

Analytical approximations for short rate models

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October 1, 2010

Abstract

In this article, we present the analytical approximation of zero-coupon bonds and swaption prices for general short rate models. The approximation is based on regular and singular expansions with respect to the small volatility and contains a low-dimensional integration. The model in hand assumes the short rate is an arbitrary function of a multi-dimensional Gaussian underlying process. The high approximation accuracy is confirmed by numerical experiments. We have treated two special classes of the model. The first one is a Generalized multi-factor Black-Karasinski (BK) model. The second one is a new Bounded short rate model where the rates evolve between certain user-defined limits. This model is particularly attractive for scenario generation and, due to the proposed swaption approximation, can be easily calibrated to the implied market.

1 Introduction

Short rate models underlie the first steps of quantitative finance development. The main feature of such models consists of postulating the short rate process. Vasicek [7] and Hull and White [3] pioneered a Gaussian short rate model still popular among practitioners due its analytical tractability and transparency. Black and Karasinski [1] have proposed a log-normal short rate model. Both models share the same Gaussian mean-reverting process but with different interpretations in terms of the short rate. Later, there appeared multi-factor generalizations (see, for example, [4]) as well as functional generalizations, for example, generalized Black-Karasinski (BK) [5] and quadratic short rate model [6]. One can also mention the exactly solvable Cox-Ingersoll-Ross (CIR) short rate model [2].

In this article, we consider a multi-factor mean-reverting Gaussian underlying process and view the short rate as being an arbitrary function of it. We refer to such models as generalized short rate models with Gaussian underlying (GSRG). Note that the CIR model remains slightly apart from the GSRG framework because its short rate cannot be represented as a deterministic function of a Gaussian process.

In general, the GSRG models do not permit exact analytical interpretation for both zero-coupon bonds and swaption prices¹ but have attractive features of implied volatility skew control. Note that the classic HW model has a fixed normal skew and the BK model possesses the almost-flat implied volatility form due its quasi log-normality.

The current post-crisis market favors simple models having, however, several essential elements such as the presence of a multi-factor underlying (to treat correlations) and the implied volatility skew control. A usage of the arbitrage free models was recently extended to economic scenario generation, especially in

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¹Except the Hull and White model and the Piterbarg quadratic model [6].

the insurance industry. However, the standard models lead to undesirably extreme rates. Thus, a short rate model with *bounded* evolution of the rates can be attractive for practitioners.

The GSRG models can satisfy these properties and thus become adequate for the current market environment. What has been lacking is analytical tractability. Indeed, in spite of the simplicity of the lattice implementation for the GSRG model, analytical expressions for swaption prices necessary for efficient calibration remain obscure. A step forward in this direction is the analytical expression for zero-coupon bonds for the generalized BK model. Tourruco, Hagan and Schleiniger [5] have derived the analytical approximation for the one-factor case.

In this article, we go further and come up with the multi-dimensional case analytical formulas for *both* zero-coupon bond and European swaption prices. Differences with the Tourruco, Hagan and Schleiniger approach, such as the underlying process drift adaptation to the yield curve, will be explained in the main body.

We illustrate the method on two examples. The first one is our version of the Generalized multi-factor Black-Karasinski (BK) model allowing for skew control. The second example is a new Bounded short rate model where the rates evolve between certain user-defined limits.

The paper is organized as follows. In Section 2, we introduce notations, present swaption pricing logic, and derive the main PDE's. Section 3 contains the main results—quadrature formulas for the zero-coupon bond and the Arrow-Debreu (AD) price. In Section 4, we give two examples of the short rate models: our version of the Generalized multi-factor Black-Karasinski (BK) model and the Bounded short rate model. Finally, Section 5 contains a numerical confirmation of the approximation accuracy, followed by conclusions in Section 6.

2 Preliminary remarks

In this section, we introduce the main notations, derive the PDEs and determine the expansion strategy.

Denote an underlying F -factor mean-reverting process as

$$dx_i(t) = (\phi_i(t) - a_i(t)x_i(t))dt + \lambda_i(t) \cdot dW(t) \quad \text{for } i = 1, \dots, F \quad (1)$$

for the vector volatilities $\lambda_i(t)$, vector uncorrelated Brownian motions $dW(t)$, mean-reversions $a_i(t)$, and drifts $\phi_i(t)$, serving to link the short rate process to a given yield curve. The dot operation \cdot defines the inner vector product.

An equivalent, alternative form of the underlying can contain correlated scalar Brownian motions $dW_i(t)$ and scalar volatilities $\gamma_i(t)$, i.e., $dx_i(t) = (\phi_i(t) - a_i(t)x_i(t))dt + \gamma_i(t)dW_i(t)$.

We postulate the short rate process as an arbitrary function of the underlying processes,

$$r(t) = f(t, x). \quad (2)$$

Having sufficient freedom, we choose the zero initial values $x_i(0) = 0$.

Now we define the numeraire process, a savings account,

$$N_t = e^{\int_0^t ds r(s)}.$$

A zero-coupon bond with maturity T equals

$$P(t, T) = N_t E \left[\frac{1}{N_T} \mid F_t \right] = E \left[e^{-\int_t^T ds r(s)} \mid F_t \right].$$

In other words, the zero-coupon bond is a function of the underlying process. Here, and throughout the paper, we refer to the operator $E[\dots]$ as the expectation in the *risk-neutral* measure.

The discount factor $D(T) \equiv P(0, T)$ is a zero-coupon bond maturing at T as seen from the origin. Denote today's forward rate as $R(t)$,

$$D(T) = e^{-\int_0^T ds R(s)}.$$

Then, without loss of generality, we can always represent² the short rate “moving” around today's forward rate,

$$f(t, 0) = R(t). \quad (3)$$

The discount factor is considered as a model input and the drift parameters $\phi_i(t)$ should be chosen to fit it,

$$D(T) = E \left[e^{-\int_0^T ds f(s, x(s))} \right]. \quad (4)$$

Note that the drifts will have the second order in volatilities due to (3). Indeed, one should expand the short rate in the underlying variables,

$$f(t, x) = R(t) + \sum_i \frac{\partial f(t, 0)}{\partial x_i} x_i + \dots,$$

substitute it into (4), and use the fact that the average does not change with the sign of the volatility.

It is convenient to remove the mean-reverting term from the underlying x 's using an appropriate multiplier,

$$y_i(t) = x_i(t) A_i(t) \quad \text{for} \quad A_i(t) = e^{\int_0^t ds a_i(s)}.$$

Normal process y 's satisfy an SDE,

$$dy_i(t) = \theta_i(t) dt + \sigma_i(t) \cdot dW(t), \quad (5)$$

where

$$\theta_i(t) = \phi_i(t) A_i(t) \quad \text{and} \quad \sigma_i(t) = \lambda_i(t) A_i(t).$$

Obviously, the process y 's start with zero values, $y_i(0) = 0$.

The short rate (2) in terms of the Gaussian y 's will look as follows,

$$r(t) = r(t, y) \equiv f(t, \{A_i(t)^{-1} y_i(t)\}). \quad (6)$$

To calculate a generalized option price with pay-off at time T ,

$$\left(\sum_n \pi_n P(T, T_n) \right)^+,$$

one should be able to evaluate the expectation

$$V = E \left[\frac{(\sum_n \pi_n P(T, T_n))^+}{N_T} \right].$$

Note that the payment coefficients π_n could have opposite signs for non-trivial optionality.

After some experiments with the measure choices, we have concluded that the most efficient way is to proceed in the risk-neutral measure. We use the Arrow-Debreu price $q(t, u)$,

$$q(t, u) \equiv E \left[\delta(y(t) - u) e^{-\int_0^t ds r(s)} \right] = E \left[e^{-\int_0^t ds r(s)} \mid y(t) = u \right] E [\delta(y(t) - u)],$$

²One can use a proper shift, $x_i(t) \rightarrow x_i(t) + \int_0^t ds \mu_i(s)$, for some functions $\mu_i(s)$.

where u is a vector and the multi-dimensional delta function is defined by the product $\delta(y(t) - u) = \prod_{i=1}^F \delta(y_i(t) - u_i)$. It permits us to present the option price as a multi-dimensional integral,

$$V = \int du \left(\sum_n \pi_n P(T, T_n; u) \right)^+ q(T, u). \quad (7)$$

Thus, our goal is to approximate the zero-coupon bond price $P(t, T; u)$ and the Arrow-Debreu price $q(t, u)$ as functions of arbitrary u . Then, the integration (7) will be performed numerically.

At the end of this section, we present differential equations for both components of the interest. The zero-coupon bond price will correspond to the following PDE,

$$\left(\partial_t - r(t, u) + \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} \sigma_i(t) \cdot \sigma_j(t) \partial_i \partial_j \right) P(t, T; u) = 0, \quad (8)$$

with the final conditions $P(T, T; u) = 1$ and $\partial_i \equiv \partial_{u_i}$. The PDE can be easily derived by Ito's formula requiring zero drift of the martingale $e^{-\int_0^t ds r(s)} P(t, T; y(t))$.

The Arrow-Debreu price, $q(t, u)$, satisfies the conjugate forward PDE

$$\partial_t q(t, u) = \left(-r(t, u) - \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} \sigma_i(t) \cdot \sigma_j(t) \partial_i \partial_j \right) q(t, u), \quad (9)$$

with the initial condition $q(0, u) = \prod_i \delta(u_i - y_i(0))$. Indeed, the integral $P(0, T) = \int du P(t, T; u) q(t, u)$ is independent of time t . After its differentiation over time, we obtain the desired PDE (9).

The corresponding parabolic operator for the backward PDE $(\partial_t + L(t, u))P(t, T; u) = 0$, reads

$$L(t, u) = -r(t, u) + \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j, \quad (10)$$

where we denoted the instantaneous covariance matrix as

$$C_{ij}(t) = \sigma_i(t) \cdot \sigma_j(t). \quad (11)$$

The conjugate operator for the forward PDE has the form $(\partial_t - L^*(t, u))q(t, u) = 0$,

$$L^*(t, u) = -r(t, u) - \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j. \quad (12)$$

As we will see below, the drifts θ_i have orders of σ^2 . This we will use to apply the perturbation technique to the PDE's above, considering $\theta \rightarrow \varepsilon\theta$ and $\sigma^2 \rightarrow \varepsilon\sigma^2$.

3 Main results

In this section, we present the main results—quadrature formulas for the zero-coupon bond and the AD price. We use two types of the expansion technique: regular for the zero-coupon bonds and singular for the AD price.

3.1 Zero-coupon bond

Introducing ε in the backward parabolic operator

$$L(t, u) = -r(t, u) + \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j, \quad (13)$$

we denote the unperturbed operator as $L_0(t)$ and a perturbation as $L_1(t)$,

$$\begin{aligned} L_0(t) &= -r(t, u), \\ L_1(t) &= \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j. \end{aligned}$$

The solution of the unperturbed equation

$$(\partial_t + L_0(t)) P_0(t, T, u) = 0 \quad (14)$$

for the unit final value $P_0(T, T, u) = 1$ is very simple,

$$P_0(t, T, u) = \exp \left(- \int_t^T d\tau r(\tau, u) \right). \quad (15)$$

The first and second corrections can be obtained using the regular perturbation technique³,

$$\begin{aligned} P_1(t, T, u) &= \varepsilon P_0(t, T, u) \\ &\times \int_t^T ds \left\{ - \sum_i \theta_i(s) g_i(s, T, u) + \frac{1}{2} \sum_{i,j} C_{ij}(s) (g_i(s, T, u) g_j(s, T, u) - g_{ij}(s, T, u)) \right\} \end{aligned} \quad (16)$$

with

$$g_i(s, T, u) = \int_s^T d\tau \partial_i r(\tau, u) \quad \text{and} \quad g_{ij}(s, T, u) = \int_s^T d\tau \partial_i \partial_j r(\tau, u).$$

Let us stress that the short rate $r(t, u)$ and its derivatives are functions of the normal underlyings y ; in order to handle the initial dependence of the underlying mean-reverting process x one should use formula (6).

Note that Tourrucôo, Hagan and Schleiniger have presented the zero-coupon bond in *exponential* form,

$$P(t, T; u) = P_0(t, T, u) \exp \{ \varepsilon \Phi_1(t, T; u) + \varepsilon^2 \Phi_2(t, T; u) + O(\varepsilon^3) \},$$

which can sometimes deliver explosive values, especially for large maturities and volatilities.

To finalize the zero-coupon bond approximation, we should fix the drifts, unknown so far, to reproduce the yield curve or discount factor $D(T)$. Equivalently, we can calculate the y 's averages, $E[y_i(t)] = m_i(t) = \int_0^t ds \theta_i(s)$, requiring the fit

$$D(T) = P_0(0, T, 0) + P_1(0, T, 0)$$

to the leading order in the volatilities. Noticing that

$$P_0(0, T, 0) = \exp \left(- \int_0^T d\tau r(\tau, 0) \right) = \exp \left(- \int_0^T d\tau R_\tau \right) = P(0, T),$$

³See Appendix A.

we conclude that, in order to fit the yield curve, we should choose the averages $m_i(t)$ s.t. the second-order correction equals zero, i.e.,

$$P_1(0, T, 0) = 0$$

or

$$\int_0^T ds \sum_i \theta_i(s) g_i(s, T, 0) = \frac{1}{2} \int_0^T ds \sum_{i,j} C_{ij}(s) (g_i(s, T, 0) g_j(s, T, 0) - g_{ij}(s, T, 0)).$$

Differentiating over T , we obtain

$$\sum_i \partial_i r(T, 0) m_i(T) = \sum_{i,j} \partial_i r(T, 0) \int_0^T d\tau \partial_j r(\tau, 0) V_{ij}(\tau) - \frac{1}{2} \sum_{i,j} \partial_i \partial_j r(T, 0) V_{ij}(T), \quad (17)$$

where $V_{ij}(T)$ is the y 's covariance

$$E[(y_i(t) - m_i(t))(y_j(t) - m_j(t))] = V_{ij}(t) = \int_0^t ds C_{ij}(s).$$

Either solution of equation (17) for individual averages $m_i(T)$ is suitable for the approximation. Note that Tourrucôo, Hagan and Schleiniger have matched the yield curve by a rough multiplicative adjustment that does not fix the underlying drifts.

In Appendix A, we also present the second order correction, although, according to our experiments, the *second* order approximation (precision $O(\varepsilon^2)$) does not deliver a reasonable quality increase but can substantially slow down the procedure.

3.2 AD price

In the case of the AD price, we deal with a singular perturbation technique for the parabolic operator

$$L^*(t, u) = -r(t, u) - \varepsilon \sum_i \theta_i(t) \partial_i + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j. \quad (18)$$

The AD price satisfies the forward PDE

$$(\partial_t - L^*(t, u)) q(t, u) = 0$$

with the initial condition $q(0, u) = \prod_i \delta(u_i - y_i(0))$.

Applying the Dyson technique (Appendix A) would require careful calculation of *all* terms in ε due to *singularity* of the initial value. Instead, we use the Heat-Kernel singular expansion method.

It is easy to check that the Gaussian density

$$q_0(t, u) = \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}(u - \varepsilon m(t))^T V^{-1}(t)(u - \varepsilon m(t))\right)}{(2\pi\varepsilon)^{\frac{n}{2}} |V(t)|^{\frac{1}{2}}},$$

where $m(t)$ is the $y(t)$ center and $V(t)$ is the covariance in vector/matrix notations, satisfies the PDE

$$\partial_t q_0(t, u) = -\varepsilon \sum_i \theta_i(t) \partial_i q_0(t, u) + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j q_0(t, u). \quad (19)$$

Now, following the Heat-Kernel expansion logic, we look for the solution as

$$q(t, u) = q_0(t, u) \Omega(t, u),$$

where $\Omega(t, u)$ is a *regular* function of ε with the unit initial condition $\Omega(0, u) = 1$. After substitution into the initial PDE, we get the solution for the AD price up to the first order in small volatilities,

$$\begin{aligned}
q(t, u) &= \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}(u - \varepsilon m(t))^T V^{-1}(t)(u - \varepsilon m(t))\right)}{(2\pi\varepsilon)^{\frac{n}{2}} |V(t)|^{\frac{1}{2}}} \exp\left(-\int_0^t ds r(s, V(s) V^{-1}(t) u)\right) \\
&\times \left\{ 1 + \varepsilon \int_0^t d\tau (\theta^T(\tau) - m^T(\tau) V^{-1}(\tau) C(\tau)) V^{-1}(\tau) \hat{h}^{(1)}(\tau; t, u) \right. \\
&+ \left. \frac{1}{2}\varepsilon \int_0^t d\tau \operatorname{tr} V^{-1}(\tau) C(\tau) V^{-1}(\tau) \left[\hat{h}^{(1)}(\tau; t, u) \otimes \hat{h}^{(1)}(\tau; t, u) - \hat{h}^{(2)}(\tau; t, u) \right] \right\}, \quad (20)
\end{aligned}$$

where vector

$$\hat{h}_n^{(1)}(\tau; t, u) = \sum_j \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k r(s, V(s) V^{-1}(t) u)$$

and matrix

$$\hat{h}_{nl}^{(2)}(\tau; t, u) = \sum_{km} \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k \tilde{\partial}_m r(s, V(s) V^{-1}(t) u) V_{ml}(s).$$

Note that the main formula contains vector/matrix notations including standard linear algebra products. For example, outer product $a \otimes b$ of two vectors means a matrix with elements $a_i b_j$. Here, the derivative $\tilde{\partial}_k$ denotes the short rate derivative over the *space argument*, $\tilde{\partial}_i r(s, z(u)) = \frac{\partial r(s, z(u))}{\partial z_i(u)}$.

The calculation details are given in Appendix B.

4 Model examples

In this section, we give two GSRG model examples. One of them is our version of the Generalized multi-factor Black-Karasinski model and the second one is the Bounded short rate model, which can be suitable for scenario generation where the rates evolve inside given boundaries.

We will use a *symmetric* form of the GSHG models,

$$r(t) = f\left(t, \sum_i x_i\right).$$

As mentioned in Section 2, we can always represent the short rate function $f(t, x)$ as being equal to the forward rate for zero values of the underlying process x . Similar scaling arguments can be applied to set the space derivatives of the short rate to one for $x = 0$. These two properties,

$$f(t, 0) = R(t) \quad \text{and} \quad \partial_i f(t, 0) = 1, \quad (21)$$

are convenient to analyze the calibration results. Indeed, typical underlying x parameters correspond to the *normal* case, i.e., volatilities have 1-1.5% order of magnitude for any short rate.

4.1 Generalized multi-factor Black-Karasinski model

We propose a version of Generalized multi-factor Black-Karasinski (BK) model having many attractive features, including swaption skew control and multiple factors permitting decorrelation of the rates. The short rate is presented in the form

$$r(t) = f(t, x) = R(t) + \frac{e^{S(t) \sum_i x_i} - 1}{S(t)}, \quad (22)$$

where $R(t)$ is today's forward rate and positive function $S(t)$ is a shift parameter controlling the swaption skew. For a special case of $S(t) = R^{-1}(t)$, the model degenerates to the standard multi-factor BK model,

$$r(t) = R(t) e^{R^{-1}(t) \sum_i x_i},$$

whereas when the shift parameter tends to zero, we obtain the Hull-White (HW) model,

$$r(t) = R(t) + \sum_i x_i.$$

Note that typical underlying x parameters correspond to the *normal* case, i.e., volatilities have 1-1.5% order of magnitude for any shift. This is another attractive feature of the proposed model.

The analytical expressions can be calculated using the general formulas for the zero-coupon bond (15-16) and the AD price (20).

Explicit formulas for the main derivatives are given by

$$g_i(s, T, u) = \int_s^T d\tau \partial_i r(\tau, u) = \int_s^T d\tau A_i(\tau)^{-1} \mathcal{E}(\tau, u)$$

and

$$g_{ij}(s, T, u) = \int_s^T d\tau \partial_i \partial_j r(\tau, u) = \int_s^T d\tau A_i(\tau)^{-1} A_j(\tau)^{-1} S_\tau \mathcal{E}(\tau, u),$$

where we have denoted

$$\mathcal{E}(\tau, u) = e^{S_\tau \sum_i A_i(\tau)^{-1} u}.$$

The conditions for the drifts simplify to

$$\sum_i A_i^{-1}(T) m_i(T) = \sum_{i,j} A_i^{-1}(T) \int_0^T d\tau A_j^{-1}(\tau) V_{ij}(\tau) - \frac{1}{2} S_T \sum_{i,j} A_i^{-1}(T) A_j^{-1}(T) V_{ij}(T).$$

In this case, one can choose a ‘‘symmetric’’ simplification,

$$m_i(T) = \sum_j \int_0^T d\tau V_{ij}(\tau) A_j^{-1}(\tau) - \frac{1}{2} S_T \sum_j V_{ij}(T) A_j^{-1}(T).$$

4.2 Bounded short rate model

Sometimes practitioners, especially from the insurance industry, are interested in a bounded rates evolution for scenario generation. The current short rate models do not meet that bounded requirement: There exists a probability such that rates can exceed an a priori-given (positive) value.

We propose the following Bounded short rate model,

$$r(t) = f(t, x) = \frac{U(t) e^{\beta(t) \sum_i x_i} + L(t) \alpha(t)}{e^{\beta(t) \sum_i x_i} + \alpha(t)}, \quad (23)$$

where time-dependent functions $L(t)$ and $U(t)$ are the lower and upper boundaries, respectively. As usual, the forward rate is denoted as $R(t)$. To ensure the attractive properties of $f(t, 0) = R(t)$ and $f'(t, 0) = 1$, we define the functions $\alpha(t)$ and $\beta(t)$ as

$$\alpha(t) = \frac{U(t) - R(t)}{R(t) - L(t)}$$

and

$$\beta(t) = \frac{U(t) - L(t)}{(R(t) - L(t))(U(t) - R(t))}.$$

We see that the function (23) is a *monotone* function of $X = \sum_i x_i$ and its inverse has a simple form,

$$X = \ln \left(\frac{U(t) - R(t)}{U(t) - F} \frac{F - L(t)}{R(t) - L(t)} \right) \beta^{-1}(t),$$

for $F = f(t, X)$.

Obviously, the short rate has two desired limits: $\lim_{X \rightarrow -\infty} f(t, X) = L(t)$ and $\lim_{X \rightarrow +\infty} f(t, X) = U(t)$.

A special case of the Bounded short rate model $U(t) \rightarrow +\infty$ represents the Generalized multi-factor BK model (22) for the skew parameter $S(t) = (R(t) - L(t))^{-1}$. We see that, for this special case, the skewness of the implied volatility depends on the bounds. This property holds also in the general bounded-model setup. To verify it explicitly, consider the short rate SDE,

$$dr(t) = \dots dt + \frac{(r(t) - L(t))(U(t) - r(t))}{(R(t) - L(t))(U(t) - R(t))} \sum_i dx_i(t).$$

The diffusion coefficient $D(t, r) \sim \frac{(r(t) - L(t))(U(t) - r(t))}{(R(t) - L(t))(U(t) - R(t))}$, or local volatility, can give an approximate form of the implied volatility smile as a function of strike K ,

$$\sigma_I(K) \sim \frac{D\left(t, \frac{R(t)+K}{2}\right)}{\frac{R(t)+K}{2}}.$$

We see that the implied volatility form depends on the boundaries. Thus, if one fixes the boundaries, one loses the skew control and the model can be calibrated only to one point of the implied volatility for a given swaption (say ATM). On the other hand, fixing, say, the *lower boundary*, one can still calibrate to the option skew using a residual freedom in the *upper boundary*.

5 Numerical experiments

In this section, we present numerical results for swaption prices comparing our analytical approximation (“Approx”) with the exact answer (“Exact”) obtained in a high-quality least-squares Monte Carlo simulation.

The approximation is performed in the *first* order in variances,

$$P(t, T; u) = P_0(t, T; u) + P_1(t, T; u), \quad \text{from (15-16),}$$

for the zero-coupon bond and $q(t, u)$ AD price defined in (20). Note that, according to our experiments, the *second* order approximation does not deliver a reasonable quality increase but can substantially slow down the procedure. The integration

$$V = \int du \left(\sum_n \pi_n P(T, T_n; u) \right)^+ q(T, u)$$

is taken numerically using, for example, an adaptive grid. Note that it can be useful to perform a multiplicative adjustment for the zero-coupon bond prices to obtain the known forwards.

5.1 Generalized multi-factor BK model

We consider a two-factor (2F) case of the Generalized multi-factor BK model (22),

$$r(t) = R(t) + \frac{e^{S(t)(x_1+x_2)} - 1}{S(t)}, \quad (24)$$

with the underlying processes

$$dx_i(t) = (\phi_i(t) - a_i(t) x_i(t)) dt + \gamma_i(t) dW_i(t), \quad \text{for } i = 1, 2,$$

starting from zero, $x_i(0) = 0$, with a certain correlation between Brownian motions,

$$\rho(t) = E[dW_1(t) dW_2(t)].$$

We use standard data, summarized in the table below.

Rate	$R(t)$	5.00%
Correlation	$\rho(t)$	-90.00%
Shift	$S(t)$	10
Volatility 1	$\gamma_1(t)$	1.00%
Volatility 2	$\gamma_2(t)$	1.75%
Reversion 1	$a_1(t)$	50.00%
Reversion 2	$a_2(t)$	5.00%

We have chosen particular time-independent parameters for transparency, however, the proposed approximation works for general time-independent parameters. Note that the shift parameter corresponds to a skew situated between normal and log-normal ones. Recall that the normal model is described with the zero shift parameter and the log-normal model with the shift equal to the inverse rate, $S(t) = R^{-1}(t) = 20$.

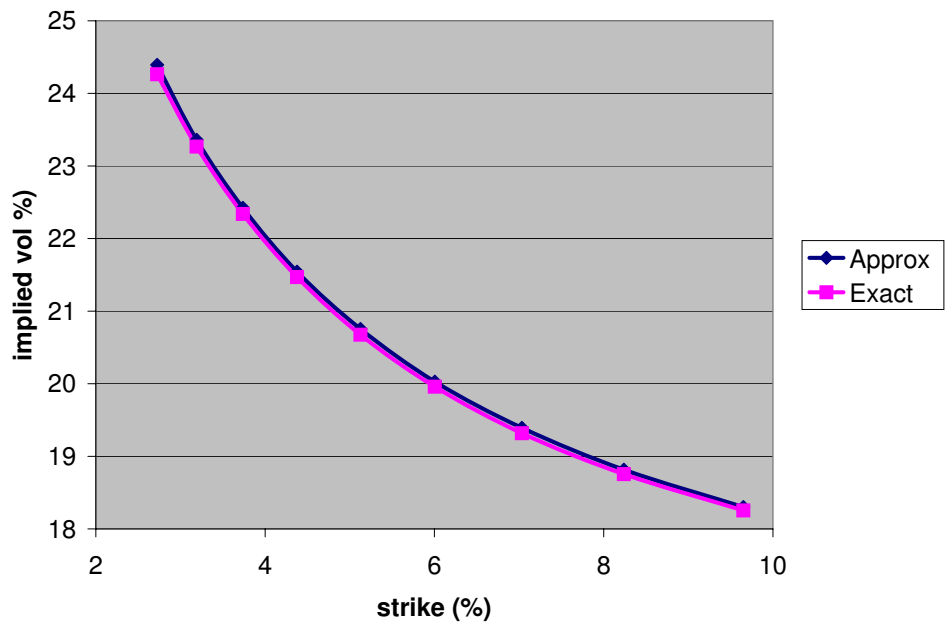


Figure 1: Comparison of the implied vols for 10Y10Y European swaption (Generalized 2F BK model).

We have tested European swaptions with annual period, 10Y length, multiple strikes and maturities 5Y, 10Y and 20Y. The results for Black implied volatilities are presented in the table below. The ATM swaptions are marked in bold.

Exer (Y)	Length (Y)	Strike (%)	Implied vol (%)		Error (bps)
			Approx	Exact	
5	10	3.28	25.77	25.71	5
5	10	3.67	25.04	24.99	6
5	10	4.10	24.34	24.31	3
5	10	4.59	23.69	23.67	2
5	10	5.13	23.09	23.09	0
5	10	5.73	22.53	22.54	-1
5	10	6.41	22.01	22.01	0
5	10	7.17	21.52	21.50	2
5	10	8.02	21.10	21.06	4
10	10	2.72	24.39	24.26	13
10	10	3.19	23.36	23.27	9
10	10	3.74	22.42	22.34	8
10	10	4.38	21.54	21.47	7
10	10	5.13	20.75	20.67	8
10	10	6.01	20.03	19.96	7
10	10	7.03	19.39	19.32	7
10	10	8.24	18.82	18.75	6
10	10	9.65	18.30	18.25	5
20	10	2.10	21.59	21.24	35
20	10	2.62	20.20	19.89	31
20	10	3.28	18.94	18.68	26
20	10	4.10	17.87	17.62	26
20	10	5.13	16.91	16.68	23
20	10	6.41	16.05	15.86	19
20	10	8.02	15.35	15.15	20
20	10	10.03	14.72	14.57	15
20	10	12.54	14.19	14.09	11

5.2 Bounded short rate model

We consider a one-factor (1F) case of the Bounded short rate model (23),

$$r(t) = f(t, x) = \frac{U(t)e^{\beta(t)x} + L(t)\alpha(t)}{e^{\beta(t)x} + \alpha(t)}, \quad (25)$$

with the underlying process

$$dx(t) = (\phi(t) - a(t)x(t)) dt + \gamma(t) dW(t),$$

starting from zero, $x(0) = 0$.

We use standard data, summarized in the table below.

Rate	$R(t)$	5.00%
Volatility	$\gamma(t)$	1.5%
Reversion	$a(t)$	5.00%
Lower bound	$L(t)$	0%
Upper bound	$U(t)$	20.00%

We have chosen particular time-independent parameters for transparency, however, the proposed approximation works for general time-independent ones.

Below, we present a graphical visualization of the short rate dependence on the underlying variable.

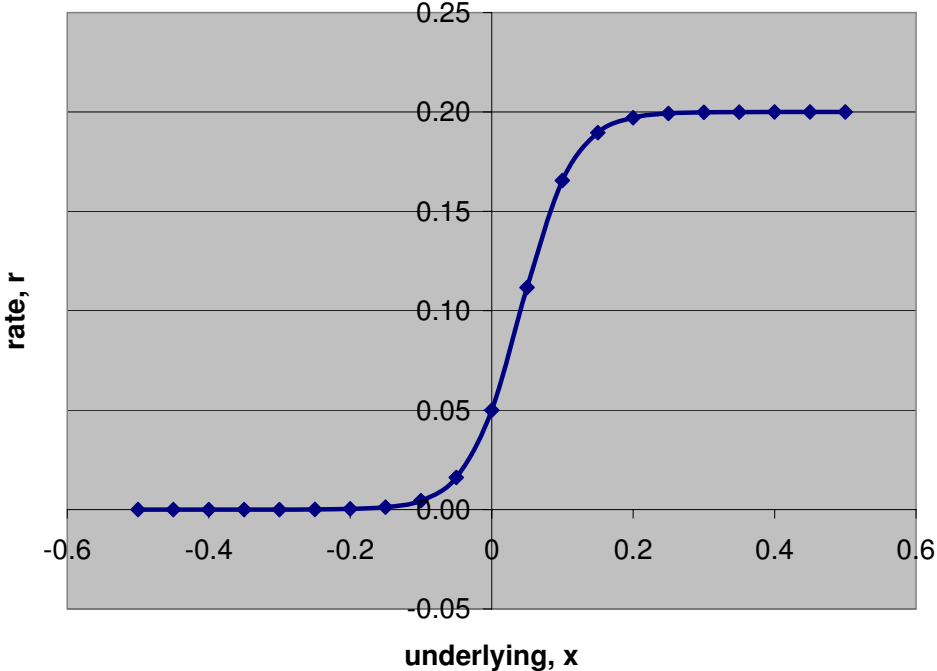


Figure 2: Rate as function of underlying, $r = f(x)$.

We have tested European swaptions with annual period, 10Y length, multiple strikes and 10Y maturity. The results for Black implied volatilities are presented in the table below.

Strike (%)	Implied vol (%)		Error (bps)
	Approx	Exact	
0.01	29.72	32.52	-280
0.5	19.98	20.54	-56
1	19.01	18.34	67
1.5	18.39	17.79	60
2	17.92	17.49	43
2.5	17.52	17.19	33
3	17.17	16.92	25
3.5	16.85	16.69	16
4	16.56	16.48	8
4.5	16.29	16.27	2
5	16.04	16.07	-3
5.5	15.8	15.87	-7
6	15.57	15.67	-10
6.5	15.35	15.49	-14
7	15.14	15.3	-16
7.5	14.93	15.11	-18
8	14.74	14.93	-19
8.5	14.54	14.74	-20
9	14.34	14.55	-21
9.5	14.15	14.36	-21
10	13.96	14.17	-21
10.5	13.77	13.98	-21
11	13.58	13.79	-21
11.5	13.38	13.6	-22
12	13.19	13.39	-20
12.5	12.99	13.18	-19
13	12.79	12.96	-17
13.5	12.58	12.74	-16
14	12.37	12.52	-15
14.5	12.14	12.31	-17
15	11.92	12.11	-19
15.5	11.69	11.91	-22
16	11.44	11.72	-28
16.5	11.19	11.52	-33
17	10.92	11.32	-40
17.5	10.63	11.12	-49
18	10.33	10.93	-60
18.5	10.01	10.73	-72
19	9.654	9.972	-31.8
19.5	9.263	9.724	-46.1
20	8.824	8.254	57
20.2	8.632	5.852	278
20.4	8.425	4.662	376.3
20.5	8.315	4.63	368.5
21	0	3.99	-399

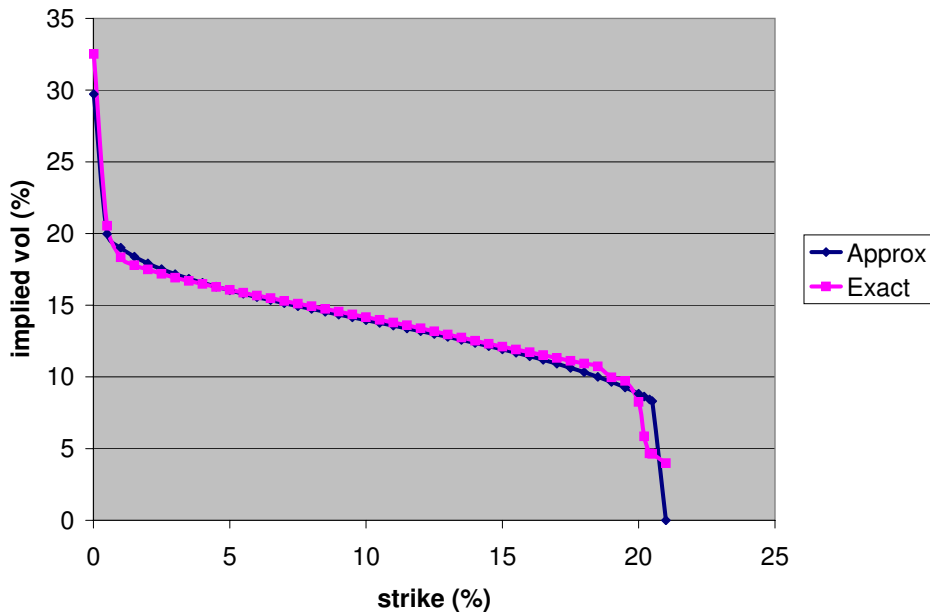


Figure 3: Comparison of the implied vols for 10Y10Y European swaption (Bounded short rate model).

6 Conclusion

In this article, we have treated generalized short rate models with Gaussian underlying (GSRG) and derived an efficient swaption price approximation. The result is based on a regular and singular expansion with respect to the small volatility and contains a low-dimensional integration.

We also presented two special cases of the GSRG model. The first one is the Generalized multi-factor Black-Karasinski (BK) model widely used by practitioners. The second one, Bounded short rate model, is new. Its rates evolve inside arbitrary thresholds. This model is especially attractive for scenario generation. The implied volatility skew for the Bounded short rate model is convex for small/moderate strikes and concave for bigger ones.

We have confirmed the analytical approximation quality by comparison with purely numerical results. We have detected an excellent analytics-numerics fit for short/medium maturities, and a good one for large maturities. We have observed a slight quality degeneration for extreme strikes.

The authors are indebted to Serguei Mechkov for discussions and numerical implementation help as well as to their colleagues at Numerix, especially to Gregory Whitten and Serguei Issakov for supporting this work and Patti Harris for the excellent editing. AA is grateful to Vladimir Piterbarg and Michael Konikov for stimulating discussions.

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A Zero-coupon bond

The parabolic operator underlying the zero-coupon bond PDE (8) reads

$$L(t, u) = -r(t, u) + \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j.$$

Denote an unperturbed operator as $L_0(t)$ and a perturbation as $L_1(t)$,

$$\begin{aligned} L_0(t) &= -r(t, u), \\ L_1(t) &= \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j. \end{aligned}$$

A solution of the unperturbed equation

$$(\partial_t + L_0(t)) P_0(t, T, u) = 0$$

for the unit final value $P_0(T, T, u) = 1$, will have a simple form,

$$P_0(t, T, u) = \exp \left(- \int_t^T d\tau r(\tau, u) \right).$$

Using an unperturbed evolution operator,

$$U_0(t_1, t_2) =: e^{\int_{t_1}^{t_2} d\tau L_0(\tau)} := e^{- \int_{t_1}^{t_2} d\tau r(\tau, u)},$$

the bond price $P_0(t, T, u)$ can be presented as a result of the action of the evolution operator on 1,

$$P_0(t, T, u) = U_0(t, T) 1.$$

The Dyson perturbation technique permits computing the first correction to the unperturbed solution

$$P_1(t, T, u) = \int_t^T ds \hat{L}_1(t, s) P_0(t, T, u), \quad (26)$$

where $\hat{L}_1(t, s)$ is the dressed perturbation operator

$$\hat{L}_1(t, s) = U_0(t, s) L_1(s) U_0^{-1}(t, s). \quad (27)$$

Rewrite the first correction in a more convenient way,

$$P_1(t, T, u) = \int_t^T ds U_0(t, s) L_1(s) U_0(s, T) 1 = \int_t^T ds P_0(t, s, u) L_1(s) P_0(s, T, u). \quad (28)$$

Differentiating the leading term, we obtain

$$L_1(s) P_0(s, T, u) = \varepsilon \left(\sum_i \theta_i(s) \partial_i + \frac{1}{2} \sum_{i,j} C_{ij}(s) \partial_i \partial_j \right) P_0(s, T, u) = \varepsilon \Lambda(s, T, u) P_0(s, T, u) \quad (29)$$

for the multiplier

$$\Lambda(s, T, u) = - \sum_i \theta_i(s) g_i(s, T, u) + \frac{1}{2} \sum_{i,j} C_{ij}(s) (g_i(s, T, u) g_j(s, T, u) - g_{ij}(s, T, u)) \quad (30)$$

with

$$g_i(s, T, u) = \partial_i \int_s^T d\tau r(\tau, u) = \int_s^T d\tau \partial_i r(\tau, u)$$

and

$$g_{ij}(s, T, u) = \partial_i \partial_j \int_s^T d\tau r(\tau, u) = \int_s^T d\tau \partial_i \partial_j r(\tau, u).$$

Finally, the first correction of the bond price reads

$$\begin{aligned} P_1(t, T, u) &= \int_t^T ds P_0(t, s, u) L_1(s) P_0(s, T, u) \\ &= \varepsilon P_0(t, T, u) \\ &\times \int_t^T ds \left\{ -\sum_i \theta_i(s) g_i(s, T, u) + \frac{1}{2} \sum_{i,j} C_{ij}(s) (g_i(s, T, u) g_j(s, T, u) - g_{ij}(s, T, u)) \right\}. \end{aligned}$$

For completeness, we calculate the second order correction to the bond,

$$P_2(t, T, u) = \int_t^T ds_1 \int_{s_1}^T ds_2 \hat{L}_1(t, s_1) \hat{L}_1(t, s_2) P_0(t, T, u) \quad (31)$$

$$= \int_t^T ds_1 \int_{s_1}^T ds_2 P_0(t, s_1, u) L_1(s_1) P_0(s_1, s_2, u) L_1(s_2) P_0(s_2, T, u). \quad (32)$$

Using the L_1 action on the leading order bond (29), we obtain

$$P_2(t, T, u) = \int_t^T ds_1 \int_{s_1}^T ds_2 P_0(t, s_1, u) L_1(s_1) (\Lambda(s_2, T, u) P_0(s_1, T, u)).$$

Calculating the action of the L_1 on the product in the r.h.s.,

$$\begin{aligned} L_1(s_1) (\Lambda(s_2, T, u) P_0(s_1, T, u)) &= P_0(s_1, T, u) L_1(s_1) (\Lambda(s_2, T, u)) + \Lambda(s_2, T, u) L_1(s_1) (P_0(s_1, T, u)) \\ &+ \sum_{i,j} C_{ij}(s_1) \partial_j \Lambda(s_2, T, u) \partial_i P_0(s_1, T, u), \end{aligned}$$

we can further simplify the second correction to the bond price

$$\begin{aligned} P_2(t, T, u) &= P_0(t, T, u) \left\{ \frac{1}{2} \left(\int_t^T ds \Lambda(s, T, u) \right)^2 \right. \\ &+ \left. \int_t^T ds_1 \int_{s_1}^T ds_2 \left(L_1(s_1) \Lambda(s_2, T, u) - \sum_{i,j} C_{ij}(s_1) g_i(s_1, T, u) \partial_j \Lambda(s_2, T, u) \right) \right\} \quad (33) \end{aligned}$$

or

$$\begin{aligned} P_2(t, T, u) &= P_0(t, T, u) \left\{ \frac{1}{2} \left(\int_t^T ds \Lambda(s, T, u) \right)^2 \right. \\ &+ \int_t^T ds_1 \sum_i \left(\theta_i(s_1) - \sum_j C_{ij}(s_1) g_j(s_1, T, u) \right) \int_{s_1}^T ds_2 \partial_i \Lambda(s_2, T, u) \\ &+ \left. \frac{1}{2} \int_t^T ds_1 \sum_{ij} C_{ij}(s_1) \int_{s_1}^T ds_2 \partial_i \partial_j \Lambda(s_2, T, u) \right\}, \quad (34) \end{aligned}$$

where the derivatives of the multiplier are

$$\begin{aligned}\partial_i \Lambda(s, T, u) &= -\sum_k \theta_k(s) g_{ik}(s, T, u) + \frac{1}{2} \sum_{kn} C_{kn}(s) (2 g_{ki}(s, T, u) g_n(s, T, u) - g_{nki}(s, T, u)) \\ \partial_i \partial_j \Lambda(s, T, u) &= -\sum_k \theta_k(s) g_{ijk}(s, T, u) \\ &+ \frac{1}{2} \sum_{kn} C_{kn}(s) (2 g_{kij}(s, T, u) g_n(s, T, u) + 2 g_{ki}(s, T, u) g_{jn}(s, T, u) - g_{nkij}(s, T, u))\end{aligned}$$

and $g_{ijkn}(s, T, u) = \partial_i \partial_j \partial_k \partial_n g(s, T, u)$, as usual.

B AD price

In the case of the AD price, we deal with a *singular* perturbation technique. The AD price satisfies the forward PDE $(\partial_t - L^*(t, u)) q(t, u) = 0$ with the parabolic operator

$$L^*(t, u) = -r(t, u) - \varepsilon \sum_i \theta_i(t) \partial_i + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j$$

and the initial condition $q(0, u) = \prod_i \delta(u_i - y_i(0))$.

It is easy to check that the Gaussian density

$$q_0(t, u) = \frac{\exp\left(-\frac{1}{2} \varepsilon^{-1} (u - \varepsilon m(t))^T V^{-1}(t) (u - \varepsilon m(t))\right)}{(2\pi \varepsilon)^{\frac{n}{2}} |V(t)|^{\frac{1}{2}}},$$

where $m(t)$ is the $y(t)$ center and $V(t)$ is the covariance in vector/matrix notations, satisfies the PDE

$$\partial_t q_0(t, u) = -\varepsilon \sum_i \theta_i(t) \partial_i q_0(t, u) + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j q_0(t, u).$$

The Gaussian density derivatives look as follows,

$$\begin{aligned}\frac{\partial_t q_0(t, u)}{q_0(t, u)} &= \frac{1}{2} \varepsilon^{-1} (u - \varepsilon m(t))^T V^{-1}(t) C(t) V^{-1}(t) (u - \varepsilon m(t)) + \theta^T(t) V^{-1}(t) (u - \varepsilon m(t)) \\ &- \frac{1}{2} \text{tr} C(t) V^{-1}(t), \\ \frac{\partial_i q_0(t, u)}{q_0(t, u)} &= -\varepsilon^{-1} (V^{-1}(t) (u - \varepsilon m(t)))_i, \\ \frac{\partial_i \partial_j q_0(t, u)}{q_0(t, u)} &= \varepsilon^{-2} (V^{-1}(t) (u - \varepsilon m(t)))_i (V^{-1}(t) (u - \varepsilon m(t)))_j - \varepsilon^{-1} (V^{-1}(t))_{ij},\end{aligned}$$

where we have used the relations

$$\begin{aligned}\partial_t V^{-1}(t) &= -V^{-1}(t) \partial_t V(t) V^{-1}(t) = -V^{-1}(t) C(t) V^{-1}(t), \\ \partial_t \ln |V(t)| &= \text{tr} \partial_t V(t) V^{-1}(t) = \text{tr} C(t) V^{-1}(t).\end{aligned}$$

Now, following the Heat-Kernel expansion logic, we look for the solution as

$$q(t, u) = q_0(t, u) \Omega(t, u),$$

where $\Omega(t, u)$ is a *regular* function of ε with the unit initial condition $\Omega(0, u) = 1$. After substitution into the initial PDE, we get an equation on the $\Omega(t, u)$,

$$\begin{aligned}\partial_t \Omega(t, u) &= -r(t, u) \Omega(t, u) - \varepsilon \sum_i \theta_i(t) \partial_i \Omega(t, u) \\ &+ \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j \Omega(t, u) + \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \ln q_0(t, u) \partial_j \Omega(t, u).\end{aligned}$$

Or, using the derivative of the Gaussian $\partial_i \ln q_0(t, u)$, we obtain

$$\begin{aligned}\partial_t \Omega(t, u) + r(t, u) \Omega(t, u) + u^T V^{-1}(t) C(t) \partial \Omega(t, u) \\ + \varepsilon \left\{ \theta^T(t) \partial \Omega(t, u) - \frac{1}{2} \text{tr} C(t) \partial^2 \Omega(t, u) - m^T(t) V^{-1}(t) C(t) \partial \Omega(t, u) \right\} = 0,\end{aligned}\quad (35)$$

where we have used vector notations for the first derivatives $\partial = \{\partial_1, \partial_2, \dots\}$ and the drifts $\theta = \{\theta_1, \theta_2, \dots\}$, and matrix notations for the second derivatives $(\partial^2)_{ij} = \partial_i \partial_j$.

We will look for the solution Ω as a regular expansion in ε ,

$$\Omega(t, u) = b_0(t, u) + \varepsilon b_1(t, u) + \dots \quad (36)$$

with the initial conditions $b_0(t, u) = 1$ and $b_1(t, u) = 0$.

The leading-order solution will satisfy the unperturbed PDE

$$\partial_t b_0(t, u) + u^T V^{-1}(t) C(t) \partial b_0(t, u) + r(t, u) b_0(t, u) = 0. \quad (37)$$

This is a linear first-order PDE with characteristics (passing through u at time τ)

$$u_c(t, \tau, u) = V(\tau) V^{-1}(t) u,$$

which can be derived from the fact that $\{\partial_t + u^T V^{-1}(t) C(t) \partial\} V^{-1}(t) u = 0$. Thus, the solution has the following form,

$$b_0(t, u) = \exp \left(- \int_0^t ds r(s, V(s) V^{-1}(t) u) \right). \quad (38)$$

The derivatives in matrix notations read

$$\begin{aligned}\frac{\partial_t b_0(t, u)}{b_0(t, u)} &= -r(t, u) + u^T V^{-1}(t) C(t) V^{-1}(t) h^{(1)}(t, u), \\ \frac{\partial b_0(t, u)}{b_0(t, u)} &= -V^{-1}(t) h^{(1)}(t, u), \\ \frac{\partial^2 b_0(t, u)}{b_0(t, u)} &= V^{-1}(t) \left[h^{(1)}(t, u) \otimes h^{(1)}(t, u) - h^{(2)}(t, u) \right] V^{-1}(t),\end{aligned}$$

where we denoted the vector

$$h_n^{(1)}(t, u) = \sum_j \int_0^t ds V_{nk}(s) \tilde{\partial}_k r(s, V(s) V^{-1}(t) u) \quad (39)$$

and the matrix

$$h_{nl}^{(2)}(t, u) = \sum_{km} \int_0^t ds V_{nk}(s) \tilde{\partial}_k \tilde{\partial}_m r(s, V(s) V^{-1}(t) u) V_{ml}(s). \quad (40)$$

Here, the derivative $\tilde{\partial}_k$ denotes the short rate derivative over the space argument, $\tilde{\partial}_i r(s, z(u)) = \frac{\partial r(s, z(u))}{\partial z_i(u)}$.

Proceed now to the first correction, $b_1(t, u)$. After substituting the expansion (36) into the PDE (35) and retaining the first-order terms in ε , we obtain

$$\partial_t b_1(t, u) + u^T V^{-1}(t) C(t) \partial b_1(t, u) + r(t, u) b_1(t, u) = g(t, u), \quad (41)$$

where the r.h.s. reads

$$\begin{aligned} g(t, u) &= -\theta^T(t) \partial b_0(t, u) + \frac{1}{2} \text{tr} C(t) \partial^2 b_0(t, u) + m^T(t) V^{-1}(t) C(t) \partial b_0(t, u) \\ &= b_0(t, u) \left\{ (\theta^T(t) - m^T(t) V^{-1}(t) C(t)) V^{-1}(t) h^{(1)}(t, u) \right. \\ &\quad \left. + \frac{1}{2} \text{tr} C(t) V^{-1}(t) [h^{(1)}(t, u) \otimes h^{(1)}(t, u) - h^{(2)}(t, u)] V^{-1}(t) \right\}. \end{aligned}$$

It is easy to see that the first correction has the following solution,

$$b_1(t, u) = e^{-\int_0^t ds r(s, V(s) V^{-1}(t) u)} \int_0^t d\tau g(\tau, V(\tau) V^{-1}(t) u) e^{\int_0^\tau ds r(s, V(s) V^{-1}(t) u)}. \quad (42)$$

After some algebra, we arrive at

$$\begin{aligned} b_1(t, u) &= b_0(t, u) \\ &\times \left\{ \int_0^t d\tau (\theta^T(\tau) - m^T(\tau) V^{-1}(\tau) C(\tau)) V^{-1}(\tau) \hat{h}^{(1)}(\tau; t, u) \right. \\ &\quad \left. + \frac{1}{2} \int_0^t d\tau \text{tr} V^{-1}(\tau) C(\tau) V^{-1}(\tau) [\hat{h}^{(1)}(\tau; t, u) \otimes \hat{h}^{(1)}(\tau; t, u) - \hat{h}^{(2)}(\tau; t, u)] \right\}, \end{aligned}$$

where

$$\begin{aligned} \hat{h}_n^{(1)}(\tau; t, u) &= \sum_j \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k r(s, V(s) V^{-1}(t) u), \\ \hat{h}_{nl}^{(2)}(\tau; t, u) &= \sum_{km} \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k \tilde{\partial}_m r(s, V(s) V^{-1}(t) u) V_{ml}(s). \end{aligned}$$

Finally, the AD price, up to the first order in small volatilities, looks as follows,

$$\begin{aligned} q(t, u) &= \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}(u - \varepsilon m(t))^T V^{-1}(t)(u - \varepsilon m(t))\right)}{(2\pi\varepsilon)^{\frac{n}{2}} |V(t)|^{\frac{1}{2}}} \exp\left(-\int_0^t ds r(s, V(s) V^{-1}(t) u)\right) \\ &\times \left\{ 1 + \varepsilon \int_0^t d\tau (\theta^T(\tau) - m^T(\tau) V^{-1}(\tau) C(\tau)) V^{-1}(\tau) \hat{h}^{(1)}(\tau; t, u) \right. \\ &\quad \left. + \frac{1}{2}\varepsilon \int_0^t d\tau \text{tr} V^{-1}(\tau) C(\tau) V^{-1}(\tau) [\hat{h}^{(1)}(\tau; t, u) \otimes \hat{h}^{(1)}(\tau; t, u) - \hat{h}^{(2)}(\tau; t, u)] \right\}. \quad (43) \end{aligned}$$