsu manifold, we have

(3.21) $(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = -\eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\,g(X,Y)\xi$

Adding (3.20) and (3.21), We obtain

$$2(\overline{\nabla}_X\emptyset)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\,g(X,Y)\xi$$

Subtracting (3.20) from (3.21),

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + g(\phi X,\phi Y)\xi + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y] - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2\,g(X,Y)\xi$$

Corollary 3.7 If M be a$\xi -$ horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection . Then

$$\boldsymbol{2(\overline{\nabla}_X\emptyset)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(X)Y - \eta(X)\phi Y + 2\,g(X,Y)\xi}$$

And

$$\boldsymbol{2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + g(\phi X,\phi Y)\xi + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y] - \eta(X)Y - \eta(X)\phi Y + 2\,g(X,Y)\xi}$$

For any X ∈ D ,Y ∈ $D^{\perp}$

Corollary 3.8. If M be a $\xi -$vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric metric connection. Then

$$\boldsymbol{2(\overline{\nabla}_X\emptyset)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - g(\phi X,\phi Y)\xi - \nabla_Y\emptyset X - h(Y,\emptyset X) - \emptyset[X,Y] - \eta(Y)X - \eta(Y)\phi + 2\,g(X,Y)\xi}$$

And

$$\boldsymbol{2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + g(\phi X,\phi Y)\xi + \nabla_Y\emptyset X + h(Y,\emptyset X) + \emptyset[X,Y] - \eta(Y)X - \eta(Y)\phi X + 2\,g(X,Y)\xi}$$

For any X ∈ D ,Y ∈ $D^{\perp}$

IV. PARALLEL DISTRIBUTION

Definition 2. The horizontal (resp., vertical) distribution D (resp. , $D^{\perp}$) is said to be Parallel [3] with respect to the connection on M if $\nabla_X Y \in D$ (resp. , if $\nabla_Z W \in D^{\perp}$) for any vector field X,Y $\in D$ (resp. , $W, Z \in D^{\perp}$)

Theorem 4.1 Let M be a $\xi - vertical$ CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric metric connection. Then
(4.1) $\boldsymbol{h(X, \emptyset Y) = h(Y, \emptyset X)}$
$\boldsymbol{for\ any\ X, Y \in D.}$
Proof : Using Parallelism of horizontal distribution D, We have
(4.2) $\nabla_X(\emptyset Y) \in D \quad and\ \nabla_Y \emptyset X \in D \quad for\ any\ X, Y \in D.$
From (3.2), we have
(4.3) $Bh(X,Y) = -\text{g(X,Y)}\xi$
$for\ any\ X, Y \in D$
Also from (2.11) we have
(4.4) $\phi h(X,Y) = Bh(X,Y) + Ch(X,Y)$
Using (4.3) in (4.4)
(4.5) $2Ch(Y,X) = h(X, \phi Y) + h(Y, \phi X)$
Next,from (3.3) we have
From (4.4) ,Applying (4.5)
$2\phi h(X,Y) = -2g(X,Y)\xi + h(X, \phi Y) + h(Y, \phi X)$
(4.6) $h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X,Y) + 2g(X,Y)\xi$
Replacing X to ϕX
(4.7) $h(\phi X, \phi Y) + h(Y,X) = 2\phi h(\phi X, Y) + 2g(\phi X, Y)\xi$
Now replacing Y to ϕY in (4.6)
(4.8) $h(\phi X, \phi Y) + h(Y,X) = 2\phi h(X, \phi Y) + 2g(X, \phi Y)\xi$
From (4.7) and (4.8)
$h(\phi X, Y) = h(X, \phi Y)$
$for\ any\ X, Y \in D.$
Hence theorem is proved.

Theorem 4.2. Let M be a $\xi - vertical$ submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then
(4.9) $\boldsymbol{A_{\emptyset Y} X + A_{\emptyset X} Y \in D^{\perp}}$
$\boldsymbol{for\ any\ X, Y \in D^{\perp}.}$

Proof : Let, $X, Y \in D^{\perp}$.then using Weingarten Formula .We have
$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N - g(\phi X, N)\xi$
Putting N= ϕY
$\bar{\nabla}_X \phi Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - g(\phi X, \phi Y)\xi$
$(\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y) = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - g(\phi X, \phi Y)\xi$
Using Gauss formula
$(\bar{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - g(\phi X, \phi Y)\xi - \phi(\nabla_X Y + h(X,Y))$
(4.11) $(\bar{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \phi \nabla_X Y - \phi h(X,Y) - g(\phi X, \phi Y)\xi$
Interchanging X and Y
(4.12) $(\bar{\nabla}_Y \phi)X = -A_{\phi X} Y + \nabla_Y^{\perp} \phi X - \phi \nabla_Y X - \phi h(Y,X) - g(\phi Y, \phi X)\xi$
Adding (4.11) and (4.12) ,we get
$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -A_{\phi Y} X - A_{\phi X} Y + \nabla_X^{\perp} \phi Y + \nabla_Y^{\perp} \phi X - \phi \nabla_X Y - \phi \nabla_Y X - 2\phi h(X,Y) - 2g(\phi X, \phi Y)\xi$
(4.13) $(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -A_{\phi Y} X - A_{\phi X} Y + \nabla_X^{\perp} \phi Y + \nabla_Y^{\perp} \phi X - \phi \nabla_X Y - \phi \nabla_Y X - 2\phi h(X,Y)$
$-2g(X,Y)\xi + 2\eta(X)\eta(Y)\xi$
From (2.13) and (4.13)
$$- \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X + 2g(X,Y)\xi = -A_{\phi Y} X - A_{\phi X} Y + \nabla_X^{\perp} \phi Y + \nabla_Y^{\perp} \phi X$$
$$-\phi \nabla_X Y - \phi \nabla_Y X - 2\phi h(X,Y) - 2g(X,Y)\xi + 2\eta(X)\eta(Y)\xi$$
$$- \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X = -A_{\phi Y} X - A_{\phi X} Y + \nabla_X^{\perp} \phi Y + \nabla_Y^{\perp} \phi X - \phi \nabla_X Y - \phi \nabla_Y X$$
$$-2\phi h(X,Y) + 2\eta(X)\eta(Y)\xi$$
Operating g bothside w.r.to Z $\in D$

$- \eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)$
$= -g(A_{\phi Y}X,Z) - g(A_{\phi X}Y,Z) + g(\nabla_X^\perp \phi Y,Z) + g(\nabla_Y^\perp \phi X,Z) - g(\phi\nabla_X Y,Z) - g(\phi\nabla_Y X,Z)$
$- 2\phi g(h(X,Y),Z) + 2\eta(X)\eta(Y)g(\xi,Z)$
$g(A_{\phi Y}X) + g(A_{\phi X}Y) = 0$
$g(A_{\phi Y}X + A_{\phi X}Y) = 0$
This implies that
$(A_{\phi Y}X + A_{\phi X}Y) \in \boldsymbol{D}^\perp$
for any X, Y $\in \boldsymbol{D}^\perp$.

Hence theorem is proved.
Definition3.A CR-submanifold is said to be mixed-totally geodesic if$h(X,Z) = 0$
$for\ all\ X \in D\ and\ Z \in D^\perp.$
Definition 4.A Normal vector field $N \neq 0$ is called$D - parallel$ normal section if
$\nabla_X^\perp N = 0$for all $X \in D$.

Theorem 4.3. Let M be a mixed totally geodesic $\xi - vertical$ CR-submanifold of a nearly hyperbolic Kenmotsu manifold$\overline{M}$ with quarter symmetric metric connection. Then the normal section $N \in \emptyset D^\perp$ is $D - paralle$l if and only if $\nabla_X \emptyset N \in D\ for\ all\ X \in D$.

Proof. Let $N \in \emptyset D^\perp$, then from (3.2) we have
$-\eta(X)(QY) - \eta(Y)(QX) + 2\,g(X,Y)(Q\xi) + 2Bh(Y,X)$
$= (Q)\nabla_X \phi PY + (Q)\nabla_Y \phi PX - (Q)A_{\phi QY}X - (Q)A_{\phi QX}Y - g(\phi X, \phi QY)(Q\xi)$
$- g(\phi Y, \phi QX)(Q\xi)$
(4.15) $2\,g(X,Y)(\xi) + 2Bh(X,Y) = Q\nabla_Y(\emptyset X) - QA_{\emptyset Y}X.$
Using definition of mixed geodesic CR-submanifold,
$h(X,Y) = 0,\ if\ X \in D\ and\ Z \in D^\perp$
(4.16) $2\,g(X,Y)(\xi) = Q\nabla_Y(\emptyset X) - QA_{\emptyset Y}X.$
As $A_{\emptyset Y}X \in D$, for $X \in D$.
(4.17) $2\,g(X,Y)\xi = Q\nabla_Y(\emptyset X)$
Operating ϕ both side
$2\,g(X,Y)\phi\xi = \phi Q\nabla_Y(\emptyset X)$
$\phi Q\nabla_Y(\emptyset X) = 0$
In particular,
(4.18) $Q\nabla_Y(\emptyset X) = 0$
From (3.3)
$- \eta(X)(\phi QY)\_\,\eta(Y)(\phi QX) + (\phi Q)\nabla_X Y + (\phi Q)\nabla_Y X + 2Ch(Y,X)$
$= h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX$
$(\phi Q)\nabla_X Y = \nabla_X^\perp \phi Y$

Putting , Y=ϕN
$(\phi Q)\nabla_X \phi N = \nabla_X^\perp \phi^2 N$
$(\phi Q)\nabla_X \phi N = \nabla_X^\perp (N + \eta(N)\xi)$
Then by Definition of Parallelism of N, We have
$(\phi Q)\nabla_X \phi N = 0$
Consequently , We get
$\nabla_X(\phi N) \in D,\quad for\ all\ X \in D.$
This completes the Proof.

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