Testing Fuzzy Hypotheses Using Fuzzy Data
Based on Fuzzy Test Statistic

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Abstract

This paper deals with the problem of testing hypothesis when both the hypotheses and the available data are fuzzy. First, four different kinds of fuzzy hypotheses are defined. Then, a procedure is developed for constructing the fuzzy point estimation based on fuzzy data. Also, the concept of fuzzy test statistic is defined based on the α-cuts of the fuzzy null hypothesis and the α-cuts of the constructed fuzzy point estimation. Finally, by introducing a credit level, we propose a method to evaluate the fuzzy hypotheses of interest. The proposed method is employed to test the fuzzy hypotheses for the mean of a normal distribution, and the variance of a normal distribution. A practical example in lifetime testing is provided, to show the applicability of the proposed method in applied studies.
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1 Introduction and Motivation

In classical approaches to testing statistical hypotheses, it is assumed that both the underlying hypotheses and the available data are crisp. For example, if the difference between two population means is to be tested, the ordinary null hypothesis stipulates that the difference between two population means is precisely equal to zero. In addition, it is usually assumed that the collected observations are precise. However, we would sometimes like to test if two means are nearly equal or not. On the other hand, sometimes the available observations are not precise. For instance, in economic studies we may wish to test if the means of incomes of households of two interested populations are approximately equal or not. In such a case, the hypothesis of exact equality of means seems to be unrealistic. As an example of imprecise data, consider the problem of lifetime testing. In lifetime analysis, the data available are usually reported as imprecise data. For instance, measuring the lifetime of a battery may not yield an exact result. A battery may work perfectly over a certain period but be losing in power for some time, and finally go dead completely at a certain time. In this case, the data may be reported as imprecise quantities such as: about 1000 (h), approximately 1400 (h), about between 1000 (h) and 1200 (h), essentially less than 1200 (h), and so on. The classical procedures for testing hypotheses are not appropriate for dealing with such imprecise cases. After the inception of the notion of fuzzy sets by Zadeh [32], there have been attempts to analyze the problem of testing hypotheses for these situations using fuzzy set theory. See Taheri [21] for a review of some related works.

In the present work, we consider the fuzzy hypotheses instead of crisp ones, and introduce a procedure to test such hypotheses based on a fuzzy test statistic when the available data are fuzzy. Our proposed procedure is an extension of Taheri and Arefi's approach to testing statistical hypotheses [22], in which the data are assumed to be crisp. But, here we consider testing fuzzy hypotheses when the available data are fuzzy, too. In this extension, we construct a fuzzy test statistic using the α-cuts of the fuzzy null hypothesis and the α-cuts of the fuzzy point estimation. Then, we introduce a procedure for testing the fuzzy hypotheses of interest.

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Testing statistical hypotheses under imprecise (fuzzy) constraints were investigated by some authors. The problem of testing hypotheses with fuzzy data was considered by Casals and Gil [5], Casals et al. [6], Filzmoser and Viertl [9], Grzegorzewski [12], and Wu [31]. This topic, using fuzzy random variables, was studied by Körner [16] and Montenegro et al. [19]. Testing fuzzy hypotheses was discussed by Arnold [1, 2], Taheri and Arefi [22], Taheri and Behboodian [23, 24], and Watanabe and Imaizumi [29]. Testing fuzzy hypotheses with fuzzy data was investigated by Grzegorzewski [13], Kruse and Meyer [17], Taheri and Behboodian [25], and Torabi et al. [27]. For some other recent works on testing hypothesis in fuzzy environment, see Buckley [3, 4], Denoeux and Masson [7], Hryniewicz [14], Thompson and Geyer [26], and Viertl [28]. Also, the problem of point estimation in the fuzzy environment has been investigated by some authors, e.g. Gil et al. [11] and Wu [30].

This paper is organized as follows: In Section 2, we recall some preliminary concepts about fuzzy numbers and interval arithmetic. In Section 3, we introduce different kinds of fuzzy hypotheses. A new method to test fuzzy hypotheses based on fuzzy data is given in Section 4. In Section 5, we apply our method to test fuzzy hypotheses for the mean and for the variance of a normal distribution. A practical example in lifetime testing is provided in Section 6. In Section 7, we compare our method with some other works. A brief conclusion is provided in Section 8.

2   Preliminary Concepts

In this section, we recall some preliminary concepts about fuzzy numbers and interval arithmetic. For details, the reader can refer to standard texts, e.g. Klir and Yuan [15].

A fuzzy set $\widetilde{A}$ of the universe X is defined by a membership function $\widetilde{A} : X \rightarrow [0, 1]$. An α-cut of $\widetilde{A}$, written as $\widetilde{A}[\alpha]$, is defined as $\widetilde{A}[\alpha] = \{x | \widetilde{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$.

A fuzzy number $\widetilde{M}$ is a fuzzy set of the real numbers satisfying:

i) $\widetilde{M}(x) = 1$ for some x,

ii) $\widetilde{M}[\alpha]$ is a closed bounded interval for $0 < \alpha \leq 1$.

A triangular fuzzy number $\widetilde{T}_1 = (a_1, a_2, a_3)_T$ is defined by three numbers $a_1 < a_2 < a_3$ as

$$\widetilde{T}_1(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & a_1 \leq x < a_2, \\ \frac{a_3 - x}{a_3 - a_2} & a_2 \leq x < a_3. \end{cases}$$

Also, the membership functions of fuzzy sets $\widetilde{T}_2 = (a_1, a_2)_{EL}$ and $\widetilde{T}_3 = (a_2, a_3)_{ES}$ for $a_1 < a_2 < a_3$ are defined as

$$\widetilde{T}_2(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & a_1 \leq x < a_2, \\ 1 & a_2 \leq x, \end{cases} \qquad \widetilde{T}_3(x) = \begin{cases} 1 & x < a_2, \\ \frac{a_3 - x}{a_3 - a_2} & a_2 \leq x < a_3. \end{cases}$$

Let $I = [a, b]$ and $J = [c, d]$ be two closed intervals. Then, based on the interval arithmetic, we have

$$I + J = [a + c, b + d],$$

$$I - J = [a - d, b - c],$$

$$I.J = [\alpha_1, \beta_1], \quad \alpha_1 = \min\{ac, ad, bc, bd\}, \quad \beta_1 = \max\{ac, ad, bc, bd\},$$

$$I \div J = [\alpha_2, \beta_2], \quad \alpha_2 = \min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \quad \beta_2 = \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\},$$

where, zero does not belong to $J = [c, d]$ in the last case.

3   Fuzzy Hypotheses

In this section, we recall some models as fuzzy sets of real numbers for modelling the extensions of simple, one-sided, and two-sided ordinary (crisp) hypotheses to fuzzy ones (see also [22]).

Definition 1 *Let θ_0 be a known real number.*

i) Any hypothesis of the form ($H : \theta$ is approximately θ_0) is called a fuzzy simple hypothesis.
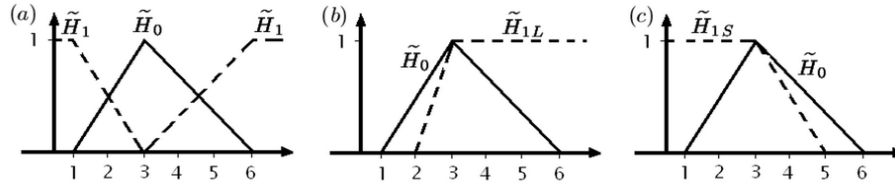
Figure 1: Three forms of fuzzy hypotheses in Example 1

ii) Any hypothesis of the form $(H : \theta$ is not approximately $\theta_0)$ or, equivalently, $(\theta$ is away from $\theta_0)$ is called a fuzzy two-sided hypothesis.

iii) Any hypothesis of the form $(H : \theta$ is essentially smaller than $\theta_0)$ is called a fuzzy left one-sided hypothesis.

iv) Any hypothesis of the form $(H_1 : \theta$ is essentially larger than $\theta_0)$ is called a fuzzy right one-sided hypothesis.

In the next section, we investigate some methods to test the following forms of hypotheses:

$$a) \begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_1 : \theta \text{ is not approximately } \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_1, \end{cases}$$

$$b) \begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1L} : \theta \text{ is essentially larger than } \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}, \end{cases}$$

$$c) \begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1S} : \theta \text{ is essentially smaller than } \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1S}. \end{cases}$$

For the above hypotheses, we suppose that

$$\widetilde{H}_0 = (a_1, \theta_0, a_3)_T, \quad \widetilde{H}_1 = \widetilde{H}_0^c, \quad \widetilde{H}_{1L} = (a_1', \theta_0)_{EL}, \quad \widetilde{H}_{1S} = (\theta_0, a_3')_{ES},$$

where $a_1 \leq a_1'$ and $a_3' \leq a_3$.

Note 1 *It should be mentioned that case (a) is a natural generalization of a crisp simple hypothesis versus a two-sided hypothesis of the form: $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta \neq \theta_0$. Moreover, case (b) is a natural generalization of the crisp simple hypothesis $H_0 : \theta = \theta_0$ v.s. one-sided hypothesis $H_1 : \theta > \theta_0$. A similar argument holds for case (c).*

Example 1 *In the above fuzzy hypotheses, suppose that $a_1 = 1$, $a_1' = 2$, $\theta_0 = 3$, $a_3' = 5$, and $a_3 = 6$, then we have some fuzzy hypotheses in which the related membership functions are shown in Fig. 1: $(a) - (c)$.*

4 Testing Fuzzy Hypotheses Based on Fuzzy Data

Let $X_1, X_2, ..., X_n$ be a random sample from a probability density function (or probability mass function) $f(x; \theta)$, where the parameter θ is unknown. Suppose that the available data of the random sample are observed as the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$ rather than the crisp data $x_1, x_2, ..., x_n$. We can obtain a fuzzy point estimation for θ as follows.

Definition 2 *Let $\theta^* = u(x_1, x_2, ..., x_n)$ be a point estimation for θ. By substituting the α-cut of the fuzzy numbers $\widetilde{X}_i$, $i = 1, ..., n$ for x_i, $i = 1, ..., n$ into θ^*, the α-cut of the fuzzy point estimation $\widetilde{\theta}^*$ is obtained as follows*

$$\widetilde{\theta}^*[\alpha] := \left\{ u(x_1, x_2, ..., x_n); \ x_i \in \widetilde{X}_i[\alpha], \ i = 1, 2, ..., n \right\}.$$

In the following, we introduce a procedure for testing a fuzzy simple hypothesis against a fuzzy two-sided hypothesis and a fuzzy one-sided hypothesis, respectively.

4.1 Testing Fuzzy Simple Hypothesis against Fuzzy Two-sided Hypothesis

Suppose that we are interested in testing the following fuzzy hypotheses

$$\begin{cases} H_0 : \theta \ is \ approximately \ \theta_0, \\ H_1 : \theta \ is \ not \ approximately \ \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \ is \ \widetilde{H}_0, \\ H_1 : \theta \ is \ \widetilde{H}_1, \end{cases}$$

where $\widetilde{H}_0 = (a_1, \theta_0, a_3)_T$ is a triangular fuzzy number and its α-cuts are $\widetilde{\theta}_0[\alpha] = [a_1 + (\theta_0 - a_1)\alpha, a_3 - (a_3 - \theta_0)\alpha]$. In the crisp case, the decision rule for testing a crisp null hypothesis $H_0 : \theta = \theta_0$ against a crisp alternative $H_0 : \theta \neq \theta_0$, at the significance level β, is

$$\begin{cases} Q_0 \geq Q_{1-\beta/2} \ or \ Q_0 \leq Q_{\beta/2} & \Rightarrow & RH_0 \ (Rejection \ of \ H_0), \\ Q_{\beta/2} < Q_0 < Q_{1-\beta/2} & \Rightarrow & AH_0 \ (Acceptance \ of \ H_0), \end{cases}$$

where Q_0 is the value of the crisp test statistic (under H_0), and $Q_{\beta/2}$ and $Q_{1-\beta/2}$ are the $\beta/2$ and $1 - \beta/2$ quantiles of the crisp test statistic. Now, we introduce an approach for testing the above fuzzy hypotheses based on the fuzzy data $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$.

i) First, we obtain a fuzzy point estimation $\widetilde{\theta}^*$ using Definition 2.

ii) By substituting α-cuts of the fuzzy point estimation $\widetilde{\theta}^*$ for the point estimation θ^*, and the α-cuts of $\widetilde{H}_0$ for θ_0 in the crisp test statistic (Q_0) and by using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$.

Subsequently, the obtained fuzzy test statistic is used to provide an approach for testing the fuzzy hypotheses based on the following quadruplet procedure [22] (see Fig. 2).

a) We calculate the total area under the graph of $\widetilde{Z}$, denoted by A_T.

b) We obtain the area under the graph of $\widetilde{Z}$, but to the right of the vertical line through $Q_{1-\beta/2}$ and to the left of the vertical line through $Q_{\beta/2}$, denoted by A_R.

c) We choose a value for the credit level ϕ from $(0, 1]$.

d) Finally, we decide to reject or accept H_0 in the following way

$$\begin{cases} A_R/A_T \geq \phi \Rightarrow RH_0, \\ A_R/A_T < \phi \Rightarrow AH_0. \end{cases}$$

Remark 1: Note that, in the underlying current environment, we come across two kinds of uncertainty. The first kind of uncertainty (probabilistic one) is related to the randomness of data and, in testing hypothesis, it is controlled by the significance level β (or the confidence level $1 - \beta$). But, the second kind of uncertainty (possibilistic one) is due to the impreciseness (fuzziness) of the data as well as due to the impreciseness of the hypotheses of interest. This kind of uncertainty may be controlled by credit level ϕ in making the decision whether to accept or reject H_0. It is obvious that the selected value of ϕ is more or less subjective so that, by increasing the credit level ϕ (as by increasing the confidence level $1 - \beta$), we guard against the rejection of H_0.

Remark 2: As a special case, suppose that we want to test the following crisp hypothesis based on the fuzzy data $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$:

$$\begin{cases} H_0 : \theta = \theta_0, \\ H_1 : \theta \neq \theta_0. \end{cases}$$

The α-cuts of the fuzzy test statistic $\widetilde{Z}$ are obtained by the proposed approach, by substituting α-cuts of the fuzzy point estimation $\widetilde{\theta}^*$ for the point estimation θ^* in the crisp test statistic (Q_0). Then, we can test these hypotheses based on the quadruplet procedure provided.
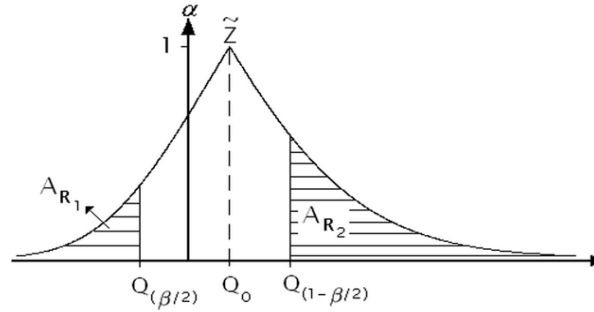
Figure 2: The related quantities in testing a fuzzy simple hypothesis versus a fuzzy two-sided hypothesis

4.2   Testing Fuzzy Simple Hypothesis against Fuzzy Right One-sided Hypothesis

Suppose that we wish to test the following fuzzy hypotheses

$$\begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1L} : \theta \text{ is essentially larger than } \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}, \end{cases}$$

where $\widetilde{H}_0 = (a_1, \theta_0, a_3)_T$ is a triangular fuzzy number for which $\widetilde{\theta}_0[\alpha] = [a_1 + (\theta_0 - a_1)\alpha, a_3 - (a_3 - \theta_0)\alpha]$. Let $\theta^* = u(x_1, x_2, ..., x_n)$ be a point estimation for θ. In the ordinary case, the decision rule for testing a crisp null hypothesis $H_0 : \theta = \theta_0$ against a crisp alternative $H_0 : \theta > \theta_0$, at the significance level β, is of the form

$$\begin{cases} Q_0 \geq Q_{1-\beta} \Rightarrow RH_0, \\ Q_0 < Q_{1-\beta} \Rightarrow AH_0, \end{cases}$$

where Q_0 is the value of the crisp test statistic (under H_0), and $Q_{1-\beta}$ is the $(1-\beta)$-quantile of the distribution of the crisp test statistic. Now, we introduce an approach for testing the above fuzzy hypotheses based on the fuzzy data $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$.

i) First, using Definition 2, we obtain a fuzzy point estimation $\widetilde{\theta}^*$.

ii) By substituting the α-cuts of the fuzzy point estimation $\widetilde{\theta}^*$ for the point estimation θ^*, and the α-cuts of $\widetilde{H}_0$ for θ_0 in the crisp test statistic (Q_0) and by using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$.

Now, we use the fuzzy test statistic thus obtained to provide an approach for testing fuzzy right one-sided hypotheses based on the following quadruplet procedure [22] (see Fig. 3).

a) We calculate the total area under the graph of $\widetilde{Z}$, denoted by A_T.

b) We obtain the area under the graph of $\widetilde{Z}$, but to the right of the vertical line through $Q_{1-\beta}$, denoted by A_R.

c) We choose a value for the credit level ϕ from $(0, 1]$.

d) Finally, we decide whether to reject or accept H_0 in the following way

$$\begin{cases} A_R/A_T \geq \phi \Rightarrow RH_0, \\ A_R/A_T < \phi \Rightarrow AH_0. \end{cases}$$

Remark 3: As a special case, suppose that we want to test the following crisp hypothesis based on the fuzzy data $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$:

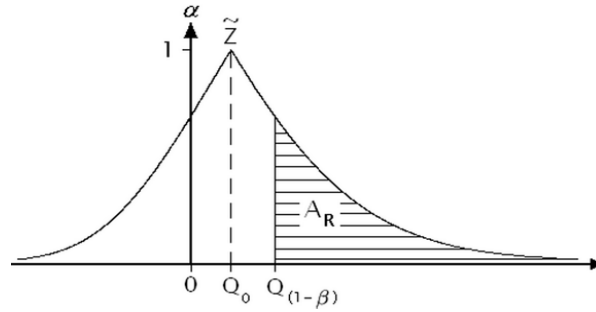$$\begin{cases} H_0 : \theta = \theta_0, \\ H_1 : \theta > \theta_0. \end{cases}$$

Figure 3: The related quantities in testing a fuzzy simple hypothesis versus a fuzzy right one-sided hypothesis
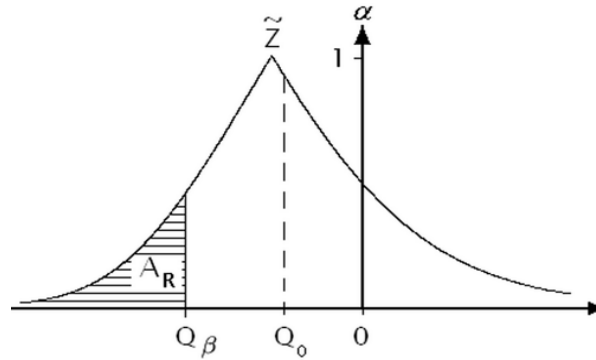

Figure 4: The related quantities in testing a fuzzy simple hypothesis versus a fuzzy left one-sided hypothesis

The α-cuts of the fuzzy test statistic $\widetilde{Z}$ are obtained using the above approach, but only by substituting the α-cuts of the fuzzy point estimation $\widetilde{\theta}^*$ for the point estimation θ^* in the crisp test statistic (Q_0). Finally, we can test these hypotheses based on the proposed quadruplet procedure.

Remark 4: We can also apply the above procedure for testing a fuzzy simple hypothesis against a fuzzy left one-sided hypothesis based on fuzzy data (see Fig. 4).

Remark 5: It is obvious that if in testing crisp hypotheses the value of the observed test statistic is close to the related quantile, then the classical methods for making the decision whether to accept or reject the null hypothesis are very sensitive. In such cases, we propose to use the following alternative methods:

I1) Testing crisp hypotheses with fuzzy test statistic based on Buckley's approach [3].

I2) Substituting the fuzzy data for the crisp data and testing the crisp hypotheses based on the method suggested in Remarks 2 and 3.

I3) Substituting the fuzzy hypotheses for the crisp hypotheses and testing them based on Taheri and Arefi's approach [22].

I4) Substituting the fuzzy hypotheses for the crisp hypotheses and using the fuzzy data instead of the crisp data to test such hypotheses based on the method proposed in Subsections 4.1 and 4.2.

Let us now illustrate the above different cases through the following numerical example.

Example 2 *Assume that, based on a random sample of size $n = 100$ from a population with the distribution $N(\theta, \sigma^2 = 9)$, we want to test some hypotheses about the mean θ at the significance level $\beta = 0.05$. The observed value of the test statistic is $z_0 = \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}}$.*

A1) *Let the mean of the random sample be $\bar{x} = 2.4934$. We want to test the following hypotheses*

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

Here, $z_0 = 1.6447 < z_{1-\beta} = 1.6448$. Hence, we accept the null hypothesis H_0.

A2) *In case **A1**, let the mean of the random sample be $\bar{x} = 2.4935$. Here, $z_0 = 1.6450 > z_{1-\beta} = 1.6448$ and, therefore, we reject the null hypothesis H_0.*

A3) *In case **A2**, let the mean of the random sample be $\bar{x} = 2.4935$, but we want to test the following hypotheses*

$$\begin{cases} H_0 : \theta = 2.0001, \\ H_1 : \theta > 2.0001. \end{cases}$$

Here, $z_0 = 1.6447 < z_{1-\beta} = 1.6448$. Hence, we accept the null hypothesis H_0.

A4) *Consider the case **A1**. Let the mean of the random sample be $\bar{x} = 2.4936$. We want to test the following hypotheses*

$$\begin{cases} H_0 : \theta = 2.0001, \\ H_1 : \theta > 2.0001. \end{cases}$$

Here, $z_0 = 1.6450 > z_{1-\beta} = 1.6448$. Hence, we reject the null hypothesis H_0.

*Consider the above cases. The pairs (**A1, A2**) and (**A3, A4**) have different results with respect to accepting or rejecting the null hypothesis H_0 with a small change in the sample mean. The pair (**A2, A3**) has different results with respect to accepting or rejecting the null hypothesis H_0 with a small change in the hypotheses. The pair (**A1, A4**) have different results for accepting or rejecting the null hypothesis H_0 with a slight change in the sample mean and the hypotheses.*

*For testing the pairs (**A1, A2**) and (**A3, A4**), we can use Buckley's approach and the method proposed in Remarks 2 and 3. For the pair (**A2, A3**) (and also for the pairs (**A1, A2**) and (**A3, A4**)), we can define the fuzzy hypotheses in a suitable manner, and then test such hypotheses based on Taheri and Arefi's approach. For the pair (**A1, A4**) (and also for the pairs (**A1, A2**), (**A2, A3**), and (**A3, A4**)), we can define the fuzzy hypotheses in a suitable manner, and then test such hypotheses with fuzzy data based on the method proposed in Subsections 4.1 and 4.2.*

5 Testing Fuzzy Hypotheses in the Normal Distribution

5.1 Testing Fuzzy Hypotheses for the Mean

Suppose that we have taken a random sample of size n from a $N(\theta, \sigma^2)$ (σ^2 known) and we have further observed the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$. Now, we want to test the following fuzzy hypotheses at the significance level β:

$$\begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1L} : \theta \text{ is essentially larger than } \theta_0, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}. \end{cases}$$

The usual point estimation for θ is $\theta^* = \bar{x}$. By substituting the α-cuts of $\widetilde{X}_i$, $i = 1, ..., n$, ($\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U]$) for x_i in the point estimation, the α-cuts of the fuzzy point estimation $\overline{\widetilde{X}}$ will be obtained as

$$\overline{\widetilde{X}}[\alpha] = \frac{1}{n} \sum_{i=1}^{n} [\widetilde{X}_i^L, \widetilde{X}_i^U] = \left[\frac{1}{n} \sum_{i=1}^{n} \widetilde{X}_i^L, \frac{1}{n} \sum_{i=1}^{n} \widetilde{X}_i^U \right] = [\overline{\widetilde{X}}^L, \overline{\widetilde{X}}^U].$$

Under the crisp null hypothesis $H_0 : \theta = \theta_0$, the value of the crisp test statistic is $z_0 = \frac{\theta^* - \theta_0}{\sigma/\sqrt{n}}$. By substituting the α-cuts of the fuzzy point estimation $\overline{\widetilde{X}}$ for θ^* and the α-cuts of $\widetilde{H}_0$ for θ_0 in z_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained to be

$$\tilde{Z}[\alpha] = \frac{\overline{\widetilde{X}}[\alpha] - \widetilde{H}_0[\alpha]}{\sigma/\sqrt{n}} = \left[\frac{\overline{\widetilde{X}}^L - a_3 + (a_3 - \theta_0)\alpha}{\sigma/\sqrt{n}}, \frac{\overline{\widetilde{X}}^U - a_1 - (\theta_0 - a_1)\alpha}{\sigma/\sqrt{n}} \right].$$
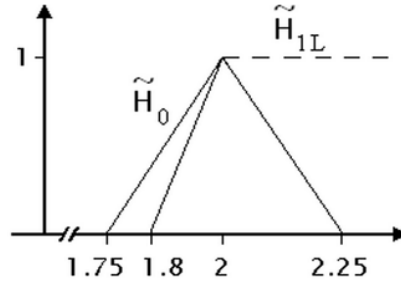
Figure 5: Fuzzy hypotheses in Example 3

For example, let $\widetilde{X}_i = (x_i - r_i, x_i, x_i + r_i)_T =: (x_i, r_i)_T$, $i = 1, ..., n$, be the symmetric triangular fuzzy numbers with $\widetilde{X}_i[\alpha] = [x_i - (1-\alpha)r_i, x_i + (1-\alpha)r_i]$. Then, the fuzzy point estimation is obtained as $\overline{\widetilde{X}} = (\overline{x} - \overline{r}, \overline{x}, \overline{x} + \overline{r})_T$ obeying $\overline{\widetilde{X}}[\alpha] = [\overline{x} - (1-\alpha)\overline{r}, \overline{x} + (1-\alpha)\overline{r}]$. Hence, the α-cuts of the fussy test statistic are obtained as

$$\widetilde{Z}[\alpha] = \frac{\overline{\widetilde{X}}[\alpha] - \widetilde{H}_0[\alpha]}{\sigma/\sqrt{n}}$$

$$= \left[\frac{\overline{x} - (1-\alpha)\overline{r} - a_3 + (a_3 - \theta_0)\alpha}{\sigma/\sqrt{n}}, \frac{\overline{x} + (1-\alpha)\overline{r} - a_1 - (\theta_0 - a_1)\alpha}{\sigma/\sqrt{n}} \right]$$

$$= \left[z_0 - (1-\alpha)(\overline{r} + a_3 - \theta_0)\frac{\sqrt{n}}{\sigma}, z_0 + (1-\alpha)(\overline{r} + \theta_0 - a_1)\frac{\sqrt{n}}{\sigma} \right],$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$. Now, using the above fuzzy test statistic, we can apply the quadruplet procedure (proposed in Subsection 4.2) for testing the fuzzy hypotheses of interest.

Example 3 *Assume that, based on a random sample of size $n = 50$ from a population $N(\theta, \sigma^2 = 9)$, we observe the fuzzy data in Table 1.*

Table 1: The fuzzy data from a normal population in Example 3

$(x_i, r_i)_T$	$(x_i, r_i)_T$	$(x_i, r_i)_T$	$(x_i, r_i)_T$	$(x_i, r_i)_T$	$(x_i, r_i)_T$
$(1.8, 0.2)_T$	$(2.8, 0.3)_T$	$(-3.4, 0.4)_T$	$(-2.1, 0.2)_T$	$(2.1, 0.2)_T$	$(-0.4, 0.2)_T$
$(2.9, 0.4)_T$	$(-4.6, 0.3)_T$	$(2.4, 0.1)_T$	$(1.0, 0.2)_T$	$(1.4, 0.1)_T$	$(0.9, 0.2)_T$
$(0.9, 0.2)_T$	$(6.8, 1.4)_T$	$(1.9, 0.3)_T$	$(0.8, 0.1)_T$	$(3.0, 0.6)_T$	$(-1.2, 0.2)_T$
$(3.7, 0.7)_T$	$(3.8, 0.4)_T$	$(6.0, 0.8)_T$	$(5.0, 1.0)_T$	$(0.3, 0.1)_T$	$(-2.8, 0.4)_T$
$(1.2, 0.2)_T$	$(4.4, 0.4)_T$	$(3.1, 0.4)_T$	$(3.0, 0.6)_T$	$(1.6, 0.2)_T$	$(1.6, 0.3)_T$
$(1.7, 0.3)_T$	$(6.0, 1.2)_T$	$(7.3, 1.5)_T$	$(6.9, 1.2)_T$	$(6.9, 1.0)_T$	$(0.9, 0.2)_T$
$(0.5, 0.1)_T$	$(1.3, 0.3)_T$	$(5.1, 1.0)_T$	$(6.2, 1.1)_T$	$(1.8, 0.2)_T$	$(5.7, 1.0)_T$
$(3.4, 0.5)_T$	$(3.4, 0.4)_T$	$(1.3, 0.2)_T$	$(5.8, 1.1)_T$	$(4.9, 1.0)_T$	$(-0.7, 0.1)_T$
$(-1.3, 0.2)_T$	$(5.8, 1.0)_T$				

A) *Suppose that we want to test the following fuzzy hypotheses at the significance level $\beta = 0.05$*

$$\begin{cases} H_0 : \theta \text{ is approximately 2,} \\ H_{1L} : \theta \text{ is essentially larger than 2,} \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}, \end{cases}$$

where $\widetilde{H}_0 = (1.75, 2, 2.25)_T$ and $\widetilde{H}_{1L} = (1.80, 2)_{EL}$ (see Fig. 5).

The fuzzy point estimation is $\overline{\widetilde{X}} = (1.924, 2.416, 2.908)_T$ and the fuzzy test statistic is obtained as $\widetilde{Z} = (-0.326\frac{\sqrt{50}}{3}, 0.416\frac{\sqrt{50}}{3}, 1.158\frac{\sqrt{50}}{3})_T = (-0.7684, 0.9805, 2.7294)_T$ with the following α-cuts

$$\widetilde{Z}[\alpha] = \left[(0.416 - 0.742(1-\alpha))\frac{\sqrt{50}}{3}, (0.416 + 0.742(1-\alpha))\frac{\sqrt{50}}{3} \right].$$
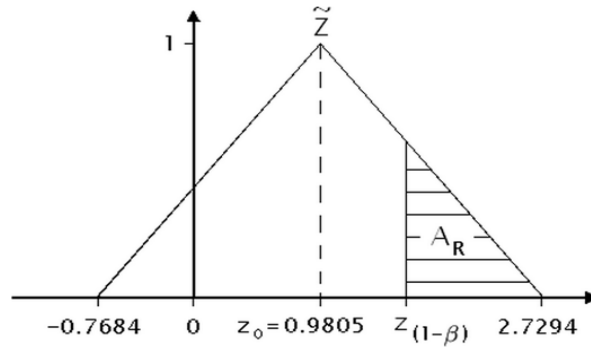
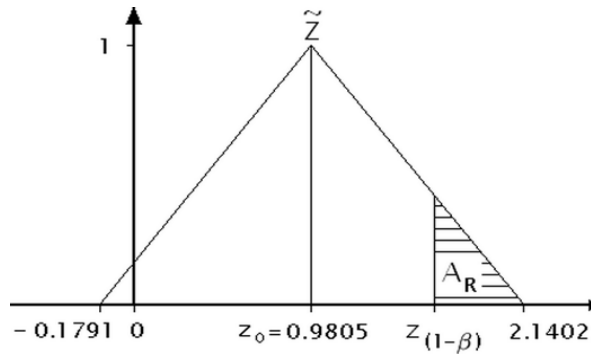Figure 6: The related quantities in Example 3 for the fuzzy hypotheses

Figure 7: The related quantities in Example 3 for the crisp hypotheses

Based on the fuzzy test statistic, we obtain $A_T = 0.742\frac{\sqrt{50}}{3} = 1.7489$ and $A_R = 0.6202$. Since $A_R/A_T = 0.3546$, we reject H_0 for every credit level $\phi \in (0, 0.3546]$ (see Fig. 6).
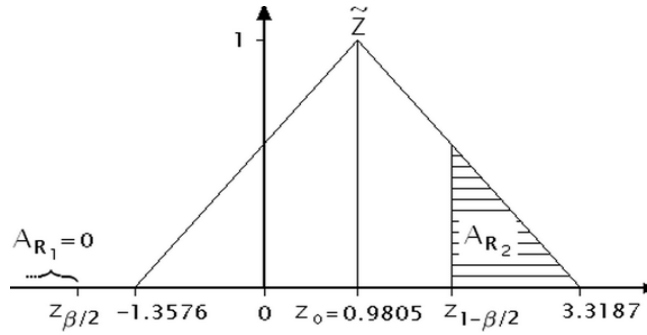
   B) *Consider the above fuzzy data. Now, suppose that we want to test the following crisp hypotheses (which is equivalent to the case $a_1 = \theta_0 = a_3$ in the above fuzzy hypotheses):*

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases} \tag{1}$$

Based on Remark 3, the α-cuts of fuzzy test statistic are calculated as

$$\begin{aligned}
\widetilde{Z}[\alpha] = \frac{\overline{\widetilde{X}}[\alpha] - \theta_0}{\sigma/\sqrt{n}} &= \left[\frac{\overline{x} - (1-\alpha)\overline{r} - \theta_0}{\sigma/\sqrt{n}}, \frac{\overline{x} + (1-\alpha)\overline{r} - \theta_0}{\sigma/\sqrt{n}}\right] \\
&= \left[z_0 - (1-\alpha)\frac{\overline{r}\sqrt{n}}{\sigma}, z_0 + (1-\alpha)\frac{\overline{r}\sqrt{n}}{\sigma}\right] \\
&= \left[(0.416 - 0.492(1-\alpha))\frac{\sqrt{50}}{3}, (0.416 + 0.492(1-\alpha))\frac{\sqrt{50}}{3}\right].
\end{aligned}$$

Hence, the fuzzy test statistic is $\widetilde{Z} = (-0.076\frac{\sqrt{50}}{3}, 0.416\frac{\sqrt{50}}{3}, 0.908\frac{\sqrt{50}}{3})_T = (-0.1791, 0.9805, 2.1402)_T$, and we obtain $A_T = 0.492\frac{\sqrt{50}}{3} = 1.1597$, $A_R = 0.1058$, and $A_R/A_T = 0.0912$. The null hypothesis in (1) is, therefore, rejected for every credit level $\phi \in (0, 0.0912]$ (see Fig. 7). Since $\overline{r} \le (\overline{r} + \theta_0 - a_1)$ and $\overline{r} \le (\overline{r} + a_3 - \theta_0)$, it is concluded that in this case, under the fuzzy hypotheses, we may reject H_0 with a higher credit level than we would the crisp hypotheses case.

Figure 8: $\widetilde{Z}$ and A_R in Example 4

Example 4 *Consider the fuzzy data in Example 3. Suppose that we wish to test the following fuzzy hypotheses at the significance level $\beta = 0.05$:*

$$\begin{cases} H_0 : \theta \text{ is approximately } 2, \\ H_1 : \theta \text{ is away from } 2, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_1, \end{cases}$$

where $\widetilde{H}_0 = (1.5, 2, 2.5)_T$ and $\widetilde{H}_1 = \widetilde{H}_0^c$. The α-cuts of the fuzzy test statistic are calculated as

$$\widetilde{Z}[\alpha] = \left[z_0 - (1 - \alpha)(\bar{r} + a_3 - \theta_0)\tfrac{\sqrt{n}}{\sigma}, z_0 + (1 - \alpha)(\bar{r} + \theta_0 - a_1)\tfrac{\sqrt{n}}{\sigma} \right]$$

$$= \left[(-0.576 + 0.992\alpha)\tfrac{\sqrt{50}}{3}, (1.408 - 0.992\alpha)\tfrac{\sqrt{50}}{3} \right].$$

Hence, the fuzzy test statistic is $\widetilde{Z} = (-0.576\tfrac{\sqrt{50}}{3}, 0.416\tfrac{\sqrt{50}}{3}, 1.408\tfrac{\sqrt{50}}{3})_T = (-1.3576, 0.9805, 3.3187)_T$, and we obtain $A_T = 0.992\tfrac{\sqrt{50}}{3} = 2.3382$ and $A_R = A_{R_1} + A_{R_2} = 0 + 0.3948 = 0.3948$. Since $A_R/A_T = 0.1688$, we reject H_0 for every credit level $\phi \in (0, 0.1688]$ (see Fig. 8).

5.2   Testing Fuzzy Hypotheses for the Variance

Assume that we have taken a random sample of size n from a population $N(\mu, \theta)$ (μ is unknown) and we have observed the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$. Suppose further that we want to test the following fuzzy hypotheses at the significance level β:

$$\begin{cases} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1L} : \theta \text{ is essentially larger than } \theta_0. \end{cases}$$

The usual point estimation for θ is $\theta^* = s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$. By substituting the α-cuts of $\widetilde{X}_i$, $i = 1, ..., n$, ($\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U]$) for x_i in the point estimation θ^*, and using the interval arithmetic, the α-cuts of the fuzzy point estimation $\widetilde{S^2}$ are obtained as follows

$$\widetilde{S^2}[\alpha] = \frac{1}{n-1} \sum_{i=1}^{n} \left([\widetilde{X}_i^L, \widetilde{X}_i^U] - [\overline{\widetilde{X}}^L, \overline{\widetilde{X}}^U] \right) \cdot \left([\widetilde{X}_i^L, \widetilde{X}_i^U] - [\overline{\widetilde{X}}^L, \overline{\widetilde{X}}^U] \right)$$

$$= \left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha] \right],$$

where

$$\begin{cases} \widetilde{\overline{X}}[\alpha] = [\widetilde{\overline{X}}^L, \widetilde{\overline{X}}^U] = \frac{1}{n}\sum_{i=1}^{n}[\widetilde{X}_i^L, \widetilde{X}_i^U] = \left[\frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i^L, \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i^U\right], \\[2mm] \widetilde{S^2}^L[\alpha] = \max\left[0, \min\left[\frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)^2, \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)(\widetilde{X}_i^U - \widetilde{\overline{X}}^L), \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)^2\right]\right], \\[2mm] \widetilde{S^2}^U[\alpha] = \max\left[\frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)^2, \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^L - \widetilde{\overline{X}}^U)(\widetilde{X}_i^U - \widetilde{\overline{X}}^L), \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{X}_i^U - \widetilde{\overline{X}}^L)^2\right]. \end{cases}$$

Under the crisp null hypothesis $H_0 : \theta = \theta_0$, the crisp test statistic $\frac{(n-1)S^2}{\theta_0}$ is distributed according to χ^2_{n-1} with $Q_0 = \frac{(n-1)\theta^*}{\theta_0} = \frac{(n-1)s^2}{\theta_0}$ as its observed value. By substituting the α-cuts of the fuzzy point estimation $\widetilde{S^2}$ for $\theta^* = s^2$ and the α-cuts of $\widetilde{H}_0 = (a_1, \theta_0, a_3)_T$ (in this case $a_1 > 0$) for θ_0 in Q_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained to be

$$\tilde{Z}[\alpha] = \frac{(n-1)\widetilde{S^2}[\alpha]}{\widetilde{H}_0[\alpha]} = \frac{(n-1)\left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha]\right]}{[a_1 + \alpha(\theta_0 - a_1), a_3 - \alpha(a_3 - \theta_0)]} = \left[\frac{(n-1)\widetilde{S^2}^L[\alpha]}{a_3 - \alpha(a_3 - \theta_0)}, \frac{(n-1)\widetilde{S^2}^U[\alpha]}{a_1 + \alpha(\theta_0 - a_1)}\right].$$

Now, based on the above fuzzy test statistic, we can test the hypotheses of interest by employing the quadruplet procedure proposed in Subsection 4.2.

Example 5 *Suppose that, based on a random sample of size $n = 20$ from a population $N(\mu, \theta)$, we observe the symmetric triangular fuzzy numbers in Table 2 as fuzzy observations.*

Table 2: The fuzzy data from a normal population in Example 5

$(r_{1i}, x_i, r_{2i})_T$	$(r_{1i}, x_i, r_{2i})_T$	$(r_{1i}, x_i, r_{2i})_T$	$(r_{1i}, x_i, r_{2i})_T$	$(r_{1i}, x_i, r_{2i})_T$
$(1.03, 1.29, 1.55)_T$	$(1.21, 1.51, 1.81)_T$	$(2.10, 2.63, 3.16)_T$	$(2.82, 3.53, 4.24)_T$	$(1.80, 2.25, 2.70)_T$
$(0.18, 0.23, 0.28)_T$	$(2.18, 2.72, 3.26)_T$	$(1.10, 1.37, 1.64)_T$	$(2.46, 3.08, 3.70)_T$	$(0.43, 0.54, 0.65)_T$
$(0.56, 0.70, 0.84)_T$	$(1.69, 2.11, 2.53)_T$	$(1.30, 1.62, 1.94)_T$	$(2.44, 3.05, 3.66)_T$	$(1.29, 1.61, 1.93)_T$
$(2.83, 3.54, 4.25)_T$	$(2.63, 3.29, 3.95)_T$	$(3.13, 3.91, 4.69)_T$	$(2.08, 2.60, 3.12)_T$	$(3.88, 4.85, 5.82)_T$

Here, the α-cuts of the fuzzy point estimation are obtained as follows (see Fig. 9)

$$\widetilde{S^2}[\alpha] = \left[\widetilde{S^2}^L[\alpha], \widetilde{S^2}^U[\alpha]\right],$$

where

$$\widetilde{S^2}^L[\alpha] = \begin{cases} \frac{1}{19}[28.09086\alpha^2 + 56.18171\alpha(1-\alpha) + 9.70434(1-\alpha)^2] & 0 \le \alpha \le 0.6942, \\ \frac{1}{19}[28.09086\alpha^2 + 44.93578\alpha(1-\alpha) + 35.23144(1-\alpha)^2] & 0.6942 < \alpha \le 1, \end{cases}$$

and

$$\widetilde{S^2}^U[\alpha] = \frac{1}{19}[28.09086\alpha^2 + 67.42764\alpha(1-\alpha) + 57.72330(1-\alpha)^2].$$

Now, suppose that we want to test the following hypotheses at the significance level $\beta = 0.05$:

$$\begin{cases} H_0 : \theta \text{ is approximately } 2, \\ H_{1L} : \theta \text{ is essentially larger than } 2, \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}, \end{cases}$$

where $\widetilde{H}_0 = (1.5, 2, 2.5)_T$ and $\widetilde{H}_{1L} = (1.75, 2)_{EL}$. The α-cuts of the fuzzy test statistic are
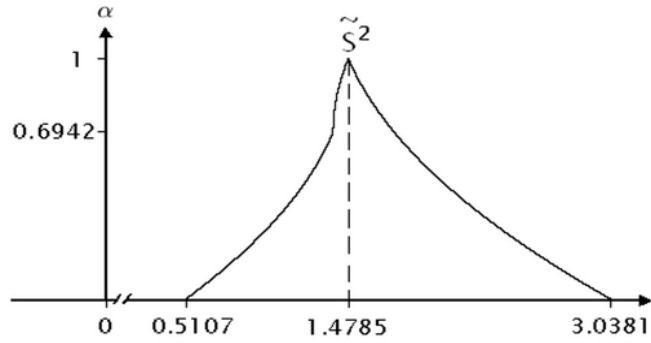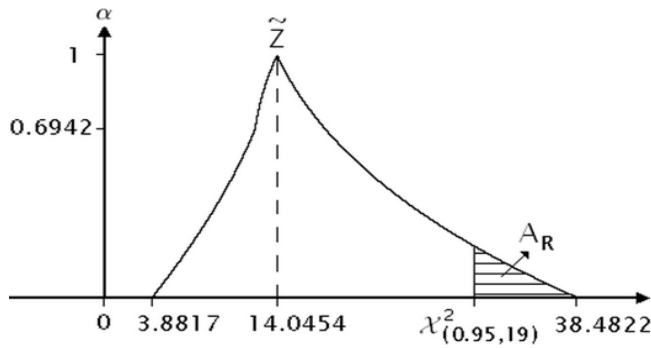
Figure 9: The fuzzy point estimation in Example 5

Figure 10: $\widetilde{Z}$ and A_R in Example 5 for the fuzzy hypotheses

$$\widetilde{Z}[\alpha] = \left[\frac{(n-1)\widetilde{S^2}^L[\alpha]}{a_3 - \alpha(a_3 - \theta_0)}, \frac{(n-1)\widetilde{S^2}^U[\alpha]}{a_1 + \alpha(\theta_0 - a_1)} \right] = \left[\frac{19\widetilde{S^2}^L[\alpha]}{2.5 - 0.5\alpha}, \frac{19\widetilde{S^2}^U[\alpha]}{1.5 + 0.5\alpha} \right] = \left[\widetilde{Z}^L[\alpha], \widetilde{Z}^U[\alpha] \right],$$

where

$$\widetilde{Z}^L[\alpha] = \begin{cases} \dfrac{28.09086\alpha^2 + 56.18171\alpha(1-\alpha) + 9.70434(1-\alpha)^2}{2.5 - 0.5\alpha} & 0 \leq \alpha \leq 0.6942, \\[2ex] \dfrac{28.09086\alpha^2 + 44.93578\alpha(1-\alpha) + 35.23144(1-\alpha)^2}{2.5 - 0.5\alpha} & 0.6942 < \alpha \leq 1, \end{cases}$$

and

$$\widetilde{Z}^U[\alpha] = \frac{28.09086\alpha^2 + 67.42764\alpha(1-\alpha) + 57.72330(1-\alpha)^2}{1.5 + 0.5\alpha}.$$

Using the trapezoidal rule [10], we obtain $A_T = 13.4699$ and $A_R = 0.8425$. Since $A_R/A_T = 0.0625$, we reject H_0 for every credit level $\phi \in (0, 0.0625]$ (see Fig. 10).

Now, suppose that, based on the fuzzy data in Example 5, we want to test the following crisp hypotheses instead of the above fuzzy ones (which is equivalent to the case $a_1 = \theta_0 = a_3$ in the above fuzzy hypotheses)

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases} \tag{2}$$

Based on Remark 3, the α-cuts of the fuzzy test statistic are calculated as follows

$$\widetilde{Z}[\alpha] = \left[\frac{(n-1)\widetilde{S^2}^L[\alpha]}{\theta_0}, \frac{(n-1)\widetilde{S^2}^U[\alpha]}{\theta_0} \right] = \left[9.5\widetilde{S^2}^L[\alpha], 9.5\widetilde{S^2}^U[\alpha] \right] = 9.5\widetilde{S^2}[\alpha].$$

Using the trapezoidal rule [10], we obtain $A_R = 0$. Hence $A_R/A_T = 0$, and we accept the null hypothesis in (2) for every credit level (see Fig. 11). Since $\theta_0 \leq a_3 - \alpha(a_3 - \theta_0)$ and $a_1 + \alpha(\theta_0 - a_1) \leq \theta_0$ for $0 < \alpha \leq 1$, it
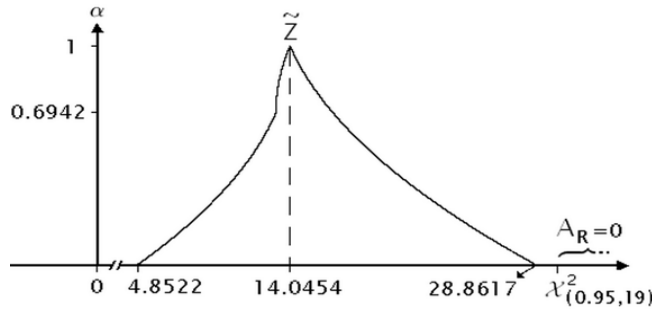
Figure 11: $\widetilde{Z}$ and A_R in Example 5 for the crisp hypothesis

is concluded that, in this case, for the fuzzy hypotheses, we may reject H_0 with a higher credit level than we did in the crisp hypotheses case.

6 Testing Fuzzy Hypotheses for Mean of an Exponential Distribution

Assume that we have taken a random sample of size n from an exponential distribution $Exp(\theta)$ with the following density

$$f(x) = \frac{1}{\theta} \, e^{-x/\theta}, \quad x > 0, \quad \theta > 0.$$

Suppose that we observe the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_n$. We want to test the following fuzzy hypotheses

$$\left\{ \begin{array}{l} H_0 : \theta \text{ is approximately } \theta_0, \\ H_{1L} : \theta \text{ is essentially larger than } \theta_0, \end{array} \right. \equiv \left\{ \begin{array}{l} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}. \end{array} \right.$$

The usual point estimation for θ is $\theta^* = \overline{x}$. By substituting the α-cuts of $\widetilde{X}_i$, $i = 1, ..., n$, $(\widetilde{X}_i[\alpha] = [\widetilde{X}_i^L, \widetilde{X}_i^U])$ for x_i in the point estimation, we obtain the fuzzy point estimation $\overline{\widetilde{X}}$ with the α-cuts $\overline{\widetilde{X}}[\alpha] = [\overline{\widetilde{X}}^L, \overline{\widetilde{X}}^U] = [\frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i^L, \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i^U]$. Under the crisp null hypothesis $(H_0 : \theta = \theta_0)$, the crisp test statistic $\frac{2n\overline{X}}{\theta_0}$ is distributed according to χ_{2n}^2 with $Q_0 = \frac{2n\theta^*}{\theta_0} = \frac{2n\overline{x}}{\theta_0}$ as its observed value. By substituting the α-cuts of the fuzzy point estimation $\overline{\widetilde{X}}$ for θ^* and the α-cuts of $\widetilde{H}_0$ for θ_0 in Q_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as follows

$$\widetilde{Z}[\alpha] = \frac{2n\overline{\widetilde{X}}[\alpha]}{\widetilde{H}_0[\alpha]} = \left[\frac{2n\overline{\widetilde{X}}^L}{a_3 - (a_3 - \theta_0)\alpha}, \frac{2n\overline{\widetilde{X}}^U}{a_1 + (\theta_0 - a_1)\alpha} \right].$$

Now, based on the above fuzzy test statistic, we can test the hypotheses of interest using the quadruplet procedure proposed in Subsection 4.2.

Example 6 Lifetime testing: *The following data (the centers of fuzzy numbers, x_i) show the lifetimes (in 1000 km) of front disk brake pads on a randomly selected set of 40 cars (same model) that were monitored by a dealer network (see, [18], pp. 337). But, in practice measuring the lifetime of a disk may not yield an exact result. A disk may work perfectly over a certain period but be braking for some time, and finally be unusable at a certain time. So, such data may be reported as imprecise quantities. Assume that the lifetimes of front disk brake pads are reported as fuzzy numbers in Table 3. In fact, imprecision is formulated by fuzzy numbers $\widetilde{X}_i = (x_i, s_i)_R$, with $s_i = 0.05x_i$, $i = 1, 2, \ldots, 40$, as follows*

$$\widetilde{X}_i(t) = \left\{ \begin{array}{ll} 1 - \frac{t - x_i}{s_i} & x_i \leq t \leq x_i + s_i, \\ 0 & otherwise. \end{array} \right.$$
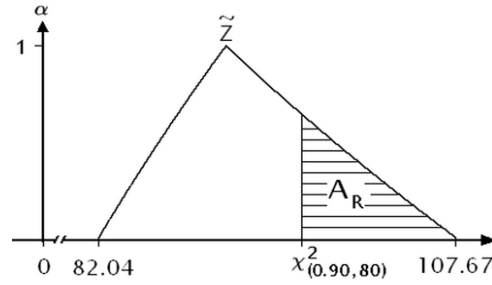
Figure 12: $\widetilde{Z}$ and A_R in Example 6

Table 3: The fuzzy data of the lifetimes (in 1000 km) of front disk brake pads in Example 6

$\widetilde{X}_i = (x_i, s_i)_R$	$\widetilde{X}_i = (x_i, s_i)_R$	$\widetilde{X}_i = (x_i, s_i)_R$	$\widetilde{X}_i = (x_i, s_i)_R$	$\widetilde{X}_i = (x_i, s_i)_R$
$(86.2, 4.3)_R$	$(45.1, 2.3)_R$	$(52.1, 2.6)_R$	$(54.2, 2.7)_R$	$(59.0, 3.0)_R$
$(38.4, 1.9)_R$	$(41.0, 2.1)_R$	$(56.4, 2.8)_R$	$(81.3, 4.1)_R$	$(62.4, 3.1)_R$
$(45.5, 2.3)_R$	$(36.7, 1.8)_R$	$(42.2, 2.1)_R$	$(51.6, 2.6)_R$	$(34.4, 1.7)_R$
$(22.7, 1.1)_R$	$(22.6, 1.1)_R$	$(40.0, 2.0)_R$	$(38.8, 1.9)_R$	$(50.2, 2.5)_R$
$(48.8, 2.4)_R$	$(81.7, 4.1)_R$	$(61.5, 3.1)_R$	$(53.6, 2.7)_R$	$(50.7, 2.5)_R$
$(42.8, 2.1)_R$	$(102.5, 5.1)_R$	$(42.7, 2.1)_R$	$(80.6, 4.0)_R$	$(64.5, 3.2)_R$
$(73.1, 3.7)_R$	$(28.4, 1.4)_R$	$(46.9, 2.3)_R$	$(45.9, 2.3)_R$	$(33.8, 1.7)_R$
$(59.8, 3.0)_R$	$(31.7, 1.6)_R$	$(33.9, 1.7)_R$	$(50.6, 2.5)_R$	$(56.7, 2.8)_R$

Suppose that the lifetime of the front disk brake pad has an exponential distribution with an unknown mean θ. The fuzzy point estimation for the parameter θ based on the fuzzy data is $\overline{\widetilde{X}} = (\overline{x}, \overline{s})_R = (51.2750, 2.5575)_R$ with α-cuts $\overline{\widetilde{X}}[\alpha] = [51.2750, 51.2750 + (1 - \alpha)2.5575]$. Suppose that we want to test the following fuzzy hypotheses at the significance level $\beta = 0.10$:

$$\begin{cases} H_0 : \theta \text{ is approximately 45,} \\ H_{1L} : \theta \text{ is essentially larger than 45,} \end{cases} \equiv \begin{cases} H_0 : \theta \text{ is } \widetilde{H}_0, \\ H_1 : \theta \text{ is } \widetilde{H}_{1L}, \end{cases}$$

where $\widetilde{H}_0 = (40, 45, 50)_T$ and $\widetilde{H}_{1L} = (43, 45)_{EL}$. The α-cuts of the fuzzy test statistic are calculated as follows

$$\widetilde{Z}[\alpha] = \left[\frac{4102}{50 - 5\alpha}, \frac{4306.6 - 204.6\alpha}{40 + 5\alpha} \right].$$

Also, the fuzzy test statistic is obtained as follows

$$\widetilde{Z}(z) = \begin{cases} \frac{10z - 820.4}{z} & 82.040 < z \le \frac{4102}{45}, \\ \\ \frac{4306.6 - 40z}{204.6 + 5z} & \frac{4102}{45} < z \le 107.665, \\ \\ 0 & otherwise. \end{cases}$$

Hence, $A_T = 12.6486$, $A_R = 3.4834$, and $A_R/A_T = 0.2754$. Therefore, H_0 is rejected for every $\phi \in (0, 0.2754]$ (see Fig. 12).

7   A Comparison Study

In this section, we compare our method with two well known methods proposed for testing statistical hypotheses in the fuzzy environment.

7.1   Comparison with Buckley's Approach

Buckley [3] studied the problem of testing crisp hypotheses based on the fuzzy test statistic. He first considered the confidence intervals for the parameter of interest as the α-cuts of a fuzzy point estimation. Then, the fuzzy test statistic could be defined based on the α-cuts of the fuzzy point estimation. Finally, the statistical hypotheses could be evaluated using a credit level.

Our proposed approach has the following two advantages over Buckley's.

1) While Buckley considers the problem of testing based on crisp data and crisp hypotheses, we assume that both the hypotheses and the data are fuzzy. Our method is, therefore, more convenient in real world studies.

2) We introduce a fuzzy point estimation based on the α-cuts of fuzzy data for obtaining the fuzzy test statistic, whereas Buckley uses a set of confidence intervals as the α-cuts of a fuzzy point estimation. The fuzzy point estimation for $\alpha = 1$ is reduced to the usual crisp point estimation in our method, but Buckley's sometimes does not yield a usual crisp point estimation. For instance, the confidence interval for σ^2 of a normal distribution is as follows

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \right].$$

For $\alpha = 1$, using Buckley's method, the point estimation of σ^2 would be obtained as $\frac{(n-1)s^2}{\chi^2_{0.5,n-1}}$, which is not a usual point estimation. On the other hand, based on our proposed method described in Subsection 5.2, we obtain $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ as the point estimation of σ^2, which is exactly the usual crisp point estimator of σ^2 (see also [8]).

7.2   Comparison with Wu's Approach

Wu [31] proposed an approach for testing the fuzzy mean of a normal distribution based on fuzzy data. He used the following notations for testing the fuzzy hypothesis $\widetilde{H}_0 : \widetilde{\mu} = \widetilde{\mu}_0$ against $\widetilde{H}_1 : \widetilde{\mu} \succ \widetilde{\mu}_0$ (where $\succ$ is an ordering between two fuzzy numbers)

$$x_\alpha^L = \frac{1}{n}\sum_{i=1}^{n}(\widetilde{x}_i)_\alpha^L - core(\widetilde{\mu}_0), \qquad x_\alpha^U = \frac{1}{n}\sum_{i=1}^{n}(\widetilde{x}_i)_\alpha^U - core(\widetilde{\mu}_0),$$

where $(\widetilde{x}_i)_\alpha^L = \inf\{t \mid \widetilde{x}_i(t) \geq \alpha\}$ and $(\widetilde{x}_i)_\alpha^U = \sup\{t \mid \widetilde{x}_i(t) \geq \alpha\}$, and $core(\widetilde{\mu}_0)$ is the center of the fuzzy number (e.g. $core(\widetilde{\mu}_0) = \mu_0$ if $\widetilde{\mu}_0(\mu_0) = 1$). Then, he proposed to accept $\widetilde{H}_0$ in the α-cut sense if $x_\alpha^L < z_{1-\beta}\frac{\sigma}{\sqrt{n}}$ and $x_\alpha^U < z_{1-\beta}\frac{\sigma}{\sqrt{n}}$, and to accept $\widetilde{H}_1$ in the α-cut sense if $x_\alpha^L \geq z_{1-\beta}\frac{\sigma}{\sqrt{n}}$ and $x_\alpha^U \geq z_{1-\beta}\frac{\sigma}{\sqrt{n}}$. Then, he introduced degrees of optimism and pessimism and also a degree of belief to evaluate the hypotheses of interest. Finally, by transferring the basic problem to an optimization problem, and by solving the problem, one could decide whether to accept or reject the hypotheses. He introduced a similar approach for testing a simple fuzzy hypothesis against a two-sided fuzzy hypothesis.

Some of the advantages of our method over Wu's method are as follows:

1) We use the α-cuts of the fuzzy null hypothesis for calculating the fuzzy test statistic while, Wu used the center of the fuzzy null hypothesis ($core(\widetilde{\mu}_0)$). Now, consider different fuzzy hypotheses with different spreads and similar centers. Based on our proposed approach, one obtains different fuzzy test statistics (and so, different results in testing the hypotheses), whereas, by applying Wu's approach, one obtains similar results for testing such different hypotheses.

2) For testing fuzzy hypotheses, we obtain a fuzzy test statistic based on all the α-cuts of the fuzzy data, but Wu's approach is based only on $(\widetilde{x}_i)_\alpha^L = \inf\{t \mid \widetilde{x}_i(t) \geq \alpha\}$ and $(\widetilde{x}_i)_\alpha^U = \sup\{t \mid \widetilde{x}_i(t) \geq \alpha\}$ of the fuzzy data.

3) In addition to the significant level β, we use one additional criterion (the credit level) to evaluate the fuzzy hypotheses. However, Wu suggested three additional criteria which may confuse the decision maker in evaluating the hypotheses of interest in practical problems.

8   Conclusion

We extended an approach to the problem of testing fuzzy hypotheses when the available data are fuzzy, too. In the proposed approach, the fuzzy hypotheses are tested based on two criteria: a significance level (coming derived from the randomness of data) and a credit level (stemming from the fuzzy viewpoint). This approach is especially suitable for testing crisp/fuzzy hypotheses when the observed value of the test statistic is close to the quantile of the distribution of the test statistic. The advantage of the proposed approach is that it is a natural analogue of the usual approach to the problem of testing statistical hypotheses, since decision making is essentially based on the so called fuzzy test statistic. The proposed approach is general and can be applied for testing fuzzy hypotheses with any type of fuzzy data.

We illustrated the proposed approach through some numerical examples. In addition, the applicability of the approach was explained by a practical example of lifetime testing.

Extensions of the proposed method to test the parameters of regression models can be considered for future work. In addition, studying the problem of testing hypothesis within the framework of granular computing [20, 33] is a potential subject for further research.

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