Robust Synthesis of Feedforward Compensators

Alvaro Giusto and Fernando Paganini

***Abstract*—The design of a feedforward compensator for robust $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance under structured uncertainty is considered. For linear time-invariant uncertainty, a convex method based on linear matrix inequalities (LMI's) across the frequency variable is given. For nonlinear or time-varying perturbations and $\mathcal{H}_\infty$ performance, the design problem is reduced exactly to a state-space LMI, and extensions to $\mathcal{H}_2$ performance are discussed. An example illustrates the application of these techniques to two-degree-of-freedom control design.**

***Index Terms*—Feedforward compensator, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance, prefilter design, robust synthesis.**

I. Introduction

One of the central problems in robust control is the design of a controller in the presence of uncertainty of spatial structure

$$\Delta = \mathrm{diag}[\delta_1 I_{k_1}, \cdots, \delta_{m_e} I_{k_{m_e}}, \Delta_1, \cdots, \Delta_{m_C}]. \tag{1}$$

While the robustness *analysis* question has been addressed with considerable success (see [3], [10]–[12], [16], [17]), the general *synthesis* problem (see [2] and [11]) is widely recognized to be of intrinsically harder complexity. This paper discusses how the robust synthesis problem simplifies under the special structure of *feedforward* compensation, depicted in Fig. 1.

We mention two motivations for such a configuration. The first, depicted in Fig. 2, is a tracking problem in which P is the feedforward part of a two-degree-of-freedom controller [7]–[9], [18]. The feedback F and the feedforward P must be designed so that the output s tracks the reference r, in the presence of disturbances w and uncertainty Δ.

Although P and F could be synthesized jointly as a controller K, their roles are quite distinct: F is the only part which affects stability and disturbance rejection, and P influences the command response; this suggests the possibility of a two-stage design. In the absence of uncertainty, this approach is validated in [14] where it is shown that the achievable closed-loop transfer functions from r to s are not restricted by the choice of a stabilizing F. While the overall controller order may increase, there is added flexibility in the specifications for each stage. One can first design F for stability and disturbance rejection by, e.g., $\mathcal{H}_\infty$ control, and then P for tracking with other criteria (fixed input response, $\mathcal{H}_2$, channel decoupling, etc.).

For systems with uncertainty Δ, this decomposition is restrictive and the feedback F (the main tool for canceling uncertainty) will affect the achievable robust tracking performance. Still, since the joint synthesis problem is hard it may be useful to break down the design in this way, allowing also for different kinds of requirements at each stage. Some references taking this approach are [9] and [15]. Given a robustly stabilizing feedback F, the subsequent synthesis of P to satisfy a tracking performance objective falls in the setup of Fig. 1.

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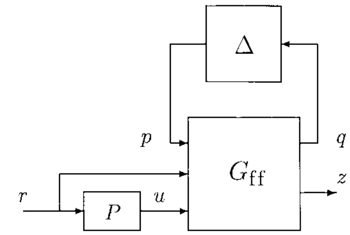

Fig. 1. Robust feedforward compensation problem.

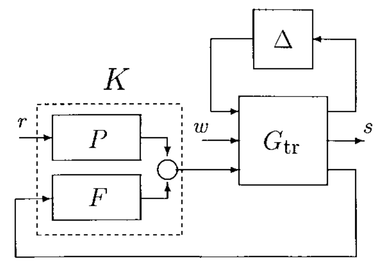

Fig. 2. Two-degree-of-freedom controller.

The second motivation for the configuration of Fig. 1 is a "disturbance feedforward" situation, where r is a disturbance which we cannot control but is available for measurement.

In this paper we study the design problem of Fig. 1 for the general case of dynamic uncertainty of spatial structure (1), and where the blocks can be linear time-invariant (LTI), linear time-varying (LTV), or nonlinear (NL). For the characterization of system performance, we will consider both the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ performance criteria on the closed-loop map from r to z. Generally speaking, we show that robust feedforward synthesis has the complexity of the corresponding robustness *analysis* problem. In particular, convex conditions for robust synthesis are derived, which parallel the robustness analysis conditions of [3], [10]–[12], and [16].

The paper is organized as follows: the problem formulation is explained in Section II. Section III presents convex frequency domain methods for prefilter design, which apply to problems with LTI uncertainty. Section IV provides state-space methods for problems with NLTV uncertainty. An application example is considered in Section V, and conclusions are given in Section VI. For a more extensive treatment of the material in this note, the reader is referred to [6].

II. Problem Formulation

We consider the configuration of Fig. 1, where the generalized plant G_{ff} is assumed to be finite-dimensional LTI, of state-space realization

$$G_{ff}(s) = \left[\begin{array}{c|ccc} A & B & E & M \\ \hline C & L & H & N \end{array}\right] \tag{2}$$

with $A \in \mathbb{R}^{n\times n}$ stable. In (2) we are using the customary notation

$$C(sI-A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The inputs of G_{ff} are partitioned as in Fig. 1. The dimensions of the signals p, r, u, q, and z are, respectively, d_p, d_r, d_u, d_q, and d_z. For simplicity let $m := d_p = d_q$.

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The uncertain component Δ is defined by a class of operators over the space $\mathcal{L}_2$ of square-integrable signals. $\mathbf{B}_{\mathbf{\Delta}}$ denotes the set of causal, possibly NL operators of spatial structure (1) with $\mathcal{L}_2$-induced norm less than 1. We can also restrict these operators to be LTV, or further to be LTI, as will be specified in each case.

The first design requirement is robust stability. This property must be assumed for the open loop, however, since the feedforward P has no stabilizing effect (in a two-degree-of-freedom context, the feedback F must be robustly stabilizing). Then a stable P will maintain closed-loop stability and give the transfer function

$$T_P(s) := G_{ff}\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & P \end{bmatrix} \tag{3}$$

between (p, r) and (q, z). Now given Δ, the closed-loop map from r to z is denoted $T_{zr}(\Delta)$ and is the object of our performance specifications. For a prespecified γ, the system is said to have *robust $\mathcal{H}_\infty$ performance* level γ if

$$\sup_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|T_{zr}(\Delta)\|_{\mathcal{L}_2\to\mathcal{L}_2} < \gamma. \tag{4}$$

For LTI uncertainty, the $\mathcal{L}_2$-induced norm is the standard $\mathcal{H}_\infty$ norm from Hardy space theory. The uncertain system will have *robust $\mathcal{H}_2$ performance* level γ, if

$$\sup_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \|T_{zr}(\Delta)\|_2 < \gamma.$$

For an LTI system T, the $\mathcal{H}_2$ norm is given by

$$\|T\|_2^2 := \int_{-\infty}^{\infty} \mathrm{trace}(T(j\omega)^* T(j\omega))\frac{d\omega}{2\pi}. \tag{5}$$

When systems are not LTI there is no universally accepted interpretation for the $\mathcal{H}_2$ criterion; in Section IV-B we will comment on alternative approaches for robust $\mathcal{H}_2$ performance design.

The problem under consideration is to find P, if it exists, such that T_{zr} satisfies the robust ($\mathcal{H}_\infty$ or $\mathcal{H}_2$) performance specification. We will provide synthesis methods based on known robust performance analysis conditions [2], [3], [10]–[12], [16]; these involve scaling matrices

$$X = \mathrm{diag}[X_1, \cdots, X_{m_e}, x_1 I_{k_1}, \cdots, x_{m_C} I_{k_{m_C}}] \tag{6}$$

which commute with Δ in (1). $\mathbb{X}$ denotes the set of positive definite matrices of this form.

III. Frequency Domain Methods for LTI Uncertainty

In this section we will state two convex conditions for the robust synthesis problem with LTI uncertainty, for the cases of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance. For proofs we refer to [6].

Let us partition T_P in (2) as $T_P = [T_0 \quad T_1] = [T_0 \quad T_{11} + T_{12}P]$ in correspondence to the inputs p and r. The main observation is that if at a certain frequency ω, (7), which is shown at the bottom of the page, holds, then for any LTI perturbation of the structure (1) we have $T_{zr}(\Delta(j\omega))^* T_{zr}(\Delta(j\omega)) \le Y(\omega)$. This follows by the methods of the structured singular value theory [11]; for details see [6].

Also notice that the left-hand side of (7) is *affine* in the unknowns $X(\omega)$, $Y(\omega)$, and $P(\omega)$; we denote it by $\Psi(\omega, X(\omega), Y(\omega), P(j\omega))$. This leads to the following statements.

Proposition 1: A stable prefilter $P(s)$ ensures a level γ of robust $\mathcal{H}_2$ performance in the presence of LTI uncertainty if there exist functions $X(\omega) \in \mathbb{X}$, $Y(\omega)$ such that

$$\int_{-\infty}^{+\infty} \mathrm{trace}(Y(\omega))\frac{d\omega}{2\pi} \le \gamma^2$$
$$\Psi(\omega, X(\omega), Y(\omega), P(j\omega)) < 0, \qquad \forall\,\omega.$$

Proposition 2: A stable prefilter $P(s)$ ensures a level γ of robust $\mathcal{H}_\infty$ performance in the presence of LTI uncertainty if there exists a function $X(\omega) \in \mathbb{X}$ such that

$$\Psi(\omega, X(\omega), \gamma^2 I, P(j\omega)) < 0, \qquad \forall\,\omega.$$

Thus prefilter synthesis is reduced to the minimization of a linear objective subject to convex infinite-dimensional constraints (stability of $P(s)$ is also a convex constraint). To obtain a finite-dimensional approximation there are two well-known approaches: frequency gridding and optimization over the span of a set of basis functions [1]. The stability of $P(s)$ is easily imposed in the latter, but for frequency gridding an approximation step is required.

Propositions 1 and 2 also can also be extended to the case of real parametric uncertainty, by extending the function Ψ in terms of a "G-scaling"; see [6].

IV. State-Space LMI Synthesis for NLTV Uncertainty

In this section we study the problem of feedforward design for robust performance against structured NLTV uncertainty. We provide a complete linear matrix inequalities (LMI) solution to the problem for the $\mathcal{H}_\infty$ performance case, based on the analysis conditions in [11] and [16]. In addition, we briefly discuss two approaches for $\mathcal{H}_2$ performance in this uncertainty class.

A. Exact Solution for the $\mathcal{H}_\infty$ Case

We will now show that the existence of dynamic prefilters that guarantee a given level γ of robust $\mathcal{H}_\infty$ performance against NLTV uncertainties is equivalent to an LMI in state space.

The following result is stated in terms of the problem formulation (2); let $\tilde{\mathcal{N}}_R$ be a matrix whose columns constitute a basis for the kernel of $[M' \quad N']$ (prime denotes transpose) and

$$\tilde{\mathcal{L}}_R := \begin{bmatrix} \tilde{\mathcal{N}}_R & 0 \\ 0 & I_{d_p+d_r} \end{bmatrix}.$$

Theorem 3: There exists a dynamic prefilter $P(s)$ satisfying robust $\mathcal{H}_\infty$ performance level γ under perturbations in $\mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}$ if and only if there exist symmetric positive definite matrices $R \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times n}$, and $X \in \mathbb{X}$ satisfying

$$\tilde{\mathcal{L}}_R' \begin{bmatrix} AR + RA' & RC' & BX & E \\ CR & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} & LX & H \\ XB' & XL' & -X & 0 \\ E' & H' & 0 & -\gamma^2 I \end{bmatrix} \tilde{\mathcal{L}}_R < 0 \tag{8}$$

$$\begin{bmatrix} ZA' + AZ & BX & ZC' \\ XB' & -X & XL' \\ CZ & LX & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0 \tag{9}$$

$$R - Z \ge 0. \tag{10}$$

$$\begin{bmatrix} T_0(j\omega)X(\omega)T_0(j\omega)^* - \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} & T_{11}(j\omega) + T_{12}(j\omega)P(j\omega) \\ (T_{11}(j\omega) + T_{12}(j\omega)P(j\omega))^* & -Y(\omega) \end{bmatrix} < 0 \tag{7}$$

Proof: It is well known [11], [16] that $\mathcal{H}_\infty$ robust performance is equivalent to the existence of a constant matrix $X \in \mathbb{X}$, such that $\|Q^{-1}T_P Q_\gamma\|_\infty < 1$, where we define

$$Q := \begin{bmatrix} X^{1/2} & 0 \\ 0 & I \end{bmatrix}, \qquad Q_\gamma := \begin{bmatrix} X^{1/2} & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}$$

and $X^{1/2}$ denotes the positive square root of the matrix X. Using (3), this is equivalent to the standard $\mathcal{H}_\infty$ synthesis problem

$$\|G_{11}^Q + G_{12}^Q P(I - G_{22}^Q P)^{-1} G_{21}^Q\|_\infty < 1 \tag{11}$$

where

$$G^Q := \left[\begin{array}{c|cc} A & [B \quad E]Q_\gamma & M \\ \hline Q^{-1}C & Q^{-1}[L \quad H]Q_\gamma & Q^{-1}N \\ 0 & [0 \quad I]Q_\gamma & 0 \end{array}\right]. \tag{12}$$

Applying the LMI formulation of [4] to this $\mathcal{H}_\infty$ synthesis, (11) is feasible if and only if there exist matrices $R > 0$ and $S > 0$ such that

$$\mathcal{L}_R' \begin{bmatrix} AR + RA' & RC'Q^{-1} & [B \quad E]Q_\gamma \\ Q^{-1}CR & -I & Q^{-1}[L \quad H]Q_\gamma \\ Q_\gamma \begin{bmatrix} B' \\ E' \end{bmatrix} & Q_\gamma \begin{bmatrix} L' \\ H' \end{bmatrix} Q^{-1} & -I \end{bmatrix} \mathcal{L}_R < 0 \tag{13}$$

$$\mathcal{L}_S' \begin{bmatrix} A'S + SA & S[B \quad E]Q_\gamma & C'Q^{-1} \\ Q_\gamma \begin{bmatrix} B' \\ E' \end{bmatrix} S & -I & Q_\gamma \begin{bmatrix} L' \\ H' \end{bmatrix} Q^{-1} \\ Q^{-1}C & Q^{-1}[L \quad H]Q_\gamma & -I \end{bmatrix} \mathcal{L}_S < 0 \tag{14}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{15}$$

where

$$\mathcal{L}_R := \begin{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} \tilde{\mathcal{N}}_R & 0 \\ 0 & I_{d_p + d_r} \end{bmatrix}$$

$$\mathcal{L}_S := \begin{bmatrix} I_n & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{d_q + d_z} \end{bmatrix}, \qquad \text{with } U = \begin{bmatrix} I_{d_p} \\ 0_{d_r \times d_p} \end{bmatrix}.$$

We now show that (13)–(15) are equivalent to (8)–(10). The equivalence of (13) and (8) follows by using the expression for $\mathcal{L}_R$ in (13) and left and right multiplying the last block row and column in (13) by $[\begin{smallmatrix} X^{1/2} & 0 \\ 0 & \gamma I \end{smallmatrix}]$. Also, $Q_\gamma U = [\begin{smallmatrix} X^{1/2} \\ 0 \end{smallmatrix}]$so substituting the expression for $\mathcal{L}_S$ in (14), we obtain

$$\begin{bmatrix} A'S + SA & SBX^{1/2} & C'Q^{-1} \\ X^{1/2}B'S & -I & X^{1/2}L'Q^{-1} \\ Q^{-1}C & Q^{-1}LX^{1/2} & -I \end{bmatrix} < 0 \tag{16}$$

which is equivalent to

$$\begin{bmatrix} S^{-1}A' + AS^{-1} & BX & S^{-1}C' \\ XB' & -X & XL' \\ CS^{-1} & LX & -Q^2 \end{bmatrix} < 0$$

and reduces to (9) if we define $Z := S^{-1}$. With this definition, the equivalence of (10) and (15) follows by a Schur complement operation. ■

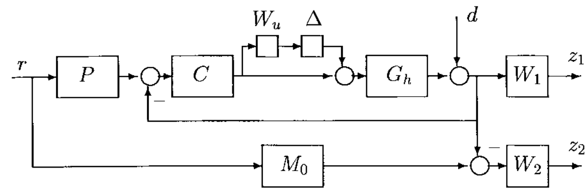

Fig. 3. System interconnection for synthesis.

We have found that the LMI conditions (8)–(10) are equivalent to the solvability of the robust $\mathcal{H}_\infty$ prefilter synthesis in the NLTV uncertainty case. Robust performance can be optimized by minimizing γ subject to these constraints, which can be solved by standard techniques [1]. Given feasible solutions Z, R, and X for a certain γ, the prefilter $P(s)$ can be obtained by the methods proposed in [4], and has the same order as the generalized plant G.

B. Considerations about the $\mathcal{H}_2$ Performance Case

The standard motivations for $\mathcal{H}_2$ performance (response to impulses and response to stationary white noise) do not lead to the same unequivocal choice if the system is not LTI. Thus there is more than one generalization of (5) for NLTV systems.

One approach [3], [17] is to define the $\mathcal{H}_2$ norm as the energy response to an impulsive input, which leads to LMI upper bounds for robust performance (see [3] and [13]). It turns out that the design of a feedforward compensator to satisfy such bounds reduces to a finite-dimensional LMI problem, analogously to Theorem 3 (see [6] for a full statement and proof). This provides a valuable tool for prefilter design in tracking problems, where reference inputs such as steps or ramps can be studied by adding suitable weighting functions.

A second approach is to interpret the $\mathcal{H}_2$ norm as a measure of the worst response to white noise, characterized in a deterministic setting [12]. Exact analysis conditions in terms of state-space LMI's are available [12], and it is natural to inquire whether the corresponding robust prefilter synthesis can also be solved by an LMI. We do not know the answer to this question; however, a basis function approach may be used, as described in [6].

V. Example

We will examine a problem with disturbance rejection and tracking in the presence of structured uncertainty. The system under consideration is extracted from [15] and corresponds to the control of pitch and angle of attack of an aircraft. The uncertain model and control configuration are depicted in Fig. 3; numerical values for the model can be found in [6]. The uncertainty is supposed to be LTI and norm-bounded.

Disturbances are modeled at the outputs of the plant, and we use an $\mathcal{H}_\infty$ performance criterion weighted by $W_1(s)$ to impose disturbance rejection,

$$\mu_{DR} := \max_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|T_{z_1 d}(\Delta)\|_\infty. \tag{17}$$

In addition the command response requirement from [15] establishes that the step response must be as close as possible to the desired response represented by $M_o(s)$. The weight $W_2(s)$ provides a stable approximation to an integrator, which converts the problem to an impulse response specification, captured by the robust $\mathcal{H}_2$ performance criterion

$$\beta_{CR} := \max_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|T_{z_2 r}(\Delta)\|_2. \tag{18}$$

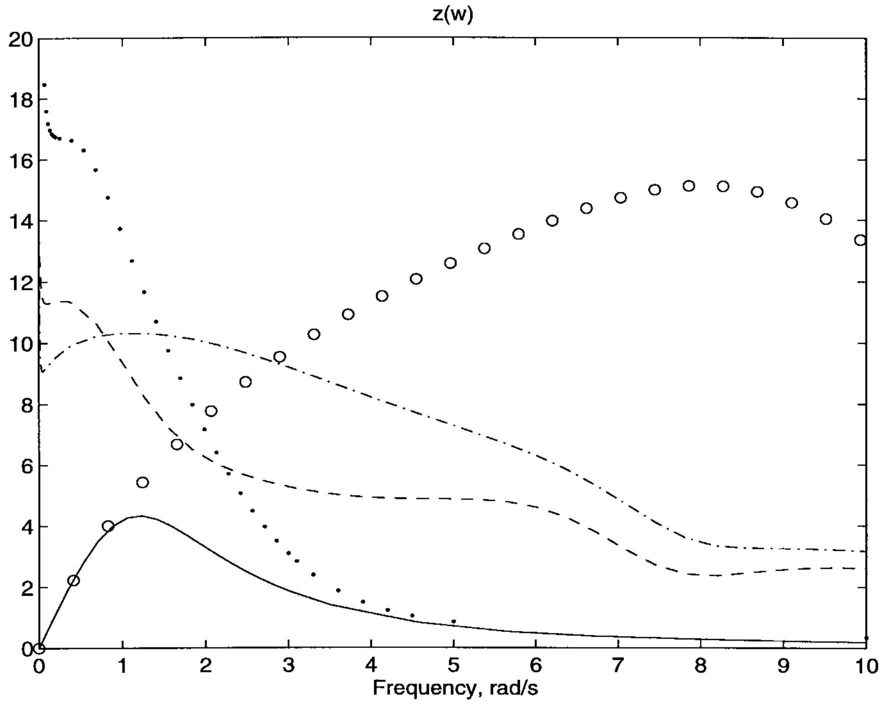

Fig. 4. Robust $\mathcal{H}_2$ analysis: P_0: solid; P_N: circles; P_1: dotted; P_{ltv}: dashed; P_{id}: dash dot.

There exists a tradeoff between objectives (17) and (18): both involve constraints on the feedback $C(s)$. Still, we will adopt a two-stage design strategy, by associating the specification μ_{DR} with C and β_{CR} exclusively with P. As mentioned, this is a restrictive choice, but it is natural when dealing with different robust performance norms. We remark that we also studied the case where both specifications employed the $\mathcal{H}_\infty$ norm (see [6]); in this case we found that the two-stage procedure outperformed the joint synthesis by $D-K$ iteration as in [2].

First, C was designed for robust stability and disturbance rejection by $D-K$ iteration. It is important to mention that the weights W_u and W_1 were selected to satisfy the singular value-type specifications in [15]; this makes our design realistic in regard to these conditions.

Once C is given and ensures robust stability, we address the prefilter design problem with the methods of this paper. We start by using a gridding approach (25 frequency points) on the condition of Proposition 1 to obtain an optimizing frequency response $P_0(\omega_i)$, $i = 0 \cdots 24$ and a performance bound of $\gamma_0 = 3.30$. This sets a limit on achievable performance since we have not imposed yet that $P(s)$ is stable. A number of choices for a stable P are now compared.

- Fitting the points $P_0(\omega_i)$ with stable transfer functions (of third order for each entry), we obtain an approximation $P_1(s)$.
- The optimal $\mathcal{H}_2$ prefilter for the nominal system was computed and named $P_N(s)$.
- A prefilter design based on NLTV uncertainty. A difficulty arose since the feedback controller C was not robustly stabilizing for this larger uncertainty set. We were forced to suitably scale down the weight W_u. With this change, the "impulsive" approach to $\mathcal{H}_2$ performance mentioned in Section IV-B was employed to obtain a prefilter $P_{ltv}(s)$.
- The absence of prefilter, i.e., $P = I_2 := P_{id}$ was also analyzed.

These choices were analyzed by means of Proposition 1; $Y(\omega)$ was obtained for each case, and Fig. 4 contains a plot of $z(\omega) := \sqrt{\mathrm{trace}(Y(\omega))}$. The performance bound is then $(\int_0^\infty z(\omega)^2 (d\omega/\pi))^{1/2}$, presented in Table I, together with the dynamical orders of the prefilters.

TABLE I
ROBUST $\mathcal{H}_2$ PERFORMANCE COST FOR VARIOUS DESIGNS

	β_{CR}	order
$P_0(j\omega_i)$	3.30	-
$P_{ltv}(s)$	10.59	15
$P_1(s)$	11.99	12
$P_{id}(s)$	14.01	0
$P_N(s)$	24.73	28
P_{ltvr}	7.20	15
P_{1r}	7.36	12
P_C	8.25	13

Fig. 4 shows that depending on the frequency range, one design or another may do a better job in approximating the ideal response.[1] Still, there seems to be room for overall improvement. We pursue this by a basis function approach where we fix A and C in the prefilter and search over B and D to reduce the cost over the grid of frequencies; this is a convex problem. Natural choices for A, C are the previous designs; starting from P_1 and P_{ltv} we obtained the improved designs P_{1r} and P_{ltvr}. Another choice is to employ the A and C from the feedback compensator $C(s)$; this has attractive implementation features (see [5]). The resulting prefilter is denoted P_C. These three choices gave very similar performance (see Table I) and significantly improved the earlier designs.

VI. CONCLUSIONS

The results in this paper show that robust feedforward design for systems with structured uncertainty can be reduced to convex optimization problems of the same nature as those available for

[1] Notice the poor performance of the nominal $\mathcal{H}_2$ design.

robustness analysis. Infinite-dimensional convex conditions apply to dynamic prefilter design under LTI uncertainty, which admit well-known methods to approximate them in finite dimensions. For NLTV uncertainty, the problem is solved exactly via an LMI in the $\mathcal{H}_\infty$ performance case. Extensions to generalized $\mathcal{H}_2$ performance measures are given in [6].

Applying these results to tracking problems, we have shown that there is no added difficulty in design when going from feedback synthesis to two-degree-of-freedom synthesis by adding a prefilter: the design problem is tractable, and the only cost is at the implementation stage. Since the performance gains can be significant, this is an attractive design choice.

A concrete case study was examined and different alternative designs were compared.

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Comments on "Robust Stability of Linear Systems with Delayed Perturbations"

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***Abstract*—This paper comments on the result of a recent paper. The estimate of the stability robustness of linear time-delay systems in that paper is compared with the one which is solved in a Riccati matrix inequality framework.**

***Index Terms*—Stability robustness, time-delay systems.**

I. Comments

In the above-mentioned paper,[1] some results about the stability robustness of linear time-delay systems are presented. To make a brief comment on the paper, let us consider the basic linear system with delayed perturbation

$$\dot{x}(t) = Ax(t) + E(t)x(t-h) \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state variable, $A \in \mathbb{R}^{n\times n}$ is a stable matrix, $E(t) \in \mathbb{R}^{n\times n}$ represents the perturbations in the delayed state, and $h > 0$ denotes the delayed interval. First we state the result that is obtained from Theorem 1 of the paper[1].

Theorem 1: System (1) with $\|E(t)\| < \eta$ is asymptotically stable if the condition

$$\eta < \eta_{k_0}(Q,\alpha) := \lambda_{\min}^{1/2}[P^{-1}(2\alpha Q - \alpha^2 I)P^{-1}] \tag{2}$$

is satisfied, where

$$0 < \alpha < 2\lambda_{\min}(Q) \tag{3}$$

and $P > O$ is the solution of

$$A^TP + PA = -2Q, \quad Q > O. \tag{4}$$

In that paper, the author also studied the selection of Q and α which maximize the bound $\eta_{k_0}(Q,\alpha)$. We comment that the problem is solved in the well-known Riccati matrix inequality framework and there is no need to introduce the auxiliary matrix Q.

Proposition 1: System (1) with $\|E(t)\| < \eta$ is asymptotically stable if

$$I + A^TP + PA + \eta^2P^2 < O \quad \text{and} \quad P > O \tag{5}$$

are solvable.

Proof: Let

$$V(x(t)) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\theta)x(\theta)\,d\theta. \tag{6}$$

Then, by a routine calculation, we have

$$\begin{aligned}\dot{V}(x(t)) &= x^T(t)(I + A^TP + PA + \eta^2P^2)x(t)\\ &\quad - x^T(t)P(\eta^2 I - EE^T)Px(t)\\ &\quad - [E^TPx(t) - x(t-h)]^T\\ &\quad \cdot [E^TPx(t) - x(t-h)].\end{aligned} \tag{7}$$

Thus the result follows. □

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