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HOMOTOPY ANALYSIS METHOD FOR MULTI-DEGREE-OF-FREEDOM NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT

In reality, the behavior and nature of nonlinear dynamical systems are ubiquitous in many practical engineering problems. The mathematical models of such problems are often governed by a set of coupled second-order differential equations to form multi-degree-of-freedom (MDOF) nonlinear dynamical systems. It is extremely difficult to find the exact and analytical solutions in general. In this paper, the homotopy analysis method is presented to derive the analytical approximation solutions for MDOF dynamical systems. Four illustrative examples are used to show the validity and accuracy of the homotopy analysis and modified homotopy analysis methods in solving MDOF dynamical systems. Comparisons are conducted between the analytical approximation and exact solutions. The results demonstrate that the HAM is an effective and robust technique for linear and nonlinear MDOF dynamical systems. The proof of convergence theorems for the present method is elucidated as well.

1 INTRODUCTION

Multi-degree-of-freedom (MDOF) nonlinear dynamical systems are omnipresent in numerous practical engineering problems. In normal circumstances, many MDOF nonlinear problems are not readily resorted to analytical approaches. However, the development of analytical methods can provide an all-embracing understanding for nonlinear dynamical systems. Hence, the homotopy analysis method (HAM), proposed by Liao [1] for solving a class of nonlinear problems, was thus emerged as a robust analytical technique at the beginning of the 1990s. The enlightened idea of the HAM was originated from the homotopy in topology. Inasmuch as the HAM is valid for analyzing the nonlinear problems having small as well as large parameters, it overcomes the foregoing restrictions of conventional asymptotic methods.

For more than one decade, a number of scholars have adopted the HAM to a variety of nonlinear problems in engineering and physical sciences, including Liao and his associates [2-4] furnished the analytical formulas for various nonlinear dynamical systems, Xu [5] derived the explicit solutions for the free convection about a vertical flat plate embedded in a porous medium, Allan and Syam [6] solved the nonhomogeneous Blasius problem, Abbasbandy [7] generalized the HAM to the problem of nonlinear heat transfer equations, Hayat et al. [8-10] deduced the solutions of grade fluid problems, Song and Zhang [11] unraveled the problem of fractional KdV-Burgers-Kuramoto differential equations, and Inc [12] considered the Laplace equation having Dirichlet and Neumann boundary conditions by using the HAM.

Besides, a coupling of the homotopy technique and the perturbation method, the homotopy perturbation method (HPM) [13, 14] was proposed to investigate several kinds of nonlinear problems. Using such a hybrid method, Cveticanin [15] presented the HPM solutions for complex-valued differential equations with strong cubic nonlinearity, Chowdhury et al. [16] derived the series solutions for the theoretical modelling of nonlinear population dynamics, Odibat and Momani [17] modified the HPM to the quadratic Riccati differential equation of fractional order, Ramos [18]

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applied the HPM to the Lane-Emden equation, which plays a significant role in astrophysics, and Yildirim [19] probed the problem of Boussinesq-like equations in nonlinear dispersive waves via the HPM.

In this paper, the application of the HAM is exploited to MDOF linear and nonlinear dynamical systems. The significance of MDOF dynamical systems is mainly due to its global bifurcation, regular and chaotic motions, the intensive research subjects are thus at the forefront of nonlinear dynamics. Recently, some achievements and fruitful outcome have been established for MDOF dynamical systems. For instance, Zhang and his associates [20, 21] derived the equations of parametrically excited structural elements (i.e. plates and beams) into two-degree-of-freedom nonlinear systems by means of von Kármán type equations with Galerkin's approach to conduct the qualitative analysis. Yagasaki [22] provided numerical evidence of the fast diffusion in the three-degree-of-freedom Hamiltonian system. Wagg and Bishop [23] resolved the dynamical problem of vibro-impact oscillators having multiple motion limiting constraints. Chen et al. [24] made use of the multidimensional Lindstedt-Poincaré method to investigate the nonlinear vibration of axially moving beams. Peng et al. [25] adopted the functions of nonlinear output frequency-response to the linear parameter estimation for MDOF nonlinear systems. Jang and Choi [26] employed the geometrical design method for MDOF vibration absorbers. More recently, Mei et al. [27] utilized the asymptotic numerical method to the treatment of MDOF nonlinear dynamic systems.

The objective of the present work is to conduct the quantitative analysis for MDOF dynamical systems, four illustrative examples are selected to substantiate the validity and accuracy of the homotopy analysis and modified homotopy analysis methods. Comparisons are carried out between the results of the present method and the exact solution. The results demonstrate that the HAM is an effective and robust technique for linear and nonlinear MDOF dynamical systems. In addition, the convergence theorems are proved for MDOF dynamical systems as well.

2 SOLUTION METHODOLOGY FOR MDOF SYSTEMS

The MDOF dynamical system is governed by the following equation

$$M\ddot{q} + G\dot{q} + Kq = F(\dot{q}, q, t) \tag{1}$$

where q is an $n-$ dimensional unknown vector, a dot denotes the differentiation with respect to time t, M, G and K are, respectively, the system mass, damping and stiffness $n \times n$ matrixes, and F is the vector function of $\dot{q}$, q and t. If $F(\dot{q}, q, t) \equiv 0$, then Eq. (1) is the MDOF autonomous dynamical system.

According to Eq. (1), a nonlinear operator is defined as

$$N[u(r,t)] = M\frac{\partial^2 u(r,t)}{\partial t^2} + G\frac{\partial u(r,t)}{\partial t} + Ku(r,t) - F\left(\frac{\partial u}{\partial t}, u, t\right) \tag{2}$$

in which

$$u(r,t) = (x_1(t), \cdots, x_n(t))^{\mathrm{T}} \tag{3}$$

$$\frac{\partial u(r,t)}{\partial t} = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}, \cdots, \frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^{\mathrm{T}} \tag{4}$$

and

$$\frac{\partial^2 u(r,t)}{\partial t^2} = \left(\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2}, \cdots, \frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2}\right)^{\mathrm{T}} \tag{5}$$

where $u(r,t)$ is an unknown vector function, r and t are spatial and temporal variables, respectively.

Following the fundamental concepts and working procedures of the HAM [1-4], the zeroth-order deformation equation is constructed as,

$$(1-p)\{L[\Phi(r,t;p) - u_0(r,t)]\} = p\hbar H(t) N[\Phi(r,t;p)] \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, $u_0(r,t)$ is the solution of initial guess, L is an auxiliary linear operator, $\hbar$ and $H(t)$ are, respectively, the auxiliary parameter and the function diagonal matrix as follows

$$\hbar = \begin{pmatrix} \hbar_1 & & 0 \\ & \ddots & \\ 0 & & \hbar_n \end{pmatrix} \tag{7}$$

$$H(t) = \begin{pmatrix} \lambda_1(t) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(t) \end{pmatrix} \tag{8}$$

in which $\hbar_i \neq 0$ and $\lambda_i(t)$ $(i = 1, \cdots, n)$ are real functions. Generally, we assume that $\hbar_i = \hbar_j$ and $\lambda_i(t) = \lambda_j(t)$ $(i \neq j)$. Therefore, $\hbar$ and $H(t)$ can be viewed as the parameter and the function, respectively. For $p = 0$ and $p = 1$, it follows from the zeroth-order deformation equation (6) that $\Phi(r,t;0) = u_0(r,t)$ and $\Phi(r,t;1) = u(r,t)$, respectively. Hence, as p increases from 0 to 1, the solution $\Phi(r,t;p)$ varies from the initial guess $u_0(r,t)$ to the exact solution $u(r,t)$.

By setting

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \Phi(r,t;p)}{\partial p^m}\bigg|_{p=0} \tag{9}$$

and expanding $\Phi(r,t;p)$ into the Taylor series expansion with respect to the embedding parameter p in accordance with the theorem of vector-valued function, one gets

$$\Phi(r,t;p) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) p^m \tag{10}$$

Provided that the auxiliary linear operator, initial guess, auxiliary parameter matrix $\hbar$ and auxiliary function matrix $H(t)$ are properly chosen, the series expansion in Eq. (10) converges at $p = 1$ to have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t) \tag{11}$$

To define the vector as

$$\mathbf{u}_m = \{u_0(r,t), u_1(r,t), \cdots, u_m(r,t)\} \tag{12}$$

Differentiating the zeroth-order deformation equation (6) m times with respect to p, then dividing the equation by $m!$ and setting $p = 0$ yield

$$L\left[u_m(r,t)-\chi_m u_{m-1}(r,t)\right]=\hbar H(t)R_m(\mathbf{u}_{m-1},r,t) \quad (13)$$

where

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} \quad (14)$$

and

$$R_m(\mathbf{u}_{m-1},r,t)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N(\Phi(r,t;p))}{\partial p^{m-1}}\right|_{p=0} \quad (15)$$

From Eq. (10), Eq. (15) can be further expressed as

$$R_m(\mathbf{u}_{m-1},r,t)=\frac{1}{(m-1)!}\left.\left\{\frac{\partial^{m-1}}{\partial p^{m-1}}N\left[\sum_{n=0}^{+\infty}u_n(r,t)p^n\right]\right\}\right|_{p=0} \quad (16)$$

Equation (13) is a linear equation and one can solve it by using the available symbolic software such as Mathematica.

3 ILLUSTRATIVE EXAMPLES AND DISCUSSION

In this section, four illustrative examples are used to demonstrate the applicability and accuracy of the HAM for MDOF linear and nonlinear systems. The algorithm is coded by the symbolic computation software Mathematica 6.0.

3.1 EXAMPLE 1

First, we consider a linear MDOF system

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}+x_1+\frac{2}{t}x_2=0 \quad (17\text{-a})$$

$$\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2}-\frac{2}{t}x_1+x_2=0 \quad (17\text{-b})$$

with the initial conditions

$$x_1(0)=0,\ x_1'(0)=1,\ x_2(0)=0,\ x_2'(0)=0. \quad (18)$$

where a prime denotes the differentiation with respect to time t. The exact solutions of Eqs. (17) subjected to the initial conditions are

$$x_1(t)=t\cos t,\ x_2(t)=t\sin t \quad (19)$$

In order to solve Eqs. (17) using the HAM, we assume that the solutions can be expressed by a set of base functions $\{t^n \mid n=0,1,2,3,\cdots\}$. The initial approximation is chosen as

$$x_{1,0}(t)=t,\ x_{2,0}(t)=t^2 \quad (20)$$

and the linear operator is expressed as

$$L_1\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2\varphi_1(t;q)}{\partial t^2}\\ \dfrac{\partial^2\varphi_2(t;q)}{\partial t^2}\end{pmatrix} \quad (21)$$

with the property

$$L_1\begin{pmatrix}C_1t+C_2\\ C_3t+C_4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad (22)$$

where C_1, C_2, C_3 and C_4 are constants. Besides, the nonlinear operator is written as

$$N_1\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2\varphi_1(t;q)}{\partial t^2}+\varphi_1(t;q)+\dfrac{2}{t}\varphi_2(t;q)\\ \dfrac{\partial^2\varphi_2(t;q)}{\partial t^2}-\dfrac{2}{t}\varphi_1(t;q)+\varphi_2(t;q)\end{pmatrix} \quad (23)$$

According to the rule of solution expression, the auxiliary function is selected as $H(t)=1$. Therefore, the zeroth-order and mth-order deformation equations can be written respectively as follows

$$(1-q)L_1\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\end{pmatrix}=q\hbar N_1\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix} \quad (24)$$

$$L_1\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\end{pmatrix}=\hbar\begin{pmatrix}R_{1,m}(\mathbf{x}_{1,m-1})\\ R_{2,m}(\mathbf{x}_{2,m-1})\end{pmatrix} \quad (25)$$

By virtue of the initial conditions and the initial approximation, we have

$$x_{1,m}(0)=0,\ x_{2,m}(0)=0,\ x_{1,m}'(0)=0,\ x_{2,m}'(0)=0\ (m\ge 1) \quad (26)$$

and

$$R_{1,m}(\mathbf{x}_{1,m-1})=x_{1,m-1}''(t)+x_{1,m-1}(t)+\frac{2}{t}x_{2,m-1}(t) \quad (27\text{-a})$$

$$R_{2,m}(\mathbf{x}_{2,m-1})=x_{2,m-1}''(t)-\frac{2}{t}x_{1,m-1}(t)+x_{2,m-1}(t) \quad (27\text{-b})$$

Solving the mth-order deformation equation (25) yields,

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{1,m}(\mathbf{x}_{1,m-1})\mathrm{d}s\right)\mathrm{d}\tau \quad (28\text{-a})$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{2,m}(\mathbf{x}_{2,m-1})\mathrm{d}s\right)\mathrm{d}\tau \quad (28\text{-b})$$

in which

$$x_{1,1}(t)=\frac{\hbar}{2}t^3 \quad (29\text{-a})$$

$$x_{2,1}(t)=\frac{\hbar}{12}t^4 \quad (29\text{-b})$$

$$x_{1,2}(t)=\left(\frac{\hbar^2}{2}+\frac{\hbar}{2}\right)t^3+\frac{\hbar^2}{30}t^5 \quad (29\text{-c})$$

$$x_{2,2}(t)=\frac{\hbar}{12}t^4+\frac{\hbar^2}{360}t^6 \quad (29\text{-d})$$

$$x_{1,3}(t)=\left(\frac{\hbar^3}{2}+\hbar^2+\frac{\hbar}{2}\right)t^3+\left(\frac{7}{120}\hbar^3+\frac{1}{15}\hbar^2\right)t^5+\frac{1}{1080}\hbar^3t^7 \quad (29\text{-e})$$

$$x_{2,3}(t)=-\left(\frac{\hbar^3}{12}-\frac{\hbar}{12}\right)t^4+\left(\frac{\hbar^3}{1800}+\frac{\hbar^2}{180}\right)t^6+\frac{\hbar^3}{20160}t^8 \quad (29\text{-f})$$

$$x_{1,4}(t)=\left(\frac{\hbar^4}{2}+\frac{3\hbar^3}{2}+\frac{3\hbar^2}{2}+\frac{\hbar}{2}\right)t^3+\left(\frac{3\hbar^4}{40}+\frac{7}{40}\hbar^3+\frac{1}{10}\hbar^2\right)t^5+\left(\frac{59\hbar^4}{25200}+\frac{\hbar^3}{360}\right)t^7+\frac{31\hbar^4}{2177280}t^9 \quad (29\text{-g})$$

$$x_{2,4}(t)=-\left(\frac{\hbar^4}{6}+\frac{\hbar^3}{4}-\frac{\hbar}{12}\right)t^4-\left(\frac{11\hbar^4}{1800}-\frac{\hbar^3}{600}-\frac{\hbar^2}{120}\right)t^6+\left(\frac{\hbar^4}{37800}+\frac{\hbar^3}{6720}\right)t^8+\frac{\hbar^4}{1814400}t^{10} \quad (29\text{-h})$$

Thus, the mth-order analytical approximation can be expressed in terms of the summation series

$$x_1(t)=x_{1,0}(t)+x_{1,1}(t)+\cdots+x_{1,m}(t) \quad (30\text{-a})$$

$$x_2(t) = x_{2,0}(t) + x_{2,1}(t) + \cdots + x_{2,m}(t) \tag{30-b}$$

The explicit expressions given by the HAM contain the auxiliary parameter $\hbar$, which gives the convergence region and the rate of approximation for the HAM. Figure 1 depicts the $\hbar -$ curves for the 10th-order approximation, it is evident that the region of admissible values of $\hbar$ is at $-1.5 < \hbar < -0.5$. For $\hbar = -1$, the series solutions are given by

$$x_1(t) = t - \frac{t^3}{2} + \frac{t^5}{24} - \frac{t^7}{720} + \frac{t^9}{40320} - \frac{t^{11}}{3628800} + \frac{t^{13}}{479001600} - \cdots \tag{31-a}$$

$$x_2(t) = t^2 - \frac{t^4}{6} + \frac{t^6}{120} - \frac{t^8}{5040} + \frac{t^{10}}{362880} - \frac{t^{12}}{39916800} + \frac{t^{14}}{6227020800} - \cdots \tag{31-b}$$

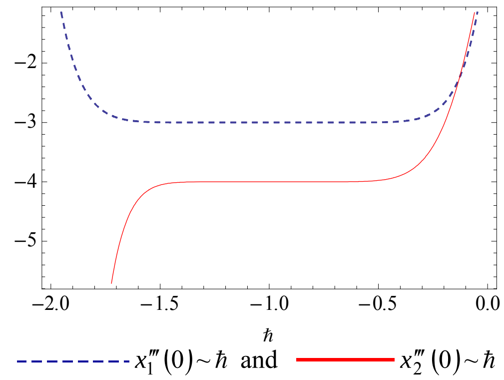


Fig. 1. The $\hbar -$ curves of $x_1'''(0)$ and $x_2^{(4)}(0)$ obtained from the 10th-order approximation for Eq. (17)

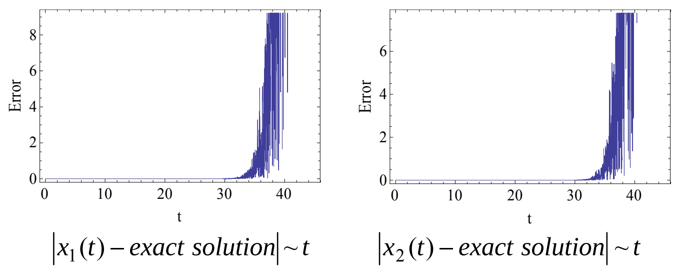


Fig. 2. Absolute errors between the 50th-order homotopy analysis and exact solutions for $\hbar = -1$

Figure 2 shows the absolute errors between the 50th-order homotopy analysis and exact solutions, which demonstrate that the 50th-order approximation provides excellent agreement with the exact solutions till $t = 37$. The convergence of approximation series can be controlled by adjusting the convergence regions when necessary.

3.2 EXAMPLE 2

In this example, the nonlinear MDOF system is governed by

$$\frac{d^2 x_1}{dt^2} + x_2 \frac{dx_1}{dt} + x_1 = \cos^2 t \tag{32-a}$$

$$\frac{d^2 x_2}{dt^2} - x_1 \frac{dx_2}{dt} + x_2 = \sin^2 t \tag{32-b}$$

with the initial conditions

$$x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 1,\ x_2'(0) = 0 \tag{33}$$

where a prime denote the differentiation with respect to time t. The exact solutions for Eqs. (32) subjected to the initial conditions are

$$x_1(t) = \sin t,\ x_2(t) = \cos t \tag{34}$$

Prior to employing the HAM method, we first expand the terms $\cos^2 t$ and $\sin^2 t$ into the Taylor series with respect to t as follows

$$\cos^2 t = \frac{1 + \cos 2t}{2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} t^{2n} = \sum_{n=0}^{\infty} a_n t^{2n} \tag{35-a}$$

$$\sin^2 t = \frac{1 - \cos 2t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1}}{(2n)!} t^{2n} = \sum_{n=0}^{\infty} b_n t^{2n} \tag{35-b}$$

where

$$a_0 = 1,\ a_n = \frac{(-1)^n 2^{2n-1}}{(2n)!},\ b_0 = 0,\ b_n = \frac{(-1)^{n-1} 2^{2n-1}}{(2n)!} \quad (n = 1, \cdots, \infty) \tag{36}$$

Suppose that the solution can be expressed by a set of base functions $\{t^n \mid n = 0,1,2,3,\cdots\}$, we choose the initial approximation as

$$x_{1,0}(t) = t,\ x_{2,0}(t) = 1 \tag{37}$$

By defining the nonlinear operator as

$$N_2\begin{pmatrix} \varphi_1(t;q) \\ \varphi_2(t;q) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 \varphi_1(t;q)}{\partial t^2} + \varphi_2(t;q)\dfrac{\partial \varphi_1(t;q)}{\partial t} + \varphi_1(t;q) \\ \dfrac{\partial^2 \varphi_2(t;q)}{\partial t^2} - \varphi_1(t;q)\dfrac{\partial \varphi_2(t;q)}{\partial t} + \varphi_2(t;q) \end{pmatrix} \tag{38}$$

Thus, the zeroth-order deformation equation can be written in the form

$$(1-q)L_1\begin{pmatrix} \varphi_1(t;q) - x_{1,0}(t) \\ \varphi_2(t;q) - x_{2,0}(t) \end{pmatrix} = q\hbar\left[N_2\begin{pmatrix} \varphi_1(t;q) \\ \varphi_2(t;q) \end{pmatrix} - \begin{pmatrix} \sum_{n=0}^{\infty} a_n t^{2n} q^n \\ \sum_{n=0}^{\infty} b_n t^{2n} q^n \end{pmatrix} \right] \tag{39}$$

and the mth-order deformation equation can be expressed as

$$L_1\begin{pmatrix} x_{1,m}(t) - \chi_m x_{1,m-1}(t) \\ x_{2,m}(t) - \chi_m x_{2,m-1}(t) \end{pmatrix} = \hbar\begin{pmatrix} R_{1,m}(\mathbf{x}_{1,m-1}) \\ R_{2,m}(\mathbf{x}_{2,m-1}) \end{pmatrix} \tag{40}$$

The linear operator L_1 is the same as given in Eq. (21). From the initial conditions and the initial approximation, we have

$$x_{1,m}(0) = 0,\ x_{2,m}(0) = 0,\ x_{1,m}'(0) = 0,\ x_{2,m}'(0) = 0 \quad (m \ge 1) \tag{41}$$

and

$$R_{1,m}(\mathbf{x}_{1,m-1}) = x_{1,m-1}''(t) + x_{1,m-1}(t) + \sum_{i+j=m-1} x_{1,i}'(t) x_{2,j}(t) - a_{m-1} t^{2m-2} \tag{42-a}$$

$$R_{2,m}(\mathbf{x}_{2,m-1}) = x_{2,m-1}''(t) + x_{2,m-1}(t)$$

$$-\sum_{i+j=m-1} x'_{2,i}(t)x_{1,j}(t)-b_{m-1}t^{2m-2} \quad (42\text{-b})$$

Solving the mth-order deformation equation (40) yields,

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{1,m}(\mathbf{x}_{1,m-1})\mathrm{d}s\right)\mathrm{d}\tau \quad (43\text{-a})$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{2,m}(\mathbf{x}_{2,m-1})\mathrm{d}s\right)\mathrm{d}\tau \quad (43\text{-b})$$

in which

$$x_{1,1}(t)=\frac{\hbar}{6}t^3 \quad (44\text{-a})$$

$$x_{2,1}(t)=\frac{\hbar}{2}t^2 \quad (44\text{-b})$$

$$x_{1,2}(t)=\left(\frac{\hbar}{6}+\frac{\hbar^2}{6}\right)t^3+\frac{(\hbar+1)\hbar}{12}t^4+\frac{\hbar^2}{120}t^5 \quad (44\text{-c})$$

$$x_{2,2}(t)=\left(\frac{\hbar}{2}+\frac{\hbar^2}{2}\right)t^2-\frac{\hbar\left(1+\frac{\hbar}{2}\right)}{12}t^4 \quad (44\text{-d})$$

The mth-order analytical approximation is given by

$$x_1(t)=x_{1,0}(t)+x_{1,1}(t)+\cdots+x_{1,m}(t) \quad (45\text{-a})$$

$$x_2(t)=x_{2,0}(t)+x_{2,1}(t)+\cdots+x_{2,m}(t) \quad (45\text{-b})$$

To determine the auxiliary parameter $\hbar$ of the modified HAM, the characteristics of the $\hbar$ – curves for the 10th-order approximation are plotted in Fig. 3. We clearly observe that the region of admissible values of $\hbar$ is in the range of $-1.4<\hbar<-0.6$. By selecting $\hbar=-1$, the series solutions of the HAM are

$$x_1(t)=t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}+\frac{t^9}{362880}-\frac{t^{11}}{39916800}+\frac{t^{13}}{6227020800}-\cdots \quad (46\text{-a})$$

$$x_2(t)=1-\frac{t^2}{2}+\frac{t^4}{24}-\frac{t^6}{720}+\frac{t^8}{40320}-\frac{t^{10}}{3628800}+\frac{t^{12}}{479001600}-\cdots \quad (46\text{-b})$$

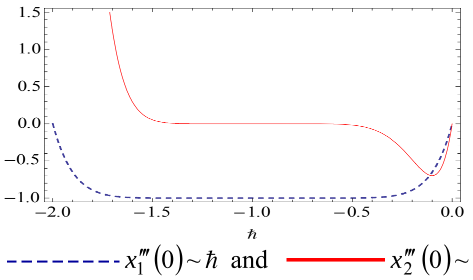


Fig. 3. The $\hbar$ – curves of $x_1'''(0)$ and $x_2'''(0)$ obtained from the 10th-order approximation for Eqs. (32)

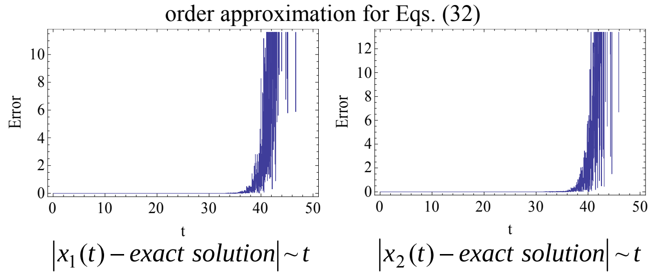


Fig. 4. Absolute errors between the 100th-order homotopy analysis and exact solutions for $\hbar=-1$

In Fig. 4, it is shown the absolute errors between the 100th-order modified HAM and exact solutions, both of them agree well until $t=37$. The validity of the series solutions can be maintained and improved in three ways, including the selection of different auxiliary parameters $\hbar$, the increase of the number of terms in series solutions and the alteration of the base functions.

3.3 EXAMPLE 3

The third example of the nonlinear MDOF system is given by

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}=-2x_1x_2^2,\ \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2}=-x_2^3+x_1^2x_2 \quad (47)$$

with the initial conditions

$$x_1(0)=0,\ x_1'(0)=1,\ x_2(0)=1,\ x_2'(0)=0 \quad (48)$$

where a prime denote the differentiation with respect to time t. Here, the exact solutions for Eqs. (47) with the initial conditions in Eqs. (48) are

$$x_1(t)=\tanh t,\ x_2(t)=\operatorname{sech} t \quad (49)$$

For the sake of simplicity, the nonlinear system (47) is firstly reduced into four first-order differential equation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_3,\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=x_4,\ \frac{\mathrm{d}x_3}{\mathrm{d}t}=-2x_1x_2^2,\ \frac{\mathrm{d}x_4}{\mathrm{d}t}=-x_2^3+x_1^2x_2 \quad (50)$$

with the initial conditions

$$x_1(0)=0,\ x_2(0)=1,\ x_3(0)=1,\ x_4(0)=0 \quad (51)$$

In this example, we define $\left\{\mathrm{e}^{-nt}\mid n=0,1,2,3,\cdots\right\}$ as the base functions. According to the rules of solution expression and the coefficient ergodicity, the initial approximation can be expressed as

$$x_{1,0}(t)=1-\mathrm{e}^{-t},\ x_{2,0}(t)=2\mathrm{e}^{-t}-\mathrm{e}^{-2t},\ x_{3,0}(t)=\mathrm{e}^{-t}$$

$$x_{4,0}(t)=-2\mathrm{e}^{-t}+2\mathrm{e}^{-2t} \quad (52)$$

The linear operator is

$$L_2\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\\ \varphi_4(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial\varphi_1(t;q)}{\partial t}\\ \dfrac{\partial\varphi_2(t;q)}{\partial t}\\ \dfrac{\partial\varphi_3(t;q)}{\partial t}\\ \dfrac{\partial\varphi_4(t;q)}{\partial t}\end{pmatrix} \quad (53)$$

and the nonlinear operator is

$$N_3\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\\ \varphi_4(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial\varphi_1(t;q)}{\partial t}-\varphi_3(t;q)\\ \dfrac{\partial\varphi_2(t;q)}{\partial t}-\varphi_4(t;q)\\ \dfrac{\partial\varphi_3(t;q)}{\partial t}+2\varphi_1(t;q)[\varphi_2(t;q)]^2\\ \dfrac{\partial\varphi_4(t;q)}{\partial t}+[\varphi_2(t;q)]^3-[\varphi_1(t;q)]^2\varphi_2(t;q)\end{pmatrix} \quad (54)$$

Hence, the zeroth-order deformation equation can be written as

$$(1-q)L_2\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\\ \varphi_3(t;q)-x_{3,0}(t)\\ \varphi_4(t;q)-x_{4,0}(t)\end{pmatrix}=q\hbar H(t)N_3\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\\ \varphi_3(t;q)\\ \varphi_4(t;q)\end{pmatrix} \quad (55)$$

and the mth-order deformation equation can be expressed as

$$L_2\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\\ x_{3,m}(t)-\chi_m x_{3,m-1}(t)\\ x_{4,m}(t)-\chi_m x_{4,m-1}(t)\end{pmatrix}=\hbar H(t)\begin{pmatrix}R_{1,m}(\mathbf{x}_{1,m-1})\\ R_{2,m}(\mathbf{x}_{2,m-1})\\ R_{3,m}(\mathbf{x}_{3,m-1})\\ R_{4,m}(\mathbf{x}_{4,m-1})\end{pmatrix} \quad (56)$$

where

$$H(t)=e^{-t} \quad (57)$$

Making use of the initial conditions and the initial approximation yields

$$x_{1,m}(0)=0,\ x_{2,m}(0)=0,\ x_{3,m}(0)=0,\ x_{4,m}(0)=0\ (m\ge 1) \quad (58)$$

and

$$R_{1,m}(\mathbf{x}_{1,m-1})=x'_{1,m-1}(t)-x_{3,m-1}(t) \quad (59\text{-a})$$

$$R_{2,m}(\mathbf{x}_{2,m-1})=x'_{2,m-1}(t)-x_{4,m-1}(t) \quad (59\text{-b})$$

$$R_{3,m}(\mathbf{x}_{3,m-1})=x'_{3,m-1}(t)+2\sum_{j+s=m-1}\left[x_{1,j}(t)\left(\sum_{i=0}^{s}x_{2,i}x_{2,s-i}\right)\right] \quad (59\text{-c})$$

$$R_{4,m}(\mathbf{x}_{4,m-1})=x'_{4,m-1}(t)+\sum_{j+s=m-1}\left[x_{2,j}(t)\left(\sum_{i=0}^{s}x_{2,i}x_{2,s-i}\right)\right]-\sum_{j+s=m-1}\left[x_{2,j}(t)\left(\sum_{i=0}^{s}x_{1,i}x_{1,s-i}\right)\right] \quad (59\text{-d})$$

Solving the mth-order deformation equation (56), one obtains

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t e^{-s}R_{1,m}(\mathbf{x}_{1,m-1})ds \quad (60\text{-a})$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t e^{-s}R_{2,m}(\mathbf{x}_{2,m-1})ds \quad (60\text{-b})$$

$$x_{3,m}(t)=\chi_m x_{3,m-1}(t)+\hbar\int_0^t e^{-s}R_{3,m}(\mathbf{x}_{3,m-1})ds \quad (60\text{-c})$$

$$x_{4,m}(t)=\chi_m x_{4,m-1}(t)+\hbar\int_0^t e^{-s}R_{4,m}(\mathbf{x}_{4,m-1})ds \quad (60\text{-d})$$

in which

$$x_{1,1}(t)=0 \quad (61\text{-a})$$

$$x_{2,1}(t)=0 \quad (61\text{-b})$$

$$x_{1,2}(t)=\frac{16}{105}\hbar^2+\frac{e^{-7t}}{21}\hbar^2-\frac{e^{-6t}}{3}\hbar^2+\frac{4e^{-5t}}{5}\hbar^2-\frac{2e^{-4t}}{3}\hbar^2+\frac{e^{-3t}}{6}\hbar^2-\frac{e^{-t}}{6}\hbar^2 \quad (61\text{-c})$$

$$x_{2,2}(t)=\frac{74}{105}e^{-4t}\hbar^2\left[\sinh(\frac{t}{2})\right]^2+\frac{113}{105}e^{-4t}\hbar^2\cosh(t)\left[\sinh\left(\frac{t}{2}\right)\right]^2-\frac{2}{35}e^{-4t}\hbar^2\cosh(2t)\left[\sinh\left(\frac{t}{2}\right)\right]^2+\frac{29}{105}e^{-4t}\hbar^2\cosh(3t)\left[\sinh\left(\frac{t}{2}\right)\right]^2+\frac{4}{5}e^{-4t}\hbar^2\sinh(2t)\left[\sinh\left(\frac{t}{2}\right)\right]^2+\frac{2}{15}e^{-4t}\hbar^2\sinh(3t)\left[\sinh\left(\frac{t}{2}\right)\right]^2 \quad (61\text{-d})$$

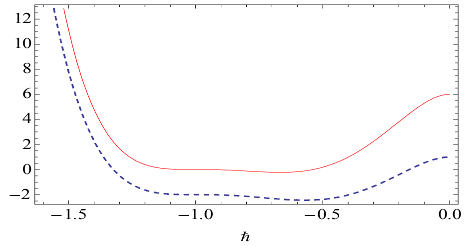
The mth-order analytical approximation is

$$x_1(t)=x_{1,0}(t)+x_{1,1}(t)+\cdots+x_{1,m}(t) \quad (62\text{-a})$$

$$x_2(t)=x_{2,0}(t)+x_{2,1}(t)+\cdots+x_{2,m}(t) \quad (62\text{-b})$$

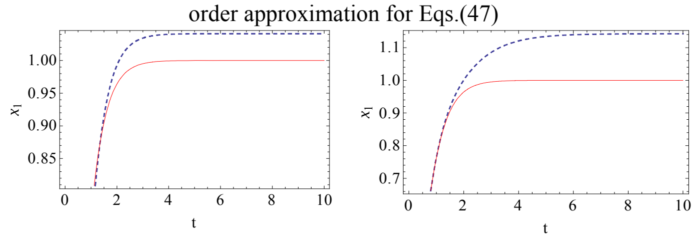
$$x_3(t)=x_{3,0}(t)+x_{3,1}(t)+\cdots+x_{3,m}(t) \quad (62\text{-c})$$

$$x_4(t)=x_{4,0}(t)+x_{4,1}(t)+\cdots+x_{4,m}(t) \quad (62\text{-d})$$



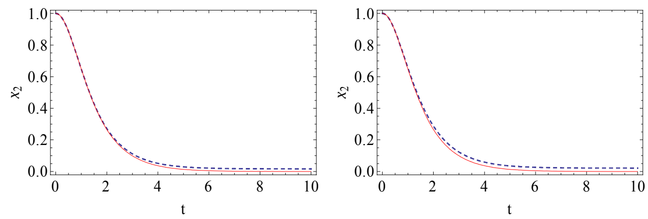
$x_1'''(0)\sim\hbar$ and $x_2'''(0)\sim\hbar$

Fig. 5. The $\hbar$ – curves of $x_1'''(0)$ and $x_2'''(0)$ obtained from the 5th-order approximation for Eqs.(47)



approximation solution and exact solution

Fig. 6. Comparison of the approximate and exact solutions for $x_1(t)$



approximation solution and exact solution

Fig. 7. Comparison of the approximate and exact solutions for $x_2(t)$

In Fig. 5, the $\hbar$ – curves are displayed for the 5th-order analytical approximation. It is obvious that the region of admissible values of $\hbar$ is $-1.1<\hbar<-0.9$. For $\hbar=-1$, the 20th-order series solution agrees well with the exact solution for $x_1(t)$, the relative error for the whole region (i.e. $t\in[1/4,10000]$) is less than 4.004% as shown in Figs. 6. In Figs. 6 and 7, the approximate ($x_1(t)$, $x_2(t)$) and exact solutions are contrasted. The deviation of the 20th-order solution of $x_1(t)$ with respect to the exact solution is comparatively large, but the 10th-order solution of $x_2(t)$ is good for comparison.

3.4 EXAMPLE 4

In this example, the nonlinear MDOF system is governed by

$$\frac{d^2x_1}{dt^2}+x_2\frac{dx_1}{dt}-x_1=1\,,\ \frac{d^2x_2}{dt^2}-x_1\frac{dx_2}{dt}-x_2=1 \quad (63)$$

with the initial conditions

$$x_1(0)=1, x_1'(0)=1, x_2(0)=1, x_2'(0)=-1 \quad (64)$$

where a prime denote the differentiation with respect to time t.

The exact solutions for Eqs. (63) subjected to the initial conditions are

$$x_1(t)=e^t\,,\ x_2(t)=e^{-t} \quad (65)$$

Suppose that the solution can be expressed by a set of base functions $\left\{t^n \mid n=0,1,2,3,\cdots\right\}$, we choose the initial approximation as

$$x_{1,0}(t)=1+t, x_{2,0}(t)=1-t \quad (66)$$

And the auxiliary linear operators are

$$L\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2\varphi_1(t;q)}{\partial t^2}\\ \dfrac{\partial^2\varphi_2(t;q)}{\partial t^2}\end{pmatrix} \quad (67)$$

By defining the nonlinear operator as

$$N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}=\begin{pmatrix}\dfrac{\partial^2\varphi_1(t;q)}{\partial t^2}+\varphi_2(t;q)\dfrac{\partial\varphi_1(t;q)}{\partial t}-\varphi_1(t;q)\\ \dfrac{\partial^2\varphi_2(t;q)}{\partial t^2}-\varphi_1(t;q)\dfrac{\partial\varphi_2(t;q)}{\partial t}-\varphi_2(t;q)\end{pmatrix} \quad (68)$$

Thus, the zeroth-order deformation equation can be written in the form

$$(1-q)L\begin{pmatrix}\varphi_1(t;q)-x_{1,0}(t)\\ \varphi_2(t;q)-x_{2,0}(t)\end{pmatrix}=q\hbar\left[N\begin{pmatrix}\varphi_1(t;q)\\ \varphi_2(t;q)\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right] \quad (69)$$

and the mth-order deformation equation can be expressed as

$$L\begin{pmatrix}x_{1,m}(t)-\chi_m x_{1,m-1}(t)\\ x_{2,m}(t)-\chi_m x_{2,m-1}(t)\end{pmatrix}=\hbar\begin{pmatrix}R_{1,m}(\mathbf{x}_{1,m-1})\\ R_{2,m}(\mathbf{x}_{2,m-1})\end{pmatrix} \quad (70)$$

From the initial conditions and the initial approximation, we have

$$x_{1,m}(0)=0, x_{2,m}(0)=0, x_{1,m}'(0)=0, x_{2,m}'(0)=0 \quad (m\geq 1) \quad (71)$$

and

$$R_{1,m}(\mathbf{x}_{1,m-1})=x_{1,m-1}''(t)-x_{1,m-1}(t)+\sum_{i+j=m-1}x_{1,i}'(t)x_{2,j}(t)-(1-\chi_m) \quad (72a)$$

$$R_{2,m}(\mathbf{x}_{2,m-1})=x_{2,m-1}''(t)-x_{2,m-1}(t)-\sum_{i+j=m-1}x_{2,i}'(t)x_{1,j}(t)-(1-\chi_m) \quad (72b)$$

Solving the mth-order deformation equation (70), one obtains

$$x_{1,m}(t)=\chi_m x_{1,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{1,m}(\mathbf{x}_{1,m-1})\mathrm{d}s\right)\mathrm{d}\tau+C_{1,m}t+C_{2,m} \quad (73a)$$

$$x_{2,m}(t)=\chi_m x_{2,m-1}(t)+\hbar\int_0^t\left(\int_0^\tau R_{2,m}(\mathbf{x}_{2,m-1})\mathrm{d}s\right)\mathrm{d}\tau+C_{3,m}t+C_{4,m} \quad (73b)$$

$$R_{1,1}(\mathbf{x}_{1,0})=x_{1,0}''(t)-x_{1,0}(t)+x_{1,0}'(t)x_{2,0}(t)-1 = -t-1+1-t-1=-2t-1 \quad (74a)$$

$$R_{2,1}(\mathbf{x}_{2,0})=x_{2,0}''(t)-x_{2,0}(t)-x_{2,0}'(t)x_{1,0}(t)-1 = t-1+1+t-1=2t-1 \quad (74b)$$

$$x_{1,1}(t)=-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right) \quad (75a)$$

$$x_{2,1}(t)=\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right) \quad (75b)$$

$$R_{1,2}(\mathbf{x}_{1,1})=x_{1,1}''(t)-x_{1,1}(t)+x_{1,0}'(t)x_{2,1}(t)+x_{1,1}'(t)x_{2,0}(t)$$
$$=\hbar(-2t-1)+\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right)+\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right)-\hbar\left(t^2+t\right)(1-t)$$
$$=\hbar\left(\frac{5t^3}{3}-3t-1\right) \quad (76a)$$

$$R_{2,2}(\mathbf{x}_{2,1})=x_{2,1}''(t)-x_{2,1}(t)-x_{2,0}'(t)x_{1,1}(t)-x_{2,1}'(t)x_{1,0}(t)$$
$$=\hbar(2t-1)-\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right)-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right)-\hbar\left(t^2-t\right)(1+t)$$
$$=\hbar\left(-\frac{5t^3}{3}+3t-1\right) \quad (76b)$$

$$x_{1,2}(t)=-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right)+\hbar^2\left(-\frac{t^2}{2}-\frac{t^3}{2}+\frac{t^5}{12}\right) \quad (77a)$$

$$x_{2,2}(t)=\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right)+\hbar^2\left(-\frac{t^2}{2}+\frac{t^3}{2}-\frac{t^5}{12}\right) \quad (77b)$$

We now successively obtain the second-order analytical approximation by HAM as the following

$$x_1(t)\approx 1+t-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right)-\hbar\left(\frac{t^3}{3}+\frac{t^2}{2}\right)+\hbar^2\left(-\frac{t^2}{2}-\frac{t^3}{2}+\frac{t^5}{12}\right)$$
$$=1+t-\left(\hbar+\frac{\hbar^2}{2}\right)t^2-\left(\frac{2\hbar}{3}+\frac{\hbar^2}{2}\right)t^3+\frac{\hbar^2}{12}t^5 \quad (78a)$$

$$x_2(t)\approx 1-t+\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right)+\hbar\left(\frac{t^3}{3}-\frac{t^2}{2}\right)+\hbar^2\left(-\frac{t^2}{2}+\frac{t^3}{2}-\frac{t^5}{12}\right)$$
$$=1-t-\left(\hbar+\frac{\hbar^2}{2}\right)t^2+\left(\frac{2\hbar}{3}+\frac{\hbar^2}{2}\right)t^3-\frac{\hbar^2}{12}t^5 \quad (78b)$$

By selecting $\hbar=-1$, the tenth-order analytical approximation series solutions of the HAM are

$$x_1(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\frac{t^8}{40320}$$
$$+\frac{t^9}{362880}+\frac{t^{10}}{3628800}+\frac{t^{11}}{39916800}+\frac{t^{13}}{47900160}-\frac{t^{14}}{10897286400}$$

$$-\frac{t^{15}}{108972864000}+\frac{29119t^{16}}{46702656000}-\frac{10886489t^{17}}{12703122432000}$$
$$+\frac{909977417t^{18}}{533531142144000}-\frac{25070839867t^{19}}{5529322745856000}$$
$$-\frac{153073227809t^{20}}{76028187755520000}+\frac{2253389787941t^{21}}{1021818843434188800} \quad (79a)$$

$$x_2(t)=1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}+\frac{t^6}{720}-\frac{t^7}{5040}$$
$$+\frac{t^8}{40320}-\frac{t^9}{362880}+\frac{t^{10}}{3628800}-\frac{t^{11}}{39916800}-\frac{t^{13}}{47900160}$$
$$-\frac{t^{14}}{10897286400}+\frac{t^{15}}{108972864000}+\frac{29119t^{16}}{46702656000}$$
$$+\frac{10886489t^{17}}{12703122432000}+\frac{909977417t^{18}}{533531142144000}-\frac{25070839867t^{19}}{5529322745856000}$$
$$-\frac{153073227809t^{20}}{76028187755520000}-\frac{2253389787941t^{21}}{1021818843434188800} \quad (79b)$$

The $[5,5]$ homotopy Pade approximation solutions of $x_1(t)$ and $x_2(t)$ are

$$x_1(t)=\frac{1+\frac{t}{2}+\frac{t^2}{9}+\frac{t^3}{72}+\frac{t^4}{1008}+\frac{t^5}{30240}}{1-\frac{t}{2}+\frac{t^2}{9}-\frac{t^3}{72}+\frac{t^4}{1008}-\frac{t^5}{30240}} \quad (80a)$$

$$x_2(t)=\frac{1-\frac{t}{2}+\frac{t^2}{9}-\frac{t^3}{72}+\frac{t^4}{1008}-\frac{t^5}{30240}}{1+\frac{t}{2}+\frac{t^2}{9}+\frac{t^3}{72}+\frac{t^4}{1008}+\frac{t^5}{30240}} \quad (80b)$$

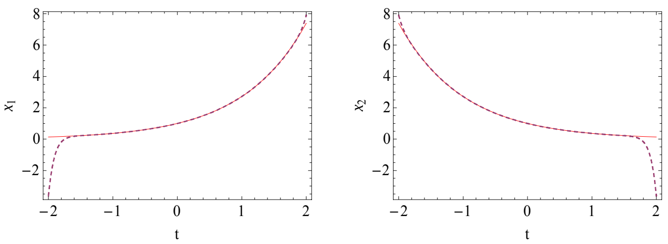


Fig. 8. Comparison of the approximate and exact solutions for $x_1(t)$ and $x_2(t)$

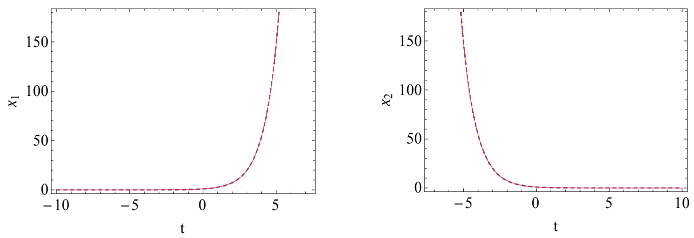


Fig. 9. Comparison of the [5,5] homotopy pade approximate and exact solutions for $x_1(t)$ and $x_2(t)$

Currently, some optimal HAM approach are developed, which can get faster convergent homotopy series solution [28, 29]. The tenth-order homotopy analysis approximation and exact solutions of $x_1(t)$ and $x_2(t)$ as show in Figs. 8. From Figs. 9, we can see that the [5,5] homotopy Pade approximat solutions and the Runge Kutta method virtually coalesce.

4. CONVERGENCE THEOREMS

In this section, the detailed proof of the convergence of the HAM solution for the MDOF dynamical system (1) is given.

Theorem 1 If the series in Eq. (11) converges, then $\sum_{m=1}^{\infty}R_m(\mathbf{u}_{m-1},r,t)=0$.

Proof. Since the series $\sum_{m=0}^{\infty}u_m(r,t)$ converges, then it can be written as

$$S(t)=\sum_{m=0}^{\infty}u_m(r,t) \quad (81)$$

and

$$\lim_{m\to\infty}u_m(r,t)=0 \quad (82)$$

In view of Eq. (14), the left-hand side of Eq. (13) satisfies

$$\sum_{m=1}^{n}[u_m(r,t)-\chi_m u_{m-1}(r,t)]$$
$$=u_1(r,t)+u_2(r,t)-u_1(r,t)+u_3(r,t)-u_2(r,t)$$
$$+\cdots+u_n(r,t)-u_{n-1}(r,t)=u_n(r,t) \quad (83)$$

From Eq. (82), one obtains

$$\sum_{m=1}^{\infty}[u_m(r,t)-\chi_m u_{m-1}(r,t)]=\lim_{m\to\infty}u_m(r,t)=0 \quad (84)$$

By virtue of the properties of linear operator L, we arrive at

$$\sum_{m=1}^{\infty}L[u_m(r,t)-\chi_m u_{m-1}(r,t)]=L\sum_{m=1}^{\infty}[u_m(r,t)-\chi_m u_{m-1}(r,t)]=0 \quad (85)$$

From Eqs. (13) and (84), we have

$$\sum_{m=1}^{\infty}L[u_m(r,t)-\chi_m u_{m-1}(r,t)]=\hbar H(t)\sum_{m=1}^{\infty}R_m(\mathbf{u}_{m-1},r,t)=0 \quad (86)$$

Supposing $\hbar\neq 0$ and $H(t)\neq 0$ implies,

$$\sum_{m=1}^{\infty}R_m(\mathbf{u}_{m-1},r,t)=0 \quad (87)$$

Theorem 2 If the series in Eq. (11) converges, then it must be the solution of system (1).

Proof. Substituting Eq. (16) into Eq. (87), we obtain

$$\sum_{m=1}^{\infty}R_m(\mathbf{u}_{m-1},r,t)=\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial p^{m-1}}N\left[\sum_{n=0}^{\infty}u_n(r,t)p^n\right]\right\}\Bigg|_{p=0}=0 \quad (88)$$

Assume

$$\varepsilon(r,t;p) = N\left[\sum_{n=0}^{\infty} u_n(r,t)p^n\right] \quad (89)$$

be the residual error of Eq. (1), which can be expanded by the Taylor series expansion at $p=0$ as follows

$$\varepsilon(r,t;p) = \sum_{m=0}^{\infty} \frac{\partial^m \varepsilon(r,t;p)}{\partial p^m}\Bigg|_{p=0} \frac{p^m}{m!} = \sum_{m=0}^{\infty} \frac{\partial^m N\left[\sum_{n=0}^{\infty} u_n(r,t)p^n\right]}{\partial p^m}\Bigg|_{p=0} \frac{p^m}{m!} \quad (90)$$

Setting $p=1$ yields,

$$\varepsilon(r,t;1) = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m \varepsilon(r,t;p)}{\partial p^m}\Bigg|_{p=0} = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m N\left[\sum_{n=0}^{\infty} u_n(r,t)p^n\right]}{\partial p^m}\Bigg|_{p=0} = 0 \quad (91)$$

Thus, the series in Eq. (11) is the solution of system (1).

5. CONCLUSIONS

In summary, the HAM is applied to obtain analytical approximation solutions for a class of MDOF dynamical systems. Four illustrative examples are selected to verify the validity and accuracy of the HAM and the modified HAM. The method provides an ingenious avenue for controlling the convergences of approximation series, the proof of convergence theorems is presented. The fundamental ideas of the HAM are essentially different from other existing analytical methods. Numerical comparisons demonstrate that the HAM is an effective and robust analytical method for MDOF dynamical systems.

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