Gravity Flow of Frictional-Cohesive Solids—Convergence to Radial Stress Fields

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Evidence is presented to the effect that general stress fields in the steady, gravity flow of frictional-cohesive solids in converging channels in plane strain and in axial symmetry approach geometrically similar radial stress fields in the region of the vertex. Radial stress fields were discussed in an earlier paper [1].[2]

Introduction

In an earlier paper [1],[2] Jenike discussed the stress fields which occur in steady gravity flow of frictional-cohesive particulate solids in converging channels in plane strain and in axial symmetry. Using the principles of plasticity, he described the fields by means of the following relations:

(*a*) Two equations of equilibrium in plane-polar/spherical coordinates

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r}\,[\sigma_r - \sigma_\theta + m(\sigma_r - \sigma_\alpha) + m\tau_{r\theta}\cot\theta] + \gamma\cos\theta = 0 \quad (1)$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial_\theta} + \frac{1}{r}\,[m(\sigma_\theta - \sigma_\alpha)\cot\theta + (2+m)\tau_{r\theta}] - \gamma\sin\theta = 0 \quad (2)$$

Here, $m = 0$ for plane strain and $m = 1$ for axial symmetry. The coordinates and the stresses are defined in Fig. 1. Compressive stresses are taken positive.

(*b*) The equation of state, called the "effective yield function" [2]

$$(\sigma_r + \sigma_\theta)\sin\delta - [(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2]^{1/2} = 0 \quad (3)$$

where δ is the effective angle of friction of a solid, Fig. 2, and is assumed constant for a given solid.

(*c*) In axial symmetry, the Haar and von Karman hypothesis [3] which, in converging flow, sets the value of the circumferential pressure σ_α equal to the major pressure σ_1 of the meridian plane r, θ

$$\sigma_\alpha = \sigma_1 \quad (4)$$

(*d*) A one-to-one relation between mean pressure

$$\sigma = \tfrac{1}{2}(\sigma_r + \sigma_\theta) \quad (5)$$

and density

$$\gamma = \gamma(\sigma) \quad (6)$$

These equations, together with appropriate stress boundary conditions, define a general stress field.

Jenike then numerically computed the particular stress fields, called the "radial stress fields," for which σ and ψ have the particular forms

$$\sigma = \gamma r s(\theta) \quad (7)$$

$$\psi = \psi(\theta) \quad (8)$$

in which $s(\theta)$ is a stress function and ψ is the angle between the direction of the major pressure σ_1 and the coordinate ray, Fig. 1. Density

$$\gamma = \gamma_0 \quad (9)$$

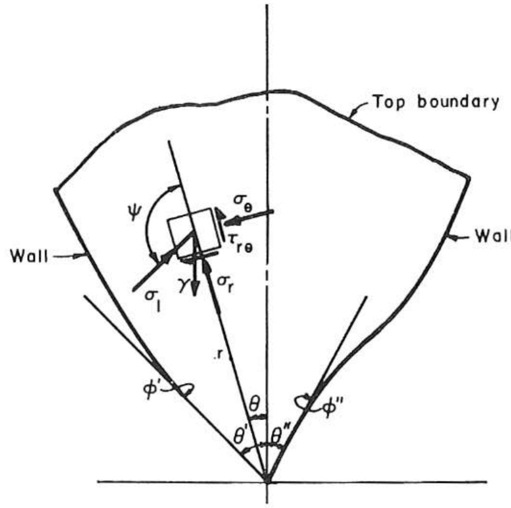

Fig. 1 A general channel

is constant; i.e., the solid is incompressible.

In channels with straight walls, radial stress fields are defined by one pair (θ', ϕ'), when the walls are symmetric with respect to the vertical axis, and by two pairs (θ', ϕ') and (θ'', ϕ''), when the walls are not symmetric. θ' and θ'' measure the slopes of the walls with respect to the vertical, while ϕ' and ϕ'' are the kinematic angles of friction between the flowing solid and the walls, Fig. 1.

Convergence of General Stress Fields to Radial Stress Fields

The significance of radial stress fields is greatly enhanced by the fact that real stress fields in plane strain and in axial symmetry converge to the radial stress fields as the vertex of the channel is approached; i.e., for $r \to 0$. This convergence is of fundamental importance in the derivation of flow-no-flow criteria for particulate solids (bulk solids) [4] because most obstructions to flow originate at the outlets of channels; i.e., in the vicinity of the vertex. Since, in real channels, convergence to a radial stress field is very rapid, the stresses in the region of an outlet are usually sufficiently well represented by the radial stress field. Therefore, they can be predicted on the basis of the slope and angle of friction of the walls at the outlet without reference to the top boundary conditions, such as the head of the solid over the outlet, the shape of the top traction-free boundary, and the shape of the walls away from the outlet.

This independence of the head and boundaries away from the vertex has been observed by experimenters and engineers for a long time [6, 7, 8]. An analysis of the equations shows how this convergence takes place.

It is compatible with experience and with the equation of state (3) to assume that density is represented by the function

$$\gamma = \gamma_0 + \gamma(\sigma) \quad (10)$$

where $\gamma(0) = 0$, and the function is continuous. In general fields, expressions (7) and (8) for σ and ψ do not hold but can be written in the form

$$\sigma = r[\gamma_0 + \gamma(\sigma)]s(r, \theta) \quad (11)$$

$$\psi = \psi(r, \theta) \quad (12)$$

The component stresses are

$$\sigma_r = \sigma(1 + \sin\delta\cos 2\psi) \quad (13)$$

$$\sigma_\theta = \sigma(1 - \sin\delta\cos 2\psi) \quad (14)$$

$$\tau_{r\theta} = \sigma\sin\delta\sin 2\psi \quad (15)$$

$$\sigma_\alpha = \sigma_1 = \sigma(1 + \sin\delta) \quad (16)$$

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[2] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, April 9, 1964.

Expression (11) is substituted for σ in the foregoing expressions, which are, in turn, substituted for the component stresses and their derivatives in the equations of equilibrium (1) and (2), leading, after transformations, to

$$\frac{\partial s}{\partial \theta} + sf(r, \theta) + g(r, \theta) = 0 \tag{17}$$

$$r\frac{\partial s}{\partial r} + sh(r, \theta) + j(r, \theta) = 0 \tag{18}$$

where

$$f(r, \theta) = 2\left(\frac{\partial \psi}{\partial \theta} + 1\right)\frac{\sin\delta}{\cos^2\delta}\sin 2\psi + 2r\frac{\partial \psi}{\partial r}\frac{\sin\delta}{\cos^2\delta}(\sin\delta + \cos 2\psi) + \frac{1}{\gamma}\frac{\partial \gamma}{\partial \theta} + m\frac{\sin\delta}{\cos^2\delta}(1 + \sin\delta)[\sin 2\psi - \cot\theta(1 + \cos 2\psi)] \tag{19}$$

$$g(r, \theta) = -\frac{\sin\delta}{\cos^2\delta}\sin(\theta + 2\psi) - \frac{\sin\theta}{\cos^2\delta} \tag{20}$$

$$h(r, \theta) = 1 + 2\left(\frac{\partial \psi}{\partial \theta} + 1\right)\frac{\sin\delta}{\cos^2\delta}(\cos 2\psi - \sin\delta) - 2r\frac{\partial \psi}{\partial r}\frac{\sin\delta}{\cos^2\delta}\sin 2\psi + \frac{r}{\gamma}\frac{\partial \gamma}{\partial r} + m\frac{\sin\delta}{\cos^2\delta} \times (1 + \sin\delta)(\cot\theta\sin 2\psi + \cos 2\psi - 1) \tag{21}$$

$$j(r, \theta) = -\frac{\sin\delta}{\cos^2\delta}\cos(\theta + 2\psi) + \frac{\cos\theta}{\cos^2\delta} \tag{22}$$

Equations (17) and (18) have real characteristics whose directions are

$$\frac{dr}{d\theta} = r\cot\left[\psi \pm \left(\frac{\pi}{4} - \frac{\delta}{2}\right)\right] \tag{23}$$

The equations are hyperbolic for $r \neq 0$ but parabolic for $r = 0$ ($\psi = \pi/4 - \delta/2$ and $\psi = 3\pi/4 + \delta/2$ lie outside of real channels, because ϕ' and $\phi'' < \delta$, Fig. 2). A typical field of characteristics for the general channel in Fig. 1 is shown in Fig. 3 in orthogonal axes r, θ. It is apparent that the effect of the variable r decreases as $r \rightarrow 0$. At $r = 0$, the parabolic equations are independent of the derivatives with respect to r and, in fact, become those of the radial stress field. Hence, barring discontinuities, the general stress field converges to a unique radial stress field at the vertex.

The concepts of plasticity permit some stress discontinuities. In particular, a discontinuity can follow a streamline. This type of discontinuity does not affect the present discussion because such a discontinuity can be looked upon as a wall and the regions on each side of the discontinuity can be considered as separate channels. Any other discontinuity would reach a wall, reflect from it, and continue reflecting from wall to wall since the field is hyperbolic. A solid flowing across these lines of stress discontinuity would undergo successive, discontinuous expansions and contractions. It is unlikely that these conditions can satisfy the second law of thermodynamics.

In physical channels, discontinuities do not seem to appear in flowing solids. If a discontinuity is introduced at the wall of a channel, the solid either does not flow at all or remains rigid (elastic) in a part of the channel and forms new walls across itself. Surface discontinuities along a top boundary are invariably contained within rigid regions.

Results of numerical solutions of general fields were presented by Johanson [5], and an example from this paper is shown in Fig.

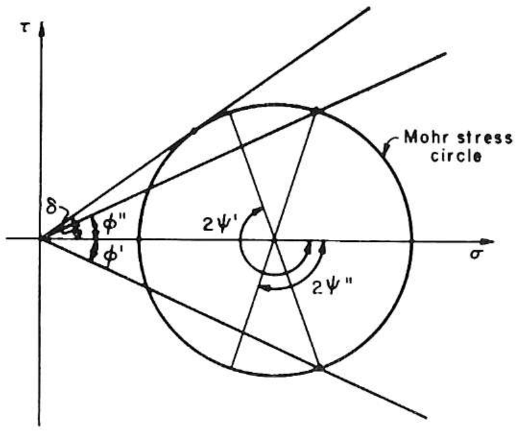

Fig. 2 Angles of friction

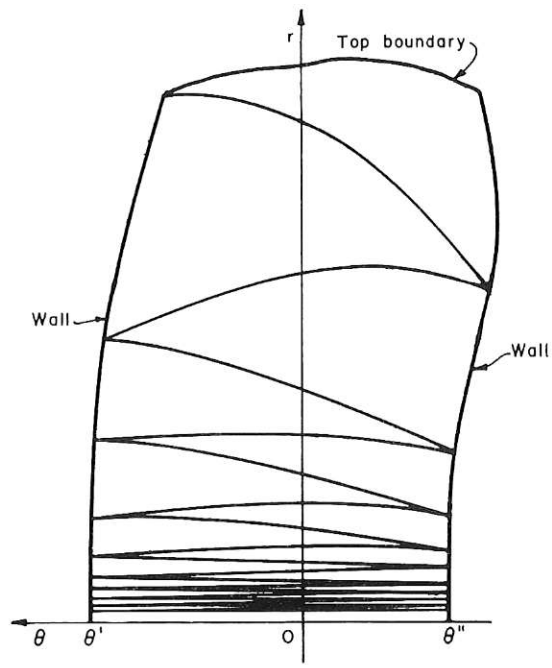

Fig. 3 Field of characteristics

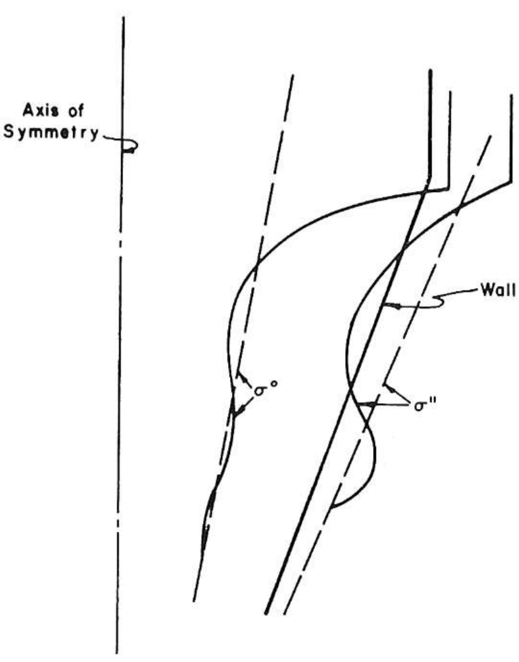

Fig. 4 Example of convergence

4. Mean pressures, equation (5), are plotted: σ^0 is the value along the axis of symmetry, and σ'' along the wall. The dashed lines represent radial stress values; the continuous wavy lines represent the values of a general stress field which arises from a high vertical head of the solid above the wedge. The convergence is very rapid.

It should be noted that the effective yield function used in this analysis applies to cohesive as well as cohesionless solids. The relevant relations are discussed in [1, 2, 4, 9, 10, 11].

Acknowledgments

Some of the results presented in this paper were obtained in the course of work conducted at the University of Utah under a National Science Foundation grant to the project, "Flow of Rigid-Plastic Solids in Converging Channels Under the Action of Body Forces." The author gratefully acknowledges the contributions of his former students, J. R. Johanson and J. L. Taylor.

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Finite Wave Propagation in Deformable Tubes

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Most studies of transient flow in deformable tubes have been made in connection with the analysis of water hammer. They have dealt therefore with small elastic deformations of tubes which are so stiff that the compressibility of the fluid must be taken into account. This Note is concerned with tubes which allow such large deformations that the fluid may be regarded as incompressible.

It is shown that, in such systems, strong compression waves will normally travel appreciably faster than weak ones. This prediction has been confirmed by experiment.

Theoretical Treatment

Fig. 1(a) shows a pressure pulse propagating with speed a relative to the fluid in front of it (which has velocity u_1, pressure p_1, and cross-sectional area A_1). In Fig. 1(b), the flow is made steady by the overall superposition of the flow velocity $-(a + u_1)$. Quantities behind the pulse are denoted by the subscript 2. Referring to the steady flow, the continuity equation is

$$A_1 a = A_2[a - (u_2 - u_1)] \tag{1}$$

and if the average pressure on the inside of the wall at the transition section is taken to be $\frac{1}{2}(p_1 + p_2)$, the momentum principle gives

$$\rho[a - (u_2 - u_1)]^2 A_2 - \rho a^2 A_1 = \frac{1}{2}(p_1 - p_2)(A_1 + A_2) \tag{2}$$

where ρ is the density of the fluid. Viscosity of the fluid and the wall material are ignored, and it is therefore unnecessary to consider energy conservation explicitly.

Equations (1) and (2) yield the following result for the wave velocity:

$$a^2 = \frac{1}{2\rho}\cdot\frac{A_2}{A_1}\cdot(p_2 - p_1)\cdot\frac{A_2 + A_1}{A_2 - A_1} \tag{3}$$

There will be a pressure-area relationship associated with any particular tube, and this may or may not be linear and elastic. For a given A_1 and p_1, compatible values of A_2 and p_2 must be chosen for the tube and equation (3) solved for a.

For small disturbances, A_2 and p_2 may be put equal to $(A + dA)$ and $(p + dp)$, respectively. To the first order in small quantities, equation (3) gives the usual expression[3] for the speed of propagation a of a weak pulse

$$a^2 = \frac{A}{\rho}\cdot\frac{dp}{dA} \tag{4}$$

Sharp finite discontinuities are most likely to occur for compression waves. This is because an increase in pressure is associated with an increase in velocity, so that each successive increment in pressure will be traveling relative to successively greater flow velocities. If the tube is such that the wave speed either remains constant or increases with pressure, compression waves will converge to form a sharp discontinuity.

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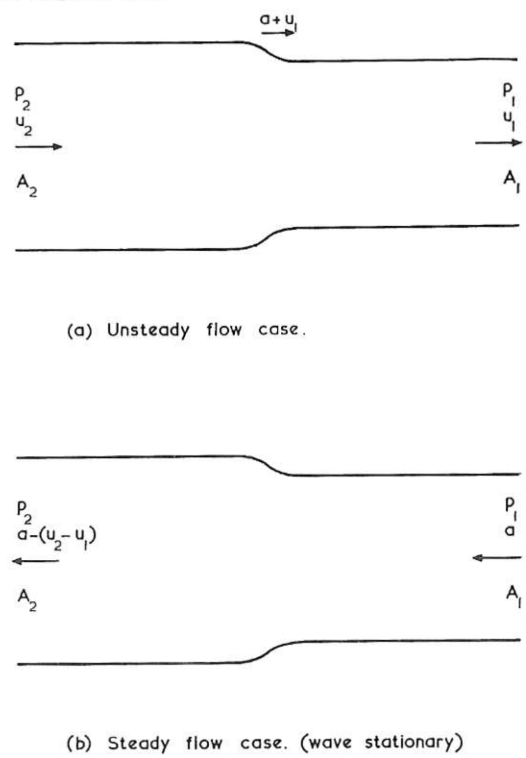

Fig. 1 Propagation of pressure pulse in deformable tube

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Manuscript received by ASME Applied Mechanics Division, July 2, 1964.