ON THE CARDINALITY OF HOMOGENEOUS COMPACTA OF COUNTABLE TIGHTNESS

Ahmed O.Elnubi

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Abstract. We prove that every homogeneous compacta of countable tightness and d(X)$\leq 2^{\aleph_o}$, is first countable. A relevant conjecture is raised by Arhangel'skiĩ, conjecture 1.17 in [1], see also van Mill [11], which says: every homogeneous compacta of countable tightness is first countable.

1 INTRODUCTION

For all undefined notions, see Engelking[6], Kunnen[10],and Juhasz[9]. Recall that $\pi\chi(X)$, $\pi\chi(A)$,$\pi\omega(X)$, $\omega(X)$, $d(X)$ and $t(X)$ denote the $\pi - character$, $\pi - character$ of A,$\pi - weight$, weight,density and tightness of X. A space X is homogeneous iff for every x,y $\in$ X there is a homeomorphism f of X onto X with f(x) = y. A space is hereditarily separable (HS) iff every subspace is separable. A space is power homogeneous if X^k is homogeneous for some k. All spaces under discussion are Tychonoff.

In this paper, we prove that if a space X is homogeneous compactum of countable tightness and d(X)$\leq 2^{\aleph_o}$, then it is first countable. Results of the same flavour were obtained by Bell [4],and Arhangel'skiĩ [2]. Bell proved that if X is a continuous image of a compact ordered space and X is power homogeneous, then X is first countable. Arhangel'skiĩ proved that if X is Corson compact and power homogeneous then X is first countable, and a compact scattered power homogeneous space is countable. A recently interesting result was obtained by van Mill [12]. He constructed a compactum of countable $\pi - weight$ and character $\aleph_1$ with the property that it is homogeneous under MA+$\dashv$CH whereas CH implies that it is not.

2 Homogeneous compacta of countable tightness.

Lemma 1 : (Šapirovskii[13]) If X is a compactum and t(X) = $\aleph_o$, then we have $\pi\chi(A)$ $\leq\aleph_o$ for every A$\subseteq$ X .

Lemma 2 : If X has $\pi\chi(A)\leq\aleph_o$ for all A$\subseteq$ X, then every dense subspace of X is separable.

Proof : Let Y be dense in X . We can take Y = A, then for every open neighbourhood N of Y there exists $V \in \upsilon =$

$$\{V_i : i = 1, 2, 3, ...\}$$

a countable local base for Y. Using the fact that if Y is dense and V is open, then cl(Y$\cap$V)=cl(V). Choose a point x(V) in the intersection, i.e.x(V)$\in$ Y $\cap$ V_i, then $\{x(V) : V \in \upsilon\}$ is the desired countabl dense set in Y.□

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Lemma 3 : If d(X)$\leq\aleph_o$ and $\pi\chi(A)\leq\aleph_o$ for every A$\subseteq$X, then $|X|\leq 2^{\aleph_o}$.

Proof : Assume that D is dense in X and $|D|\leq\aleph_o$. Associate with each subset A of X a countable sequence of neighbourhoods $g(x)=\{U_ix : i=1,2,3,....\}$ for every x$\in$ X such that $\{x\}=\cap\{U_ix : i=1,2,3,....\}$ and cl(U_{i+1}x) $\subset U_ix$ (here we have made use of the condition $\pi\chi(A)=\aleph_o$ for each A$\subseteq$ X and the regularity of X, also by regularity of X and density of D we have by theorem (3.9)(c) in Hodel [7],$\pi\chi(p,D)=\pi\chi(p,X)$). Let D_ix = U_ix$\cap$D. Clearly, D_ix$\subset$D, $|D_ix|\leq\aleph_o$, x$\in$ cl(D_ix). Now associate x with the sequence $\gamma(x)$ = $\{D_ix : i=1,2,3,....\}$ of countable sets D_ix. Denote by $\Im(X)$ the family $\{\gamma(x) : x\in X\}$ of all sequences $\gamma(x)$ constructed for each x$\in$ X and A$\subset$X . Since $|D|\leq\aleph_o$, $|D^{\aleph_o}|\leq c$ and $|{D^{\aleph_o}}^{\aleph_o}|\leq c$; thus we have

$|\Im(X)|\leq|{D^{\aleph_o}}^{\aleph_o}|\leq c$.

We need to show that the correspondence γ :X$\rightarrow\Im(X)$ is one-to-one. Let x_1,$x_2$$\in$ X, $x_1\neq x_2$. Let i_1 be such that $x_2\notin$ cl($U_{i_1}x_1$), where $U_{i_1}x_1\in$ g(x_1). Take i_2 such that $x_1\notin$ cl($U_{i_2}x_2$), where $U_{i_2}x_2\in$g(x_2).

Suppose $i_2\geq i_1$. Then it is clear that $x_1\notin$cl($U_{i_2}x_2$) and $x_2\notin$ cl($U_{i_2}x_1$). Consider ($D_{i_2}x_2$)= ($U_{i_2}x_2$)$\cap$ D, ($D_{i_2}x_1$)= ($U_{i_2}x_1$)$\cap$ D . Since $x_1\in$ cl($D_{i_2}x_1$),

$x_2\in$ cl($D_{i_2}x_2$) and $x_1\notin$ cl($U_{i_2}x_2$),$x_2\notin$ cl($U_{i_2}x_1$), then we have $x_1\notin$ cl($D_{i_2}x_2$), $x_2\notin$ cl($D_{i_2}x_1$); thus we have ($D_{i_2}x_1$)$\neq$($D_{i_2}x_2$), that is γ (x_1) and γ (x_2) are distinct sequences . Thus the correspondence γ :X$\rightarrow\Im(X)$ is one-to-one. Hence $|X|\leq|\Im(X)|\leq c$. □

Lemma 4 : If X has $\pi\chi(A)\leq\aleph_o$ for every A$\subseteq$ X, t(X) =$\aleph_o$ and d(X)$\leq 2^{\aleph_o}$, then $|X|\leq 2^{\aleph_o}$.

Proof : Let D$\subset$ X be such that cl(D) =X and $|D|\leq c$. Since t(X) = $\aleph_0$) then for for every point x$\in$ X there exists $D_x\subset$ D such that $|D_x|\leq\aleph_0$ and x$\in$ clD_x. Since $\pi\chi(A)\leq\aleph_o$ for every A$\subseteq$ X and $d(clD_x)\leq\aleph_0$, then $|clD_x|\leq c$,by Lemma 3. Denote by $\Xi(D)$ the collection of all finite or countable sets belonging to D. Since $|D|\leq c$ and $c^{\aleph_0}=c$, we have $|\Xi(D)|\leq c$. Also, since for each x$\in$X there exists a countable set $D_x\in\Xi(D)$ for which x$\in$clD_x, we have X =$\cup$(cl(B):B$\in\Xi(D)$). But $|\Xi(D)|\leq c$. and $|cl(B)|\leq c$ for every B$\in\Xi(D)$. Hence $|X|\leq c$. □

Theorem 5 ($2^{\aleph_o}\prec 2^{\aleph_1}$) : If X is a homogeneous compactum, t(X)=$\aleph_o$and d(X)$\leq 2^{\aleph_o}$, then it is first countable.

Proof : From Ismail[8], and using Lemma 1 and 4. □
Van Douwen [3] proved that if X has a countable$\pi-base$, then $|X|\leq 2^{\aleph_o}$.

Corollary 6 ($2^{\aleph_o}\prec 2^{\aleph_1}$) : A homogeneous compactum space of countable $\pi-base$ is first countable.

Proof : From Van Douwen [5] $|X|\leq 2^{\aleph_o}$ and from Ismail[8] we have $|X|=2^{\chi}$ and the proof follows .

Corollary 7 ($2^{\aleph_o}\prec 2^{\aleph_1}$) : A compact homogeneous sequential space is first countable.

Proof : From Arhangel'skiĩ [3], pp.134, problem 152, $|X|=2^{\aleph_o}$ and from Ismail[8], we have $|X|=2^{\chi}$.

By Šapirovskii [14], any compact HS, must have countable$\pi - weight$, so if it is also homogeneous, it must have size at most $2^{\aleph_o}$ by Van Douwen [5]. By using the inequality in Ismail [8],under CH the space must be first countable.

Corollary 8 : If there is a dense subset of X which is separable and $\pi\chi(X)\leq\aleph_o$, then $|X|\leq 2^{\aleph_o}$.

Proof : Using Theorem (3.8)(b) of Hodel [7], and van Douwen [5].

Corollary 9 ($2^{\aleph_o}\prec 2^{\aleph_1}$) : If X is a homogeneous compactum and h$\pi\chi(X)\leq\aleph_o$, and d(X)$\leq\aleph_o$, then it is first countable.

Proof : Using Theorem (3.8)(d) of Hodel [7], and theorem 5 above.

3 Examples and Conclusions

[1]The space βN is characterized by d(βN)$\leq 2^{\aleph_o}$.
[2]The space R^X, where the space X is Tychonoff compact, with the toplogy of uniform convergence or of pointwise convregence contains a dense subset of cardinality at most $2^{\aleph_o}$ if and only if $|X|\leq 2^{\aleph_o}$,see Engelking [6].
[3]From Hodel [7] theorem (3.8)(b) $\pi\omega(X)$ = d(X).$\pi\chi(X)$ and using Van Douwen result in[5],this means if d(X).$\pi\chi(X)$ $\leq 2^{\aleph_o}$ then $|X|\leq 2^{\aleph_o}$. Lemma 3 above is a better estimate than this combined result .
[4]By the Hewitt-Marczewski-Pondiczery theorem:If d(X_s)$\leq 2^{\aleph_o}$, for every s$\in$ S and $|S|$ $\leq 2^{2^{\aleph_o}}$, then d($\pi_{s\in S}X_s$)$\leq 2^{\aleph_o}$. Assuming the productivity of homogeneity, then we can conclude the productivity of of first countability within the class of homogeneous compactum spaces satisfying the conditions of theorem 5.
[5] Applications of Theorem 1.1 in van Mill [11] are that the cardinality of a homogeneous compactum which has countable spread or is hereditarily normal and satisfies the countable chain condition does not exceed c. Using Theorem 5 to this class of spaces in addition to countable tightness and d(X)$\leq 2^{\aleph_o}$, then easily we deduce the first countability of these spaces.

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Ahmed Osman Elnubi
Teacher's College,Dept. of Math.,
P.O.Box 2313,
Code 31982
Al Ahsa, Saudi Arabia.
e-mail:nubi1234@hotmail.com
or P.O.Box 12795 , Khartoum , Sudan