Bol. Soc. Paran. Mat. (3s.) **v. 32** 1 (2014): 193–205.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.13457

# Some Differential Identities in Prime Γ-rings

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ABSTRACT: Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. An additive mapping $d : M \longrightarrow M$, where $M$ is a $\Gamma$-ring, is called a derivation if for any $a, b \in M$ and $\alpha \in \Gamma$, $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$. In this paper, we investigate the commutativity of prime $\Gamma$-ring satisfying certain differential identities.

Key Words: Γ-rings, prime Γ-rings, derivations, ideals, commutativity.

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## 1. Introduction

Let $M$ and $\Gamma$ be additive abelian groups. If for any $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied, $(i)$ $a\alpha b \in M$ $(ii)$ $(a+b)\alpha c = a\alpha c+b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ $(iii)$ $(a\alpha b)\beta c = a\alpha(b\beta c)$, then $M$ is called a $\Gamma$-ring. An additive subgroup $U$ of $M$ is called a right (resp. a left) ideal of $M$ if $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). $U$ is said to be an ideal of $M$ if it is both a right as well as a left ideal of $M$. $M$ is said to be prime $\Gamma$-ring if $a\Gamma M\Gamma b = \{0\}$ implies that either $a = 0$ or $b = 0$ for $a, b \in M$. The centre of $\Gamma$-ring $M$ will be denoted by $Z(M)$ i.e.; $Z(M) = \{a \in M \mid a\alpha b = b\alpha a \text{ for all } b \in M \text{ and } \alpha \in \Gamma\}$. Following Jing [5], an additive mapping $d : M \longrightarrow M$ is called a derivation on $M$ if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$. For any $a, b \in M$ and $\gamma \in \Gamma$, we write $[a, b]_\gamma = a\gamma b - b\gamma a$ and $a \circ_\gamma b = a\gamma b + b\gamma a$.

Throughout this paper $M$ will denote a $\Gamma$-ring satisfying $a\alpha b\beta c = a\beta b\alpha c$ for all $\alpha, \beta \in \Gamma$ and for all $a, b, c \in M$. We shall use the following identities without any specific mention:
If $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then

$(i)$ $[a, b\beta c]_\gamma = [a, b]_\gamma\beta c + b\beta[a, c]_\gamma$,

$(ii)$ $a \circ_\alpha (b + c) = a \circ_\alpha b + a \circ_\alpha c$,

$(iii)$ $a \circ_\alpha (b\beta c) = (a \circ_\alpha b)\beta c + b\beta[c, a]_\alpha = b\beta(a \circ_\alpha c) + [a, b]_\alpha\beta c$.

2000 *Mathematics Subject Classification:* 16W25, 16N60,16U80
2 This research is partially supported by a grant from DST (SR/S4/MS:556/08)

The study of Γ-ring was initiated by Nobusawa in [4]. Further, the condition on Γ-ring was slightly weakened by Barnes [3] in sense of Nobusawa. Since then various analogous concepts and analogous results of ring theory have been studied in Γ-rings (for reference see [3], [5], [6] and [7], where further references can be found). In the present paper, we have obtained some analogous results in Γ-ring earlier obtained for rings.

## 2. Main Results

In the year 1992, Daif and Bell [4] obtained commutativity of semiprime ring $R$ satisfying differential identity $d([x, y]) = \pm[x, y]$ for all $x, y \in R$. Further, the first author together with Rehman [2] established the commutativity of semiprime ring $R$ satisfying the above identity for a well behaved subset of $R$ viz. Lie ideal of $R$. Later on, many authors explored commutativity of prime and semiprime rings satisfying various conditions on rings (for reference see, [2] etc. where further references can be found). In the present paper, our objective is to investigate commutativity of prime Γ-rings satisfying certain identities involving derivations on Γ-rings. We facilitate our discussion with the following results which are necessary for developing the proofs of our theorems:

**Lemma 2.1.** *[Lemma 2, [7]] Let $M$ be a prime Γ-ring and $U$ be a nonzero right ideal of $M$ such that $U \subseteq Z(M)$. Then $M$ is commutative.*

**Lemma 2.2.** *Let $M$ be a prime Γ-ring and $U$ be a commutative nonzero right ideal of $M$. Then $M$ is commutative.*

**Proof:** Since $U$ is commutative, $[x, y]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Replace $y$ by $y\alpha r$, we have

$$\begin{aligned} 0 &= [x, y\alpha r]_\gamma \\ &= [x, y]_\gamma \alpha r + y\alpha[x, r]_\gamma \\ &= y\alpha[x, r]_\gamma. \end{aligned}$$

Again replacing $y$ by $y\beta r_1$, we get $y\beta r_1 \alpha[x, r]_\gamma = 0$. Since $M$ is prime, either $y = 0$ or $[x, r]_\gamma = 0$. If $y = 0$, then $U = \{0\}$, a contradiction. Therefore $[x, r]_\gamma = 0$. This implies $x \in Z(M)$ i.e., $U \subseteq Z(M)$. Therefore, by Lemma 2.1, $M$ is commutative. □

**Theorem 2.3.** *Let $M$ be a prime Γ-ring and $U$ be a nonzero ideal of $M$. If $d$ is a nonzero derivation on $M$ satisfying $[d(x), x]_\gamma = 0$ for all $x \in U$, $\gamma \in \Gamma$, then $M$ is commutative.*

**Proof:** We have $[d(x), x]_\gamma = 0$ for all $x \in U$ and $\gamma \in \Gamma$. Replace $x$ by $x + y$, to get

$$[d(x), y]_\gamma + [d(y), x]_\gamma = 0 \text{ for all } x, y \in U, \ \gamma \in \Gamma.$$

Further, replacing $y$ by $y\alpha x$ in the above condition and using the same along with the given condition, we have

$$[y,x]_\gamma \alpha d(x) = 0 \text{ for all } x,y \in U \text{ and } \alpha,\gamma \in \Gamma.$$

Again, replacing $y$ by $y\beta z$ and using the above condition, we get $[y,x]_\gamma \beta z \alpha d(x) = 0$ for all $x,y,z \in U$ and $\alpha,\beta,\gamma \in \Gamma$. Now replacing $z$ by $r\delta z$ and using the primeness of $M$, we get either $[y,x]_\gamma = 0$ or $U\Gamma d(x) = \{0\}$. Now let $U_1 = \{x \in U \mid [y,x]_\gamma = 0 \text{ for all } y \in M, \gamma \in \Gamma\}$ and $U_2 = \{x \in U \mid U\Gamma d(x) = \{0\}\}$. Then it can be seen that $U_1$ and $U_2$ are additive subgroups of $U$ whose union is $U$. But a group can not be union of two of its proper subgroups, we find that either $U\Gamma d(x) = \{0\}$ for all $x \in U$ or $[x,y]_\gamma = 0$ for all $x,y \in U$ and $\gamma \in \Gamma$. If $U\Gamma d(x) = \{0\}$, then by primeness of $M$ either $U = \{0\}$ or $d(x) = 0$ for all $x \in U$. But $U \neq \{0\}$ implies that $d(x) = 0$ for all $x \in U$. Hence $d(x\gamma r) = 0$. Therefore $x\gamma d(r) = 0$. This implies $d(r) = 0$ by primeness of $M$. Therefore $d = 0$, a contradiction. Hence $[x,y]_\gamma = 0$ for all $x,y \in U$ and $\gamma \in \Gamma$ and $U$ is commutative. Therefore $M$ is commutative. □

**Corollary 2.4.** *Let $M$ be a prime $\Gamma$-ring and $d$ be a nonzero derivation on $M$ satisfying $x - d(x) \in Z(M)$ for all $x \in U$, then $M$ is commutative.*

**Proof:** We have $x - d(x) \in Z(M)$ i.e., $[x - d(x), x]_\gamma = 0$ for all $x \in U$ and $\gamma \in \Gamma$. Hence $[d(x), x]_\gamma = 0$ for all $x \in U$ and $\gamma \in \Gamma$. Therefore by Theorem 2.3, $M$ is commutative. □

**Theorem 2.5.** *Let $M$ be a 2-torsion free prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $M$ admits a nonzero derivation $d$ satisfying any one of the following conditions:*

*(i)* $[d(x), d(y)]_\gamma = 0$ *for all* $x,y \in U$ *and* $\gamma \in \Gamma$,

*(ii)* $[d(x), d(y)]_\gamma = [x,y]_\gamma$ *for all* $x,y \in U$ *and* $\gamma \in \Gamma$,

*(iii)* $[d(x), d(y)]_\gamma = [y,x]_\gamma$ *for all* $x,y \in U$ *and* $\gamma \in \Gamma$,

*(iv)* $d([x,y]_\gamma) = [x,y]_\gamma$ *for all* $x,y \in U$ *and* $\gamma \in \Gamma$.

*Then $M$ is commutative.*

**Proof:** $(i)$. Given that $[d(x), d(y)]_\gamma = 0$ for all $x,y \in U$ and $\gamma \in \Gamma$. Replace $y$ by $y\alpha z$ and use the given condition, we get

$$d(y)\alpha[d(x), z]_\gamma + [d(x), y]_\gamma \alpha d(z) = 0. \tag{2.1}$$

Replacing $z$ by $z\beta r$ for $r \in M$ in (2.1) and using (2.1), we have

$$d(y)\alpha z\beta[d(x), r]_\gamma + [d(x), y]_\gamma \alpha z \beta d(r) = 0.$$

Again, replacing $r$ by $d(x)$, we get $[d(x), y]_\gamma \alpha z \beta d^2(x) = 0$ for all $x, y, z \in M$ and $\alpha, \beta, \gamma \in \Gamma$. By primeness of $M$, we have either $[d(x), y]_\gamma = 0$ or $z\beta d^2(x) = 0$.
Take $U_1 = \{x \in U \mid [d(x), y]_\gamma = 0$ for all $y \in U$ and $\gamma \in \Gamma\}$ and $U_2 = \{x \in U \mid z\beta d^2(x) = 0$ for all $z \in U$ and $\beta \in \Gamma\}$. Then $U_1$ and $U_2$ are additive subgroups of $U$ such that $U_1 \cup U_2 = U$. But a group can not be the set theoretic union of its two proper subgroups, either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $[d(x), y]_\gamma = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$. Therefore in particular $[d(x), x]_\gamma = 0$ for all $x \in U$, $\gamma \in \Gamma$ and hence $M$ is commutative by Theorem 2.3. If $U_2 = U$, then $U\beta d^2(x) = \{0\}$ for all $x \in U$ and $\beta \in U$. Since $M$ is prime and $U \neq \{0\}$, we get $d^2(x) = 0$ for all $x \in U$. Replacing $x$ by $w\alpha y$, we find that $d^2(w\alpha y) = 0$ for all $w, y \in U$, $\alpha \in \Gamma$. Since $d^2(x) = 0$ for all $x \in U$ and $\Gamma$-ring is 2-torsion free, we have $d(w)\alpha d(y) = 0$ for all $w, y \in U$ and $\alpha \in \Gamma$. Further, replacing $w$ by $w\beta z$ and using this condition along with primeness of $M$, we get either $d(w)\beta z = 0$ or $d(x) = 0$. Again, since $M$ is prime, either $d(U) = \{0\}$ or $U = \{0\}$. Since it is given that $U \neq \{0\}$, $d(U) = \{0\}$. But $d(U) = \{0\}$ implies $U\Gamma d(M) = \{0\}$. Again, primeness of $M$ gives $d(M) = \{0\}$, which is a contradiction.

$(ii)$. Replacing $y$ by $y\beta z$ in $[d(x), d(y)]_\gamma = [x, y]_\gamma$, we get

$$[d(x), d(y\beta z)]_\gamma = [x, y\beta z]_\gamma \text{ for all } x, y, z \in U \text{ and } \beta, \gamma \in \Gamma.$$

This implies that for all $x, y, z \in U$ and $\beta, \gamma \in \Gamma$, we get

$$[d(x), d(y)]_\gamma \beta z + d(y)\beta[d(x), z]_\gamma + [d(x), y]_\gamma \beta d(z) + y\beta[d(x), d(z)]_\gamma = [x, y]_\gamma \beta z + y\beta[x, z]_\gamma.$$

Using the given condition, we arrive at

$$d(y)\beta[d(x), z]_\gamma + [d(x), y]_\gamma \beta d(z) = 0 \text{ for all } x, y, z \in M \text{ and } \beta, \gamma \in \Gamma.$$

Now using the same arguments as used after (2.1), we get the required result.

$(iii)$. Using the similar techniques as above, one can get the required result.

$(iv)$. Given that $d([x, y]_\gamma) = [x, y]_\gamma$ for all $x, y \in U$ and $\gamma \in \Gamma$. After the simplification, we get

$$[d(x), y]_\gamma + [x, d(y)]_\gamma = [x, y]_\gamma \text{ for all } x, y \in U \text{ and } \gamma \in \Gamma. \tag{2.2}$$

Replacing $y$ by $z\beta y$, we get

$$([d(x), z]_\gamma + [x, d(z)]_\gamma)\beta y + z\beta([d(x), y]_\gamma + [x, d(y)]_\gamma) + d(z)\beta[x, y]_\gamma + [x, z]_\gamma \beta d(y)$$
$$= [x, z]_\gamma \beta y + z\beta[x, y]_\gamma \text{ for all } x, y, z \in U \text{ and } \beta, \gamma \in \Gamma.$$

Using (2.2), we find that

$$d(z)\beta[x, y]_\gamma + [x, z]_\gamma \beta d(y) = 0 \text{ for all } x, y, z \in U \text{ and } \beta, \gamma \in \Gamma.$$

Further, replacing $y$ by $x$, we get $[x, z]_\gamma \beta d(x) = 0$ for all $x, z \in U$ and $\beta, \gamma \in \Gamma$. Again, replacing $z$ by $w\alpha z$, we get $[x, w]_\gamma \alpha z \beta d(x) = 0$ for all $x, w, z \in U$ and

$\alpha, \beta, \gamma \in \Gamma$. Since $M$ is prime, we have either $U\Gamma d(x) = \{0\}$ or $[x, z]_\gamma = 0$. Take $U_1 = \{x \in U \mid U\Gamma d(x) = \{0\}\}$ and $U_2 = \{x \in U \mid [x, w]_\gamma = 0$ for all $w \in U, \gamma \in \Gamma\}$. It can be easily seen that $U_1$ and $U_2$ are additive subgroups of $U$ such that $U_1 \cup U_2 = U$. Therefore either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $U\Gamma d(x) = \{0\}$ for all $x \in U$. Since $U \neq \{0\}$ and $M$ is prime, we arrive at a contradiction that $d = 0$. Therefore, now assume that $U_2 = U$. Hence $[x, w]_\gamma = 0$ for all $x, w \in U$ and $\gamma \in \Gamma$. This yields that $U$ is commutative. By Lemma 2.2, $M$ is commutative. □

**Corollary 2.6.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. If $d \neq 0$ is a derivation on $M$ such that $d([x, y]_\gamma) = [y, x]_\gamma$ for all $x, y \in U$ and $\gamma \in \Gamma$, then $M$ is commutative.*

**Proof:** Given that $d([x, y])_\gamma = [y, x]_\gamma$ for all $x, y \in U$ and $\gamma \in \Gamma$. This implies that $(-d)([x, y]_\gamma) = [x, y]_\gamma$ for all $x, y \in U$ and $\gamma \in \Gamma$. Since $-d$ is a derivation on $M$, by Theorem 2.5($iv$), $M$ is commutative. □

**Corollary 2.7.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $M$ admits a derivation $d$ satisfying any one of the following conditions:*

($i$) $d([x, y]_\gamma) = [d(x), y]_\gamma$ *for all* $x, y \in U$ *and* $\gamma \in \Gamma$,

($ii$) $d(x \circ_\gamma y) = d(x) \circ_\gamma y$ *for all* $x, y \in U$ *and* $\gamma \in \Gamma$.

*Then $M$ is commutative.*

**Proof:** ($i$). On simplifying the given condition, we have $x\gamma d(y) = d(y)\gamma x$ for all $x, y \in U$ and $\gamma \in \Gamma$. Replacing $x$ by $x\beta d(z)$, we have $x\gamma[d(y), d(z)]_\beta = 0$. Since $M$ is prime and $U \neq \{0\}$, we have $[d(y), d(z)]_\beta = 0$ for all $y, z \in U$ and $\beta \in \Gamma$. Hence $M$ is commutative by Theorem 2.5($i$).

($ii$). Using similar arguments as used in ($i$), we get the required result. □

**Theorem 2.8.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $M$ admits a nonzero derivation $d$ such that for all $x, y \in U$ and $\alpha, \gamma \in \Gamma$, $d$ satisfying any one of the following conditions:*

($i$) $d(x\alpha y) = d(y\alpha x)$,

($ii$) $d(x\alpha y) = -d(y\alpha x)$,

($iii$) $[d(x), y]_\gamma = [x, d(y)]_\gamma$.

*Then $M$ is commutative.*

**Proof:** ($i$). For all $x, y \in U$ and $\alpha \in \Gamma$, we have $d(x\alpha y) = d(y\alpha x)$. On simplifying, we have

$$[d(x), y]_\alpha + [x, d(y)]_\alpha = 0 \text{ for all } x, y \in U \text{ and } \alpha \in \Gamma. \tag{2.3}$$

Replacing $y$ by $y\beta z$ in (2.3) and using (2.3), we get

$$d(y)\beta[x,z]_\alpha + [x,y]_\alpha\beta d(z) = 0 \text{ for all } x,y,z \in U \text{ and } \alpha,\beta \in \Gamma.$$

Replace $z$ by $x$ to get $[x,y]_\alpha\beta d(x) = 0$ for all $x,y \in U$ and $\alpha,\beta \in \Gamma$. Again, replacing $y$ by $y\gamma w$ in the latter condition, we get

$$[x,y]_\alpha\gamma w\beta d(x) = 0 \text{ for all } x,y,w \in U \text{ and } \alpha,\beta,\gamma \in \Gamma. \tag{2.4}$$

Since $M$ is prime, we have $[x,y]_\alpha = 0$ or $U\Gamma d(x) = \{0\}$. The sets $x \in U$ for which these two properties hold forms additive subgroups of $U$ whose union is $U$. Hence by Brauer's trick, either $[x,y]_\alpha = 0$ for all $x,y \in U$ and $\alpha \in \Gamma$ or $U\Gamma d(x) = \{0\}$ for all $x \in U$. If $U\Gamma d(x) = \{0\}$, then by primeness of $M$, either $U = \{0\}$ or $d(x) = 0$ for all $x \in U$. But $d(x) = 0$ for all $x \in U$ gives $d = 0$ on $M$, a contradiction. Therefore $[x,y]_\alpha = 0$ for all $x,y \in U$, $\alpha \in \Gamma$ and hence $U$ is commutative and by Lemma 2.2, $M$ is commutative.

$(ii)$. For all $x,y \in U$ and $\alpha \in \Gamma$, we have $d(x\alpha y) = -d(y\alpha x)$. This implies that $d(x)\alpha y + x\alpha d(y) = -d(y)\alpha x - y\alpha d(x)$ for all $x,y \in U$ and $\alpha \in \Gamma$. Replace $y$ by $y\beta x$ and use the given condition, to get

$$x\alpha y\beta d(x) + y\alpha x\beta d(x) = 0 \text{ for all } x,y \in U \text{ and } \alpha,\beta \in \Gamma. \tag{2.5}$$

Now, replace $y$ by $y\gamma z$ in (2.5) and use (2.5), to get

$$[x,y]_\alpha\gamma z\beta d(x) = 0 \text{ for all } x,y,z \in U \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

Now using the same arguments, as used in proof of $(i)$ after (2.3), we get the required result.

$(iii)$. Replacing $y$ by $y\beta z$ in the given condition, we have

$$[x,y]_\gamma\beta d(z) + d(y)\beta[x,z]_\gamma = 0.$$

Replacing $z$ by $x$, we get $[x,y]_\gamma\beta d(x) = 0$ for all $x,y \in M$ and $\beta,\gamma \in \Gamma$. Again replacing $y$ by $y\alpha z$, we find that $[x,y]_\gamma\alpha z\beta d(x) = 0$. Since $M$ is prime, either $[x,y]_\gamma = 0$ or $U\Gamma d(x) = \{0\}$. By the same argument given in the proof of $(i)$ after (2.3), we get the required result. □

**Theorem 2.9.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $d$ is a derivation on $M$ satisfying any one of the following conditions:*

$(i)$ *$d(x\gamma y) - x\gamma y \in Z(M)$ for all $x,y \in U$ and $\gamma \in \Gamma$,*

$(ii)$ *$d(x\gamma y) - y\gamma x \in Z(M)$ for all $x,y \in U$ and $\gamma \in \Gamma$,*

$(iii)$ *$d(x)\gamma d(y) - x\gamma y \in Z(M)$ for all $x,y \in U$ and $\gamma \in \Gamma$.*

*Then $M$ is commutative.*

**Proof:** $(i)$. It is given that $d(x\gamma y) - x\gamma y \in Z(M)$ for all $x, y \in U$ and $\gamma \in \Gamma$. If $d = 0$, then we have $x\gamma y \in Z(M)$. Therefore $[x\gamma y, x]_\beta = 0$. Therefore $x\gamma[y, x]_\beta = 0$ for all $x, y \in U$ and $\beta, \gamma \in \Gamma$. Now replacing $y$ by $y\alpha z$, we find that $x\gamma y\alpha[z, x]_\beta = 0$ for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. By the primeness of $M$, we have either $x = 0$ or $U\Gamma[z, x]_\beta = \{0\}$. But $x = 0$ also implies that $U\Gamma[z, x]_\beta = \{0\}$. Therefore in both the cases, we get $U\Gamma[z, x]_\beta = \{0\}$. Since $M$ is prime, either $U = \{0\}$ or $[z, x]_\beta = 0$. Since $U \neq \{0\}$, $[z, x]_\beta = 0$ for all $x, z \in U$, $\beta \in \Gamma$ and $U$ is commutative. Therefore $M$ is commutative by Lemma 2.2.
Now assume that $d \neq 0$. Given that $d(x\gamma y) - x\gamma y \in Z(M)$. This implies that $d(x)\gamma y + x\gamma d(y) - x\gamma y \in Z(M)$. Replacing $y$ by $y\beta z$ and using the given condition, we have

$$\begin{aligned} 0 &= [d(x)\gamma y\beta z + x\gamma d(y\beta z) - x\gamma y\beta z, z]_\alpha \\ &= [x\gamma y\beta d(z), z]_\alpha \\ &= x\gamma y\beta[d(z), z]_\alpha + x\gamma[y, z]_\alpha\beta d(z) + [x, z]_\alpha\gamma y\beta d(z). \end{aligned} \tag{2.6}$$

Again, replacing $x$ by $w\delta x$ for $w \in U$ and $\delta \in \Gamma$ in (2.6), we get

$$w\delta(x\gamma y\beta[d(z), z]_\alpha + x\gamma[y, z]_\alpha\beta d(z) + [x, z]_\alpha\gamma y\beta d(z)) + [w, z]_\alpha\delta x\gamma y\beta d(z) = 0.$$

Using (2.6), we get $[w, z]_\alpha\delta x\gamma y\beta d(z) = 0$. Since $M$ is prime, we find that for each fixed $z \in U$, either $[w, z]_\alpha\delta x = 0$ or $U\Gamma d(z) = \{0\}$. Let $U_1 = \{z \in U \mid [w, z]_\alpha\delta x = 0$ for all $x, w \in U,\ \alpha, \delta \in \Gamma\}$ and $U_2 = \{z \in U \mid U\Gamma d(z) = \{0\}\}$. Since $U_1$ and $U_2$ are additive subgroups of $U$ whose union is $U$, we find that either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $[w, z]_\alpha\delta x = 0$ for all $x, w, z \in U$ and $\alpha, \delta \in \Gamma$. Since $M$ is prime, either $U = \{0\}$ or $[w, z]_\alpha = 0$ for all $w, z \in U$ and $\alpha \in \Gamma$. Since $U \neq \{0\}$, $U$ is commutative, and hence $M$ is commutative by Lemma 2.2. If $U_2 = U$, then $U\Gamma d(z) = \{0\}$ for all $z \in U$. This implies that either $U = \{0\}$ or $d = 0$, and hence in both the cases we arrive at contradictions.

$(ii)$. If $d = 0$, then using similar techniques as used in the beginning of the proof of $(i)$, we find that $M$ is commutative.
Now assume that $d \neq 0$. Since $d(x\gamma y) - y\gamma x \in Z(M)$ for all $x, y \in U$, $r \in M$ and $\gamma \in \Gamma$, we have $[d(x\gamma y) - y\gamma x, r]_\alpha = 0$ for all $x, y \in U$, $r \in M$ and $\alpha, \gamma \in \Gamma$. After simplification, we get

$$[d(x)\gamma y + x\gamma d(y), r]_\alpha = [y\gamma x, r]_\alpha \text{ for all } x, y \in U,\ r \in M \text{ and } \alpha, \gamma \in \Gamma. \tag{2.7}$$

Replacing $y$ by $y\beta r$ for $r \in M$, $\beta \in \Gamma$ in (2.7) and using (2.7), we get

$$[y\gamma x, r]_\alpha\beta r + [x\gamma y\beta d(r), r]_\alpha = [y\beta r\gamma x, r]_\alpha. \tag{2.8}$$

Again replacing $y$ by $x\delta y$ for $x \in U$, $\delta \in \Gamma$ in (2.8) and using (2.8), we get

$$x\delta[y\beta r\gamma x, r]_\alpha + [x, r]_\alpha\delta y\gamma x\beta r + [x, r]_\alpha\delta x\gamma y\beta d(r) = x\delta[y\beta r\gamma x, r]_\alpha + [x, r]_\alpha\delta y\beta r\gamma x.$$

After simplifying, we get

$$[x, r]_\alpha\delta y\gamma[x, r]_\beta + [x, r]_\alpha\delta x\gamma y\beta d(r) = 0. \tag{2.9}$$

Replacing $r$ by $r+x$ in (2.9) and using (2.9), we get

$$[x,r]_\alpha \delta x\gamma y\beta d(x)=0 \text{ for all } x,y\in U,\ r\in M \text{ and } \alpha,\beta,\gamma,\delta\in\Gamma.$$

Since $M$ is prime, we get $[x,r]_\alpha\delta x=0$ for all $x\in U$, $r\in M$ and $\alpha,\delta\in\Gamma$ or $U\Gamma d(x)=\{0\}$ for all $x\in U$. If $[x,r]_\alpha\delta x=0$, then $[x,r\gamma r_1]_\alpha\delta x=0$. Therefore, $[x,r]_\alpha\gamma r_1\delta x=0$. By primeness of $M$, either $x=0$ or $[x,r]_\alpha=0$. But $x=0$ also gives $[x,r]_\alpha=0$. Hence, there remain only two cases namely either $[x,r]_\alpha=0$ or $U\Gamma d(x)=\{0\}$. Take $U_1=\{x\in U \mid [x,r]_\alpha=0 \text{ for all } r\in M,\alpha\in\Gamma\}$ and $U_2=\{x\in U\mid U\Gamma d(x)=\{0\}\}$. But these are two additive subgroups of $U$ whose union is $U$. Therefore either $U_1=U$ or $U_2=U$. If $U_1=U$ then $U\subseteq Z(M)$. Therefore $M$ is commutative by Lemma 2.1. If $U_2=U$, then either $U=\{0\}$ or $d=0$, and we find contradictions in both the cases.

$(iii)$. If $d=0$, then $-x\gamma y\in Z(M)$ for all $x,y\in U$. Therefore $x\gamma y\in Z(M)$ and as above, $M$ is commutative.
Now suppose that $d\neq 0$. If we replace $y$ by $y\alpha r$, then for all $x,y\in U$, $r\in M$ and $\alpha,\gamma\in\Gamma$, we find that $(d(x)\gamma d(y)-x\gamma y)\alpha r+d(x)\gamma y\alpha d(r)\in Z(M)$. Therefore

$$[(d(x)\gamma d(y)-x\gamma y)\alpha r+d(x)\gamma y\alpha d(r),r]=0.$$

Using the given condition, we arrive at

$$[d(x)\gamma y\alpha d(r),r]_\beta=0. \tag{2.10}$$

Replacing $y$ by $d(z)\delta y$ in (2.10), we get

$$[d(x),r]_\beta\gamma d(z)\delta y\alpha d(r)=0 \text{ for all } x,y,z\in U,\ r\in M \text{ and } \alpha,\beta,\delta,\gamma\in\Gamma.$$

Since $M$ is prime, either $U\Gamma d(r)=\{0\}$ or $[d(x),r]_\beta\gamma d(z)=0$. Take $M_1=\{r\in M\mid U\Gamma d(r)=\{0\}\}$ and $M_2=\{r\in M\mid\ [d(x),r]_\beta\gamma d(z)=0 \text{ for all } x,z\in U \text{ and } \beta,\gamma\in\Gamma\}$.
But $M_1$ and $M_2$ are two additive subgroups of $M$ whose union is $M$. Therefore either $M_1=M$ or $M_2=M$. If $M_1=M$, then $U\Gamma d(r)=\{0\}$. Since $U\neq\{0\}$ and $M$ is prime, we find that $d=0$, a contradiction. Hence assume that $M_2=M$. This yields that $[d(x),r]_\beta\gamma d(z)=0$ for all $r\in M$. Hence $[d(x),r\alpha r_1]_\beta\gamma d(z)=0$. This implies that $[d(x),r]_\beta\alpha r_1\gamma d(z)=0$. By primeness of $M$, either $[d(x),r]_\beta=0$ for all $x\in U$, $r\in M$ and $\beta\in\Gamma$ or $d(z)=0$ for all $z\in U$. But $d(z)=0$ gives $d=0$, which is a contradiction. Therefore $[d(x),r]_\beta=0$. In particular, $[d(x),x]_\beta=0$ for all $x\in U$ and $\beta\in\Gamma$. Therefore by Theorem 2.3, $M$ is commutative. □

**Corollary 2.10.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. If $d$ is a derivation on $M$ satisfying $d(x\gamma y)+x\gamma y\in Z(M)$ for all $x,y\in U$ and $\gamma\in\Gamma$, then $M$ is commutative.*

**Proof:** $d(x\gamma y)+x\gamma y\in Z(M)$ implies that $-d(x\gamma y)-x\gamma y\in Z(M)$ i.e., $(-d)(x\gamma y)-x\gamma y\in Z(M)$. Since $-d$ is also a derivation on $M$, hence by Theorem 2.9$(i)$, $M$ is commutative. □

**Corollary 2.11.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. If $d$ is a derivation on $M$ satisfying $d(x\gamma y) + y\gamma x \in Z(M)$ for all $x, y \in U$ and $\gamma \in \Gamma$, then $M$ is commutative.*

**Theorem 2.12.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. If $d$ is a derivation on $M$ such that $d(x \circ_\gamma y) = x \circ_\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$, then $M$ is commutative.*

**Proof:** It is given that $d(x \circ_\gamma y) = x \circ_\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$. If $d = 0$, then $x \circ_\gamma y = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$. Replacing $y$ by $y\alpha z$, we have $x \circ_\gamma (y\alpha z) = 0$ for all $x, y, z \in U$ and $\alpha, \gamma \in \Gamma$. This yields that $y\alpha[z, x]_\gamma = 0$ for all $x, y, z \in U$ and $\alpha, \gamma \in \Gamma$. Since $M$ is prime and $U \neq \{0\}$, $U$ is commutative and by Lemma 2.2, we get the required result.
Now assume that $d \neq 0$. The given condition implies that

$$d(x) \circ_\gamma y + x \circ_\gamma d(y) = x \circ_\gamma y \text{ for all } x, y \in U \text{ and } \gamma \in \Gamma. \tag{2.11}$$

Replace $y$ by $y\alpha z$ in (2.11), we get

$$d(x) \circ_\gamma (y\alpha z) + x \circ_\gamma d(y\alpha z) = x \circ_\gamma (y\alpha z) \text{ for all } x, y, z \in U \text{ and } \alpha, \gamma \in \Gamma.$$

After simplification, we find that

$$(d(x) \circ_\gamma y + x \circ_\gamma d(y))\alpha z + y\alpha[z, d(x)]_\gamma + d(y)\alpha[z, x]_\gamma + (x \circ_\gamma y)\alpha d(z) + y\alpha[d(z), x]_\gamma$$
$$= (x \circ_\gamma y)\alpha z + y\alpha[z, x]_\gamma \text{ for all } x, y, z \in U \text{ and } \alpha, \gamma \in \Gamma.$$

Now using (2.11), we get

$$y\alpha[z, d(x)]_\gamma + d(y)\alpha[z, x]_\gamma + (x \circ_\gamma y)\alpha d(z) + y\alpha[d(z), x]_\gamma = y\alpha[z, x]_\gamma.$$

Replace $z$ by $x$ to get $(x \circ_\gamma y)\alpha d(x) = 0$. Now, replacing $y$ by $w\beta y$, we find that

$$[x, w]_\gamma \beta y \alpha d(x) = 0 \text{ for all } x, y, w \in U \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Since $M$ is prime, either $[x, w]_\gamma = 0$ or $U\Gamma d(x) = \{0\}$. Now using the similar arguments as used in Theorem 2.5($iv$), we find that $M$ is commutative. □

**Corollary 2.13.** *Let $M$ be a prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. If $d$ is a derivation on $M$ such that $d(x \circ_\gamma y) + x \circ_\gamma y = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$, then $M$ is commutative.*

**Theorem 2.14.** *Let $M$ be a 2-torsion free prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $d \neq 0$ is a derivation on $M$ such that $d$ satisfies any one of the following conditions:*

($i$) *$d(x) \circ_\gamma d(y) = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$,*

($ii$) *$d(x) \circ_\gamma d(y) = x \circ_\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$,* *(iii)* $d(x) \circ_\gamma d(y) + x \circ_\gamma y = 0$ *for all* $x, y \in U$ *and* $\gamma \in \Gamma$.

*Then* $M$ *is commutative.*

**Proof:** $(i)$. Replacing $y$ by $y\alpha z$ in the given condition, we get

$$(d(x) \circ_\gamma d(y))\alpha z + d(y)\alpha[z, d(x)]_\gamma + y\alpha(d(x) \circ_\gamma d(z)) + [d(x), y]_\gamma \alpha d(z) = 0.$$

Using the given condition, we have

$$d(y)\alpha[z, d(x)]_\gamma + [d(x), y]_\gamma \alpha d(z) = 0 \text{ for all } x, y, z \in U \text{ and } \alpha, \gamma \in \Gamma. \tag{2.12}$$

Replacing $z$ by $z\beta d(x)$, we get

$$(d(y)\alpha[z, d(x)]_\gamma + [d(x), y]_\gamma \alpha d(z))\beta d(x) + [d(x), y]_\gamma \alpha z \beta d^2(x) = 0.$$

Using (2.12), we get $[d(x), y]_\gamma \alpha z \beta d^2(x) = 0$ for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Primeness of $M$ yields that either $[d(x), y]_\gamma \alpha z = 0$ or $d^2(x) = 0$. Take $U_1 = \{x \in U \mid d^2(x) = 0\}$ and $U_2 = \{x \in U \mid [d(x), y]_\gamma \alpha z = 0$ for all $y, z \in U$ and $\alpha, \gamma \in \Gamma\}$. Since $U_1$ and $U_2$ are additive subgroups of $U$ such that $U_1 \cup U_2 = U$. Therefore by Brauer's trick either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $d^2(x) = 0$ for all $x \in U$. Therefore by using the arguments as used in the proof of Theorem 2.5$(i)$, $d = 0$ which is a contradiction. Now assume that $U_2 = U$ i.e., $[d(x), y]_\gamma \alpha z = 0$ for all $x, y, z \in U$ and $\alpha, \gamma \in \Gamma$. Since $U \neq \{0\}$ and $M$ is prime, $[d(x), y]_\gamma = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$. Hence $[d(x), x]_\gamma = 0$ for all $x \in U$ and $M$ is commutative by Theorem 2.3.

$(ii)$. If $d = 0$, then $x \circ_\gamma y = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$. Therefore $M$ is commutative by the argument used in the Theorem 2.12. Now assume that $d \neq 0$. Replace $y$ by $y\alpha z$ to get

$$d(y)\alpha[z, d(x)]_\gamma + y\alpha(x \circ_\gamma z) + [d(x), y]_\gamma \alpha d(z) - y\alpha[z, x]_\gamma = 0 \tag{2.13}$$

for all $x, y, z \in U$ and $\alpha, \gamma \in \Gamma$.
Replacing $y$ by $r\beta y$ in (2.13), we find that

$$r\beta(d(y)\alpha[z, d(x)]_\gamma + y\alpha(x \circ_\gamma z) + [d(x), y]_\gamma \alpha d(z) - y\alpha[z, x]_\gamma) + d(r)\beta y\alpha[z, d(x)]_\gamma$$
$$+ [d(x), r]_\gamma \beta y \alpha d(z) = 0 \text{ for all } x, y, z \in U,\ r \in M \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Using (2.13), the above yields that

$$d(r)\beta y\alpha[z, d(x)]_\gamma + [d(x), r]_\gamma \beta y \alpha d(z) = 0.$$

Further replacing $r$ by $d(x)$, we get $d^2(x)\beta y\alpha[z, d(x)]_\gamma = 0$ for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Take $U_1 = \{x \in U \mid d^2(x) = 0\}$ and $U_2 = \{x \in U \mid U\Gamma[z, d(x)]_\gamma = \{0\}$ for all $z \in U$ and $\gamma \in \Gamma\}$. If $U_1 = U$ then $d^2(x) = 0$ for all $x \in U$. Using similar techniques as used in Theorem 2.5$(i)$ we get $d = 0$, a contradiction. Therefore $U_2 = U$. Hence $U\Gamma[z, d(x)]_\gamma = \{0\}$ for all $x, z \in U$ and $\gamma \in \Gamma$. Since $M$ is prime and $U \neq \{0\}$, $[z, d(x)]_\gamma = 0$. Hence $[d(x), x]_\gamma = 0$ for all $x \in U$ and $\gamma \in \Gamma$ and $M$ is commutative by Theorem 2.3.

$(iii)$. By the similar arguments as used in $(ii)$, we can get the required result. □

**Theorem 2.15.** *Let $M$ be a 2-torsion free prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $d \neq 0$ is a derivation on $M$ such that $d$ satisfies any one of the following condition:*

*$(i)$ $[d(x), d(y)]_\gamma = y\gamma x$ for all $x, y \in U$ and $\gamma \in \Gamma$,*

*$(ii)$ $[d(x), d(y)]_\gamma = x\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$,*

*$(iii)$ $d([x, y]_\gamma) = x \circ_\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$,*

*$(iv)$ $d(x \circ_\gamma y) = [x, y]_\gamma$ for all $x, y \in U$ and $\gamma \in \Gamma$.*

*Then $M$ is commutative.*

**Proof:** $(i)$. Replacing $y$ by $y\alpha w$ in the given condition, we find that

$$y\gamma x\alpha w + d(y)\alpha[d(x), w]_\gamma + [d(x), y]_\gamma\alpha d(w) = 0 \text{ for all } x, y, w \in U \text{ and } \alpha, \gamma \in \Gamma.$$

Further, replacing $w$ by $w\delta r$ and using the same, we get

$$d(y)\alpha w\delta[d(x), r]_\gamma + [d(x), y]_\gamma\alpha w\delta d(r) = 0 \text{ for all } x, y, w \in U,\ r \in M \text{ and } \alpha, \gamma, \delta \in \Gamma.$$

Now, replacing $r$ by $d(x)$, we get $[d(x), y]_\gamma\alpha w\delta d^2(x)$ for all $x, y, w \in U$ and $\alpha, \gamma, \delta \in \Gamma$. Since $M$ is prime, we find that either $[d(x), y]_\gamma\alpha U = \{0\}$ or $d^2(x) = 0$. Take $U_1 = \{x \in U \mid [d(x), y]_\gamma\alpha U = \{0\}$ for all $y \in U,\ \alpha, \gamma \in \Gamma\}$ and $U_2 = \{x \in U \mid d^2(x) = 0\}$. But $U_1$ and $U_2$ are additive subgroups of $U$ such that $U_1 \cup U_2 = U$. Hence, by Brauer's trick either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $[d(x), y]_\gamma\alpha U = \{0\}$. Since $U \neq \{0\}$, $[d(x), y]_\gamma = 0$ for all $x, y \in U$ and $\gamma \in \Gamma$. In particular $[d(x), x]_\gamma = 0$ for all $x \in U$ and $\gamma \in \Gamma$. Therefore $M$ is commutative by Theorem 2.3. If $U_2 = U$, then $d = 0$, a contradiction.

$(ii)$. By using the similar arguments as used in proving $(i)$, we get the required result.

$(iii)$. Given that $d([x, y]_\gamma) = x \circ_\gamma y$ for all $x, y \in U$ and $\gamma \in \Gamma$. On simplifying we get

$$[d(x), y]_\gamma + [x, d(y)]_\gamma = x \circ_\gamma y \text{ for all } x, y \in U \text{ and } \gamma \in \Gamma. \tag{2.14}$$

Further, replacing $y$ by $y\alpha x$, we find that $[x, y]_\gamma\alpha d(x) = 0$. Again replacing $y$ by $r\beta y$, we get

$$[x, r]_\gamma\beta y\alpha d(x) = 0 \text{ for all } x, y \in U, r \in M \text{ and } \alpha, \beta, \gamma \in \Gamma. \tag{2.15}$$

Take $U_1 = \{x \in U \mid [x,r]_\gamma = 0$ for all $r \in M$ and $\gamma \in \Gamma\}$ and $U_2 = \{x \in U \mid U\Gamma d(x) = 0\}$. Since $U_1$ and $U_2$ are additive subgroups of $U$ such that $U_1 \cup U_2 = U$ and by Brauer's trick either $U_1 = U$ or $U_2 = U$. If $U_1 = U$ then $[x,r]_\gamma = 0$ for all $x \in U$, $r \in M$, $\gamma \in \Gamma$, and hence $U \subseteq Z(M)$. Therefore $M$ is commutative by Lemma 2.1. Now, we assume that $U_2 = U$. Since $M$ is prime and $U \neq 0$, we find that $d = 0$, a contradiction.

$(iv)$. It is given that $d(x \circ_\gamma y) = [x,y]_\gamma$. This implies that

$$d(x) \circ_\gamma y + x \circ_\gamma d(y) = [x,y]_\gamma \text{ for all } x,y \in U \text{ and } \gamma \in \Gamma.$$

Replace $y$ by $x\alpha y$ to get

$$d(x)\gamma x\alpha y + d(x)\alpha y\gamma x = 0 \text{ for all } x,y \in U \text{ and } \alpha,\gamma \in \Gamma.$$

Again, replacing $y$ by $y\beta r$, we find that

$$d(x)\gamma y\alpha[r,x]_\beta = 0 \text{ for all } x,y \in U,\ r \in M \text{ and } \alpha,\beta,\gamma \in \Gamma.$$

Now using the similar arguments as used after (2.15), we get the required result.□

**Theorem 2.16.** *Let $M$ be a 2-torsion free prime $\Gamma$-ring and $U$ be a nonzero ideal of $M$. Suppose $d$ is a derivation on $M$ such that $d$ satisfies any one of the following condition:*

*(i) $d(x)\gamma d(y) = [x,y]_\gamma$ for all $x,y \in U$ and $\gamma \in \Gamma$,*

*(ii) $d(y)\gamma d(x) = [x,y]_\gamma$ for all $x,y \in U$ and $\gamma \in \Gamma$,*

*(iii) $d(x)\gamma d(y) = x \circ_\gamma y$ for all $x,y \in U$ and $\gamma \in \Gamma$.*

*Then $M$ is commutative.*

**Proof:** $(i)$. Replacing $y$ by $y\alpha r$ in the given condition, we get

$$d(x)\gamma d(y)\alpha r + d(x)\gamma y\alpha d(r) = [x,y]_\gamma\alpha r + y\alpha[x,r]_\gamma$$

for all $x,y \in U$, $r \in M$ and $\alpha,\gamma \in \Gamma$.
Now, using the given condition, we get

$$d(x)\gamma y\alpha d(r) = y\alpha[x,r]_\gamma \text{ for all } x,y \in U,\ r \in M \text{ and } \alpha,\gamma \in \Gamma.$$

Further, replacing $r$ by $r+x$, we get $d(x)\gamma y\alpha d(x) = 0$ for all $x,y \in U$ and $\alpha,\gamma \in \Gamma$. Since $M$ is prime and $U \neq \{0\}$, $d(x) = 0$ for all $x \in U$. Hence our hypothesis implies that $[x,y]_\gamma = 0$ for all $x,y \in U$ and $\gamma \in \Gamma$ i.e., $U$ is commutative. Therefore by Lemma 2.2, $M$ is commutative.

By the similar arguments as used in $(i)$, we get the required result in cases $(ii)$ and $(iii)$. □

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