Jordan-algebraic aspects of optimization:randomization

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Abstract

We describe a version of randomization technique within a general framework of Euclidean Jordan algebras. It is shown how to use this technique to evaluate the quality of symmetric relaxations for several nonconvex optimization problems.

1 Introduction

Starting with a seminal paper [goemans], a randomization technique plays a prominent role in evaluating the quality of semidefinite relaxations for difficult combinatorial and other optimization problems. In this paper we introduce a framework for one type of such a technique within a general context of Euclidean Jordan algebras. Within this framework the original problem is in conic form with conic constraints defined by general symmetric cones with additional constraints on the (Jordan-algebraic) rank of the solution. The symmetric (convex) relaxation is obtained from the original problem by omitting rank constraints. More precisely, we prove several measure concentration results (tail estimates) on manifolds of nonnegative elements of a fixed rank (and their closures) in a given Euclidean Jordan algebra. We illustrate a usefulness of these results by a number of examples. In particular, we consider Jordan-algebraic generalizations of several relaxation results described in [barvinok],[Ye], [Zhang2]. Jordan-algebraic technique proved to be very useful for the analysis of optimization problems involving symmetric cones [F1] and especially primal-dual algorithms [F3],[F4],[M],[SA],[Ts]. In present paper we further expand the domain of applicability of this powerful technique.

2 Jordan-algebraic concepts

We stick to the notation of an excellent book [FK]. We do not attempt to describe the Jordan-algebraic language here but instead provide detailed references to [FK]. Throughout this paper:

- V is a simple Euclidean Jordan algebra;
- $\mathrm{rank}(V)$ stands for the rank of V;
- $x \circ y$ is the Jordan algebraic multiplication for $x, y \in V$;
- $< x, y >= \mathrm{tr}(x \circ y)$ is the canonical scalar product in V; here tr is the trace operator on V;
- Ω is the cone of invertible squares in V;

• $\bar{\Omega}$ is the closure of Ω in V;
• An element $f \in V$ such that $f^2 = f$ and $\mathrm{tr}(f) = 1$ is called a primitive idempotent in V;
• Given $x \in V$, we denote by $L(x)$ the corresponding multiplication operator on V, i.e.

$$L(x)y = x \circ y, \quad y \in V;$$

• Given $x \in V$, we denote by $P(x)$ the so-called quadratic representation of x, i.e.

$$P(x) = 2L(x)^2 - L(x^2).$$

Given $x \in V$, there exist idempotents $f_1, \cdots, f_k$ in V such that $f_i \circ f_j = 0$ for $i \neq j$ and such that $f_1 + f_2 \cdots + f_k = e$, and distinct real numbers $\lambda_1, \cdots, \lambda_k$ with the following property:

$$x = \sum_{i=1}^{k} \lambda_i f_i \tag{1}$$

The numbers λ_i and idempotents f_i are uniquely defined by x. (see Theorem III. 1.1 in [FK]).

The representation (1) is called the spectral decomposition of x. Within the context of this paper the notion of rank of x is very important. By definition:

$$\mathrm{rank}(x) = \sum_{i:\lambda_i \neq 0} \mathrm{tr}(f_i). \tag{2}$$

Given $x \in V$, the operator $L(x)$ is symmetric with respect to the canonical scalar product. If f is an idempotent in V, it turns out that the spectrum of $L(f)$ belongs to $\{0, \frac{1}{2}, 1\}$. Following [FK], we denote by $V(1, f), V(\frac{1}{2}, f), V(0, f)$ corresponding eigenspaces.

It is clear that

$$V = V(0, f) \oplus V(1, f) \oplus V(\frac{1}{2}, f) \tag{3}$$

and the eigenspaces are pairwise orthogonal with respect to the scalar product $<,>$. This is the so-called Peirce decomposition of V with respect to an idempotent f. However, eigenspaces have more structure (see [FK], Proposition IV. 1.1). In particular, $V(0, f), V(1, f)$ are subalgebras in V. Let f_1, f_2 be two primitive orthogonal idempotents. It turns out that

$$\mathrm{dim} V(\frac{1}{2}, f_1) \cap V(\frac{1}{2}, f_2)$$

does not depend on the choice of the pair f_1, f_2 (see Corollary IV.2.6, p.71 in [FK]). It is called the degree of V (notation $\mathrm{d}(V)$).
Note that two simple Euclidean Jordan algebras are isomorphic if and only if their ranks and degrees coincide.

We summarize some of the properties of algebras $V(1, f)$.

Proposition 1 *Let f be an idempotent in a simple Euclidean Jordan algebra V. Then $V(1, f)$ is a simple Euclidean Jordan algebra with identity element f. Moreover,*

$$rank(V(1, f)) = rank(f)$$

$$d(V(1, f)) = d(V).$$

The trace operator on $V(1, f)$ coincides with the restriction of the trace operator on V. If $\tilde{\Omega}$ is the cone of invertible squares in $V(1, f)$ then $\bar{\tilde{\Omega}} = \bar{\Omega} \cap V(1, f)$.

Proposition 1 easily follows from the properties of Peirce decomposition on V (see section IV.2 in [FK]). Notice that if c is a primitive idempotent in $V(1, f)$, then c is primitive idempotent in V. For a proof see e.g [F1].

Let $f_1, \cdots, f_r$, where $r = \mathrm{rank}(V)$, be a system of primitive idempotents such that $f_i \circ f_j = 0$ for $i \neq j$ and $f_1 + \cdots f_r = e$. Such system is called a Jordan frame. Given $x \in V$, there exists a Jordan frame $f_1, \cdots, f_r$ and real numbers $\lambda_1, \cdots, \lambda_r$ (eigenvalues of x) such that

$$x = \sum_{i=1}^{r} \lambda_i f_i.$$

The numbers λ_i (with their multiplicities) are uniquely determined by x. (See Theorem III. 1.2 in [FK]). We will need the spectral function $x \to \lambda(x) : V \to \mathbf{R^r}$, where $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \lambda_r(x)$ are ordered eigenvalues of x with their multiplicities. The following inequality holds:

$$\|\lambda(x) - \lambda(y)\|_2 \leq \|x - y\|, \forall x, y \in V. \tag{4}$$

Here $\|\lambda\|_2$ is the standard Euclidean norm in $\mathbf{R^r}$ and $\|x\| = \sqrt{\langle x, x\rangle}, x \in V$. For a proof (4) see e.g [F1],[LKF].

It is clear that

$$\mathrm{tr}(x) = \sum_{i=1}^{r} \lambda_i, \mathrm{rank}(x) = \mathrm{card}\{i \in [1, r] : \lambda_i \neq 0\}.$$

Let

$$\Omega_l = \{x \in \bar{\Omega} : rank(x) = l\}, l = 0, \ldots r.$$

Notice that the closure

$$\bar{\Omega}_l = \cup_{k=0}^{l} \Omega_k$$

for any l.

Since primitive idempotents in $V(1, f)$ remain primitive in V, it easily follows that the rank of $x \in V(1, f)$ is the same as its rank in V.

3 Tail estimates

In this section we collected several measure concentration results on manifolds of nonnegative elements of a fixed rank in a simple Euclidean Jordan algebra.

Proposition 2 *For any $l = 1, 2, \ldots, r-1$, there exists a positive measure $\tilde{\mu}_l$ on $\bar{\Omega}$ which is uniquely characterized by the following properties:*

a) the support of $\tilde{\mu}_l$ is $\bar{\Omega}_l$.

b)

$$\int_{\bar{\Omega}} e^{-\langle x, y\rangle} d\tilde{\mu}_l(x) = \det(y)^{-ld/2} \tag{5}$$

for any $y \in \Omega$; $\tilde{\mu}_l(\bar{\Omega}_{l-1}) = 0$ $(l \geq 1)$.

This Proposition immediately follows from Corollary VII.1.3 and Proposition VII.2.3 of [FK].

It will be convenient to introduce another set of measures:

$$\mu_l = e^{-tr(x)} \tilde{\mu}_l.$$

We immediately derive from Proposition 2 that $\mu_l(\bar{\Omega}_l) = 1$ and

$$\int_{\bar{\Omega}_l} e^{-\langle x,y\rangle} d\mu_l(x) = \det(e+y)^{-ld/2}$$

for any $y \in \bar{\Omega}$. Since μ_l is a probability measure on $\bar{\Omega}_l$, we can introduce standard probabilistic characteristics of measurable functions on $\bar{\Omega}_l$:

$$E[\varphi] = \int_{\bar{\Omega}_l} \varphi(x) d\mu_l(x),$$

$$Var(\varphi) = \int_{\bar{\Omega}_l} (\varphi(x) - E[\varphi])^2 d\mu_l(x).$$

Proposition 3 *Let $f_1, \dots f_r$ be a Jordan frame in V. Consider random variables $\varphi_1 \dots, \varphi_r$, where*

$$\varphi_i(x) = \langle f_i, x\rangle, i = 1, 2, \dots, r.$$

Then $\varphi_1, \dots \varphi_r$ are mutually independent identically distributed random variables having Gamma distribution with parameters $(1, \chi)$, where

$$\chi = \frac{dl}{2}.$$

Proof Let us compute the joint moment generating function $\Phi(t_1, \dots, t_r)$ for $\varphi_1, \dots, \varphi_r$. We have:

$$\Phi(t_1, \dots, t_r) = E[e^{t_1\varphi_1 + t_2\varphi_2 + \dots + t_r\varphi_r}] =$$

$$= \int_{\bar{\Omega}_l} \exp(\langle \sum_{i=1}^{r} t_i f_i, x\rangle) d\mu_l =$$

$$= \det(e - \sum_{i=1}^{r} t_i f_i)^{-\chi} = \prod_{i=1}^{r} (1 - t_i)^{-\chi} =$$

$$\prod_{i=1}^{r} E[e^{t_i\varphi_i}]$$

for $\mid t_1 \mid < 1, \mid t_2 \mid, \dots \mid t_r \mid < 1$. The Proposition follows.

Corollary 1 *Let c be a primitive idempotent. Then for $\varphi_c(x) = \langle c, x\rangle$, we have:*

$$E[\varphi_c] = Var(\varphi_c) = \chi.$$

Let $q \in V$,

$$q = \sum_{i=1}^{r} \lambda_i c_i$$

be its spectral decomposition. Consider $\varphi_q(x) = \langle q, x\rangle$. Then

$$E[\varphi_q] = \chi tr(q),$$

$$Var(\varphi_q) = \chi \|q\|^2 = \chi \sum_{i=1}^{r} \lambda_i^2.$$

The Corollary immediately follows from Proposition 3.

Proposition 4 *Let* $q \in \bar{\Omega}$. *Consider*

$$\Gamma = \{x \in \bar{\Omega}_l : \langle q, x \rangle \leq \chi(tr(q) - \tau \|q\|)\}.$$

Then

$$\mu_l(\Gamma) \leq \exp(\frac{-\chi\tau^2}{2}), \tau \geq 0. \tag{6}$$

Remark 1 *Compare this with [barvinok],Proposition 5.6.*

<u>Proof</u> For any $\lambda > 0$ we have:

$$\Gamma = \{x \in \bar{\Omega}_l : \exp(-\lambda\langle q, x \rangle) \geq \exp(-\lambda\chi(tr(q) - \tau\|q\|)\}.$$

By Markov's inequality:

$$\mu_l(\Gamma) \leq \frac{E[e^{-\lambda\langle q,x \rangle}]}{\exp(-\lambda\chi(tr(q) - \tau\|q\|))}. \tag{7}$$

Let

$$q = \sum_{i=1}^{r} \mu_i c_i \tag{8}$$

be the spectral decomposition of q. Then, using (5),

$$E[e^{-\lambda\langle q,x\rangle}] = \prod_{i=1}^{r}(1 + \lambda\mu_i)^{-\chi}.$$

Hence, by (7)

$$\mu_l(\Gamma) \leq \exp(-\chi(\sum_{i=1}^{r}[\ln(1 + \lambda\mu_i) - \lambda\mu_i] + \lambda\tau\|q\|)).$$

Using an elementary inequality

$$\ln(1 + x) \geq x - \frac{x^2}{2}, x \geq 0,$$

we obtain:

$$\mu_l(\Gamma) \leq \exp(\chi(\frac{\lambda^2\|q\|^2}{2} - \lambda\tau\|q\|)).$$

Taking $\lambda = \frac{\tau}{\|q\|}$, we obtain (6).

Proposition 5 *Let* $q \in \bar{\Omega}$ *have the spectral decomposition (8) and set* $\delta = \max\{\mu_i : 1 \leq i \leq r\}$. *Consider*

$$\Gamma = \{x \in \bar{\Omega}_l : \langle q, x \rangle \geq \chi(tr(q) + \tau\|q\|)\}.$$

Then

$$\mu_l(\Gamma) \leq \exp\{-\frac{\chi}{4}\min\{\frac{\|q\|}{\delta}, \tau\}\tau\} \tag{9}$$

for any $\tau \geq 0$.

Remark 2 *Compare this with [Zhang1], Lemma 5.1 .*

Proof Clearly, for any $\lambda > 0$

$$\Gamma = \{x \in \bar{\Omega}_l : \exp(\lambda\langle q, x\rangle \geq \exp(\lambda\chi(tr(q) + \tau\|q\|))\}.$$

Using Markov's inequality, we obtain:

$$\mu_l(\Gamma) \leq \frac{E[e^{\lambda\langle q,x\rangle}]}{\exp(\lambda\chi(tr(q) + \tau\|q\|))} =$$

$$\exp(-\chi(\sum_{i=1}^{r}(\ln(1-\lambda\mu_i) + \lambda\mu_i) + \tau\lambda\|q\|)).$$

Using an elementary inequality $\ln(1-x) \geq -x - x^2, x \leq \frac{1}{2}$, we obtain:

$$\mu_l(\Gamma) \leq \exp(\chi(\lambda^2\|q\|^2 - \tau\lambda\|q\|),$$

provided $\lambda\mu_i \leq \frac{1}{2}$ for all i. Take

$$\lambda = \frac{1}{2}\min\{\frac{1}{\delta}, \frac{\tau}{\|q\|}\}.$$

Then $\lambda\mu_i \leq \frac{1}{2}$ for all i. Notice that $\lambda\|q\| \leq \tau/2$. Hence,

$$\mu_l(\Gamma) \leq \exp(\lambda\chi\|q\|(\lambda\|q\| - \tau)) \leq$$

$$\exp(\frac{-\chi\lambda\|q\|\tau}{2}) = \exp(-\frac{\chi}{4}\min\{\frac{\|q\|}{\delta}, \tau\}\tau).$$

Proposition 6 *Let $q \in \bar{\Omega}$ have a spectral decomposition:*

$$q = \sum_{i=1}^{s}\lambda_i c_i,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$. Let

$$\Gamma = \{x \in \bar{\Omega}_l : \langle q, x\rangle \leq \chi\beta tr(q)\}$$

for some $\beta > 0$. Then

$$\mu_l(\Gamma) \leq (\frac{5e\beta}{2})^{\chi}$$

provided $e\beta\ln s \leq \frac{1}{5}$.

Remark 3 *Compare this result with [Ye], Proposition 2.2.*

Proof We know that

$$tr(q) = \sum_{i=1}^{s}\lambda_i.$$

Let

$$\bar{\lambda}_i = \frac{\lambda_i}{\sum_{j=1}^{s}\lambda_j}.$$

Then

$$\Gamma = \{x \in \bar{\Omega}_l : \sum_{i=1}^{s}\bar{\lambda}_i\langle c_i, x\rangle \leq \chi\beta\}.$$

Denote $\mu_l(\Gamma)$ by $p(s,\bar{\lambda}_1,\dots,\bar{\lambda}_s,\beta)$. We clearly have:

$$p(s,\bar{\lambda}_1,\dots\lambda_s,\beta)\leq\mu_l\{x\in\bar{\Omega}_l:\sum_{i=1}^{s}\bar{\lambda}_s\langle c_i,x\rangle\leq\chi\beta\}$$

$$=\mu_l\{x\in\bar{\Omega}_l:\langle\sum_{i=1}^{s}c_i,x\rangle\leq\frac{\chi\beta}{\bar{\lambda}_s}\}.$$

Lemma 1 *Let c be an idempotent in V and $rank(c)=s$. Then for any $\beta\in(0,1)$:*

$$\mu_l\{x\in\bar{\Omega}_l:\langle c,x\rangle\leq s\chi\beta\}\leq\exp(\chi s(1-\beta+\ln\beta))\leq$$

$$\exp(\chi s(1+\ln\beta))=(e\beta)^{\chi s}$$

.

<u>Proof of Lemma 1</u> Let

$$\Gamma\{x\in\bar{\Omega}_l:\langle c,x\rangle\leq s\chi\beta\}.$$

Then for any $\lambda>0$

$$\Gamma=\{x\in\bar{\Omega}_l:\exp(-\lambda\langle c,x\rangle\geq\exp(-\lambda s\chi\beta)\}.$$

Using Markov's inequality, we obtain

$$\mu_l(\Gamma)\leq\frac{E[\exp(-\lambda\langle c,x\rangle)]}{\exp(-\lambda s\chi\beta)}=$$

$$(1+\lambda)^{-\chi s}\exp(\lambda s\chi\beta)=$$

$$\exp(-\chi s(\ln(1+\lambda)-\lambda\beta)). \tag{10}$$

Consider the function

$$\varphi(\lambda)=\lambda\beta-\ln(1+\lambda),\lambda>0.$$

One can easily see that φ attains its minimum at the point

$$\lambda^*=\frac{1}{\beta}-1.$$

Notice that $\lambda^*>0$ if and only if $\beta\in(0,1)$. Moreover, $\varphi(\lambda^*)=1-\beta+\ln\beta$. We have from (10) that

$$\mu_l(\Gamma)\leq\exp(\chi s\varphi(\lambda^*))$$

and the result follows.

We return to the proof of Proposition 6. By Lemma 1 we obtain:

$$\mu_l\{x\in\bar{\Omega}_l:\langle\sum_{i=1}^{s}c_i,x\rangle\leq\frac{\chi\beta}{\bar{\lambda}_s}\}\leq(\frac{e\beta}{\bar{\lambda}_s s})^{\chi s}. \tag{11}$$

Notice that Lemma 1 requires that

$$\frac{\beta}{\bar{\lambda}_s s}<1.$$

However, if it does not hold true, the estimate (11) is trivial, since the left-hand side of (11) is less or equal than one. We thus obtain:

$$p(s,\bar{\lambda}_1,\dots,\bar{\lambda}_s,\beta)\leq(\frac{e\beta}{\bar{\lambda}_s})^{\chi s}.$$

Notice, further, that

$$p(s,\bar{\lambda}_1,\dots\bar{\lambda}_s,\beta)\leq\mu_l\{x\in\bar{\Omega}_l:\sum_{i=1}^{s-1}\bar{\lambda}_i\langle c_i,x\rangle\leq\chi\beta\}=$$

$$\mu_l\{x\in\bar{\Omega}_l:\sum_{i=1}^{s-1}\frac{\bar{\lambda}_i}{1-\bar{\lambda}_s}\langle c_i,x\rangle\leq\frac{\chi\beta}{1-\bar{\lambda}_s}\}=$$

$$p(s-1,\frac{\bar{\lambda}_1}{1-\bar{\lambda}_s},\dots\frac{\bar{\lambda}_{s-1}}{1-\bar{\lambda}_s},\beta)\leq[\frac{(\frac{e\beta}{1-\bar{\lambda}_s})}{\frac{\bar{\lambda}_{s-1}(s-1)}{1-\bar{\lambda}_s}}]^{\chi(s-1)}=$$

$$[\frac{e\beta}{\bar{\lambda}_{s-1}(s-1)}]^{\chi(s-1)},$$

where the last inequality follows by (11). Continuing in this way, we obtain:

$$p(s,\bar{\lambda}_1,\dots\bar{\lambda}_s,\beta)\leq min\{(\frac{e\beta}{i\bar{\lambda}_i})^{i\chi}:1\leq i\leq s\}. \tag{12}$$

Let $\delta=p(s,\bar{\lambda}_1,\dots\bar{\lambda}_s,\beta)^{\frac{1}{\chi}}$. Notice that $0<\delta<1$ and by (12):

$$\delta\leq\min\{(\frac{e\beta}{i\bar{\lambda}_i})^i:i=1,\dots,s\}.$$

Hence,

$$\bar{\lambda}_i\leq\frac{e\beta}{\delta^{1/i}i},i=1,\dots s.$$

Taking into account that

$$\sum_{i=1}^{r}\bar{\lambda}_i=1,$$

we obtain:

$$\sum_{i=1}^{s}\frac{1}{i\delta^{\frac{1}{i}}}\geq\frac{1}{e\beta}. \tag{13}$$

Reasoning now exactly as in [Ye], we obtain:

$$\frac{1}{e\beta}\leq\frac{2}{\delta}+\ln s.$$

Hence, if $e\beta\ln s\leq\frac{1}{5}$, then

$$\delta\leq\frac{5e\beta}{2}.$$

4 Sample of results

In this section we show how measure concentration results of section 3 can be used to estimate the quality of symmetric relaxations for a variety of nonconvex optimization problems.

Theorem 1 *Let $a_i\in\bar{\Omega},\alpha_i\geq 0,i=1,2,\dots k$. Suppose that the system of linear equations*

$$\langle a_i,x\rangle=\alpha_i, \tag{14}$$

$i=1,2,\dots k$, has a solution $\bar{x}\in\bar{\Omega}$. Then, given $1>\epsilon>0$, there exists $x_0\in\bar{\Omega}_l$ such that

$$\mid\langle a_i,x\rangle-\alpha_i\mid\leq\epsilon\alpha_i, \tag{15}$$

$i=1,2,\dots,k$, provided

$$l\geq\frac{8\ln(k)}{\epsilon^2 d}.$$

Remark 4 *Theorem 1 generalizes the result of A. Barvinok who considered the case, where Ω is the cone of positive definite real symmetric matrices. See [barvinok], Proposition 6.1.*

Lemma 2 *Let $a, b \in \bar{\Omega}$. Then*

$$rank(P(a)b) \leq \min(rank(a), rank(b))$$

.

Proof of Lemma 2 Let

$$a = \sum_{i=1}^{s} \lambda_i c_i$$

be the spectral decomposition of a with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_s > 0$. Then $ImP(a) = V(c_1 + c_2 + \ldots c_s, 1)$. Hence, the maximal possible rank of an element in $ImP(a)$ does not exceed $rankV(c_1 + c_2 + \ldots c_s, 1) = s$. Hence, $rank(P(a)b) \leq s = rank(a)$. If a is invertible in V, then $P(a)$ belongs to the group of automorphisms of the cone Ω and consequently preserves the rank (see [[FK]],). In general, if $a \in \bar{\Omega}$ and $\epsilon > 0$, then $a + \epsilon e \in \Omega$ and consequently $P(a + \epsilon e)$ preserves the rank, i.e.

$$r(\epsilon) = rank(P(a + \epsilon e)b) = rank(b)$$

for any $\epsilon > 0$. Notice that $P(a)b = \lim P(a + \epsilon e)b, \epsilon \to 0$. It suffices to prove the following general fact. Let $a_i, i = 1, 2, \ldots$ be a sequence in $\bar{\Omega}$ such that $rank(a_i)$ does not depend on i and is equal to s. Let, further, $a_i \to a_*, i \to \infty$. Then $rank(a_*) \leq s$. Let $x \in \bar{\Omega}$ and $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots \lambda_r(x))$ be the spectrum of x with $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \lambda_r(x)$. Notice that $rank(x)$ is equal the largest k such that $\lambda_k(x) > 0$. Let k be the largest k such that $\lambda_k(a_*) > 0$. Then $k = rank(a_*)$. Since $\lambda_k(a_i) \to \lambda_k(a_*), i \to \infty$, (see 4), we conclude that $\lambda_k(a_i) > 0$ for sufficiently large i. Hence, $k \leq rank(a_i) = s$. The result follows.
Proof of Theorem 1 Consider the following system of linear equations:

$$\langle P(\bar{x}^{1/2})a_i, y\rangle = \alpha_i, \tag{16}$$

$i = 1, 2, \ldots k, y \in \bar{\Omega}$. Notice that $y = e$ satisfies (16). More generally if

$$\mid \langle P(\bar{x})^{1/2}a_i, y_0\rangle - \alpha_i \mid < \epsilon\alpha_i,$$

$i = 1, 2, \ldots k, y_0 \in \bar{\Omega}$, then

$$\mid \langle a_i, P(\bar{x}^{1/2})y_0\rangle - \alpha_i \mid < \epsilon\alpha_i,$$

$i = 1, \ldots k, P(\bar{x}^{1/2})y_0 \in \bar{\Omega}$ and $rank(P(\bar{x}^{1/2}y_0) \leq rank(y_0)$ by Lemma 2. Hence, without loss of generality we can assume that $\bar{x} = e$ is a solution to (14) and consequently $tr(a_i) = \alpha_i$, for $i = 1, 2, \ldots k$. Consider

$$A_l = \{x \in \bar{\Omega}_l : \mid \langle a_i, x\rangle - \chi tr(a_i) \mid \leq \epsilon\chi tr(a_i), i = 1, \ldots k\}.$$

It is clear that if $A_l \neq \emptyset$, then x/χ satisfies (16). Consider also

$$B_{il} = \{x \in \bar{\Omega}_l : \langle a_i, x\rangle - \chi tr(a_i) > \epsilon\chi tr(a_i)\},$$

$$C_{il} = \{x \in \bar{\Omega}_l : \langle a_i, x\rangle - \chi tr(a_i) < -\epsilon\chi tr(a_i)\},$$

$i = 1, \ldots, k$. Notice that

$$\bar{\Omega}_l \setminus A_l = \cup_{i=1}^{k}(B_{il} \cup C_{il}).$$

Hence,

$$1 - \mu_l(A_l) \leq \sum_{i=1}^{k} [\mu_l(B_{il}) + \mu_l(C_{il})]$$

and consequently

$$\mu_l(A_l) \geq 1 - \sum_{i=1}^{k} [\mu_l(B_{il}) + \mu_l(C_{il})]. \tag{17}$$

Since $tr(a_i) \geq \|a_i\|, i = 1, \dots k, (a_i \in \bar{\Omega})$, we have:

$$B_{il} \subset \{x \in \bar{\Omega}_l : \langle a_i, x \rangle - \chi tr(a_i) > \epsilon \chi \|a_i\|\},$$

$$C_{il} \subset \{x \in \bar{\Omega}_l : \langle a_i, x \rangle - \chi tr(a_i) < -\epsilon \chi \|a_i\|\},$$

$i = 1, \dots, l$. Hence,

$$\mu_l(B_{il}) \leq \exp(-\frac{\epsilon^2 \chi}{2}), i = 1, \dots, k, \tag{18}$$

by Proposition 4 and

$$\mu_l(C_{il}) \leq \exp(-\frac{\epsilon^2 \chi}{4}), i = 1, \dots, k, \tag{19}$$

by Proposition 5 (notice that $0 < \epsilon < 1$ and $\|q\|/\delta \geq 1$ in Proposition 5). Take

$$l \geq \frac{8 \ln(2k)}{\epsilon^2 d}.$$

Then

$$\chi \geq \frac{4ln(2k)}{\epsilon^2}.$$

By (18), (19):

$$\mu_l(B_{il}) \leq \frac{1}{4k^2}, \mu_l(C_{il}) \leq \frac{1}{2k},$$

$i = 1, \dots, k$. By (17)

$$\mu_l(A_l) \geq 1 - (\frac{1}{2} + \frac{1}{4k}) > 0.$$

Thus , $A_l \neq \emptyset$ and the result follows.

Consider the following optimization problem:

$$\langle c, x \rangle \to \max, \tag{20}$$

$$\langle a_i, x \rangle \leq 1, i = 1, \dots k, \tag{21}$$

$$x \in \bar{\Omega}_l. \tag{22}$$

Notice that the case $l = 1$ corresponds to homogeneous quadratic constraints in case where Ω is one of the cones of positive definite Hermitian matrices over real, complex or quaternion numbers. See [F1]. Here we assume that $c, a_1, \dots a_k \in \bar{\Omega}$. We wish to compare the optimal value for (20)-(22) with its symmetric relaxation:

$$\langle c, x \rangle \to \max, \tag{23}$$

$$\langle a_i, x \rangle \leq 1, i = 1, \dots, k \tag{24}$$

$$x \in \bar{\Omega}. \tag{25}$$

We assume that (23)-(25) has an optimal solution x_0. Let v_{max} and v_R be optimal values for (20)-(22) and (23)-(25), respectively. Then, of course, $v_{max} \leq v_R$. Consider the optimization problem

$$\langle \tilde{c}, y \rangle \rightarrow \max, \tag{26}$$

$$\langle \tilde{a}_i, y \rangle \leq 1, i = 1, \ldots k, \tag{27}$$

$$y \in \bar{\Omega}, \tag{28}$$

where $\tilde{c} = P(x_0^{1/2})c, \tilde{a}_i = P(x_0^{1/2})a_i, i = 1, \ldots k$. Then $y = e$ is an optimal solution to (24)-(26). Indeed, $\tilde{a}_i, e\rangle = \langle a_i, x_0\rangle \leq 1, i = 1, 2, \ldots k$, $\langle \tilde{c}, e\rangle = tr(\tilde{c}) = \langle c, x_0 \rangle = v_R$. On the other hand, if y is feasible for (26)-(28), then $P(x_0^{1/2})y$ is feasible for (23)-(25) and $\langle c, P(x_0^{1/2})y\rangle = \langle \tilde{c}, y\rangle$. Thus, if $\tilde{v}_R$ is an optimal value for (26)-(28), we have $\tilde{v}_R \leq v_R$. Since $\langle \tilde{c}, e\rangle = v_R$, the result follows and $\tilde{v}_R = v_R$. Consider now the problem

$$\langle \tilde{c}, y \rangle \rightarrow \max, \tag{29}$$

$$\langle \tilde{a}_i, y \rangle \leq 1, \tag{30}$$

$$y \in \bar{\Omega}_l. \tag{31}$$

If $\tilde{v}_{max}$ is an optimal value for the problem (29)-(31), then the reasoning above shows that $\tilde{v}_{max} \leq v_{max}$. (Notice that $P(x_0^{1/2})\bar{\Omega}_l \subset \bar{\Omega}_l$ by Lemma 2). Thus,

$$\tilde{v}_{max} \leq v_{max} \leq v_R = \tilde{v}_R. \tag{32}$$

Lemma 3 *Let $\gamma > 0, \mu > 0$ be such that the set*

$$A = \{y \in \bar{\Omega}_l : \langle \tilde{a}_i, y\rangle \leq \chi\gamma, i = 1, \ldots, m, \langle \tilde{c}, y\rangle \geq \chi\mu tr(\tilde{c})\} \neq \emptyset.$$

Then

$$v_{max} \geq \frac{\gamma}{\mu} v_R. \tag{33}$$

Proof of Lemma 3 Let $y \in A$. Then $\frac{y}{\chi\gamma}$ is feasible for (29)-(31). Consequently,

$$\tilde{v}_{max} \geq \langle \tilde{c}, \frac{y}{\chi\gamma}\rangle \geq \frac{\chi\mu}{\chi\gamma} tr(\tilde{c}) = \frac{\mu}{\gamma} v_R.$$

Combining this with (32), we obtain (33).

The result follows.

Theorem 2 *For problem 20-(22) and its relaxation (23)-(25) the estimate (33) holds with $\gamma = \xi + 1 + \frac{4\ln k}{\chi}, \mu = 1 - \frac{1}{2\sqrt{\chi}}$, where $\xi > 0$ is chosen large enough so that $e^{-\xi/8} < 1 - e^{-1/8}, \gamma - 1 > 1$.*

Remark 5 *Theorem 2 generalizes the results of [BN],[BN1], who considered the case $l = 1, \Omega$ is the cone of positive definite real symmetric matrices. In [Zhang1] the case $l = 1, \Omega$ is the cone of positive definite Hermitian complex matrices was considered.*

Proof We will show that with this choice of μ and γ the corresponding set A in Lemma 3 is not empty. Let

$$A_i = \{y \in \bar{\Omega}_l : \langle \tilde{a}_i, y\rangle > \chi\gamma\},$$

$$C = \{y \in \bar{\Omega}_l : \langle \tilde{c}, y\rangle < \mu\chi tr(\tilde{c})\}.$$

Then

$$A^c = (\cup_{i=1}^k A_i) \cup C$$

and consequently

$$\mu_l(A) \geq 1 - \mu_l(C) - \sum_{i=1}^k \mu_l(A_i). \tag{34}$$

Notice that $\gamma > 1$ and $tr(\tilde{a}_i) \leq 1, \forall i$. Choose

$$\tau = \frac{\gamma - tr(\tilde{a}_i)}{\|\tilde{a}_i\|}.$$

According to Proposition 5

$$\mu_l(A_i) \leq \exp\{-\frac{\chi}{4} \min\{\frac{\|\tilde{a}_i\|}{\delta_i}, \tau\}\tau\},$$

where δ_i is the maximal eigenvalue of $\tilde{a}_i$. Notice that

$$\frac{\|\tilde{a}_i\|}{\delta_i} \geq 1, \tau \geq \frac{\gamma - 1}{\|\tilde{a}_i\|}.$$

Hence,

$$\mu_l(A_i) \leq \exp\{-\frac{\chi}{4} \min\{1, \frac{\gamma - 1}{\|\tilde{a}_i\|}\}\tau\}.$$

Since $\gamma - 1 > 1$ and $\|\tilde{a}_i\| \leq tr(\tilde{a}_i) \leq 1$, we have:

$$\mu_l(A_i) \leq \exp\{-\frac{\chi}{4}\tau\}.$$

Notice that $\tau \geq \gamma - 1$. Hence, according to our choice of γ

$$\mu_l(A_i) \leq \exp\{-\frac{\chi}{4}(\gamma - 1)\} = \frac{\exp(\frac{-\chi\xi}{4})}{k}.$$

Consequently,

$$\sum_{i=1}^k \mu_l(A_i) \leq \exp(-\frac{\chi\xi}{4}) \leq e^{-\xi/8},$$

since $\chi \geq 1/2$. Since $\|\tilde{c}\| \leq tr(\tilde{c}), \mu < 1$, we have:

$$C \subset \{y \in \bar{\Omega}_l : \langle \tilde{c}, y \rangle < \chi(tr(\tilde{c}) - (1 - \mu)\|\tilde{c}\|)\}.$$

Hence, by Proposition 4

$$\mu_l(C) \leq \exp(-\frac{\chi}{2}(1 - \mu)^2) = e^{-\frac{1}{8}}.$$

Using (34), we obtain:

$$\mu_l(A) > 1 - e^{-\frac{1}{8}} - e^{-\frac{\xi}{8}} > 0$$

according to the choice of ξ.The result follows.

Consider the following optimization problem:

$$\langle c, x \rangle \to \min, \tag{35}$$

$$\langle a_i, x \rangle \geq 1, i = 1, \ldots, k, \tag{36}$$

$$x \in \bar{\Omega}_l \tag{37}$$

. We assume that $c, a_1, \ldots a_k \in \bar{\Omega}$. Consider also a symmetric relaxation of (35)-(37):

$$\langle c, x \rangle \to \min, \tag{38}$$

$$\langle a_i, x \rangle \geq 1, \tag{39}$$

$$x \in \Omega. \tag{40}$$

We will assume that (38)-(40) has an optimal solution. Using the same trick as we used in analyzing (23)-(25), we can assume without loss of generality that $tr(a_i) \geq 1, i = 1, \ldots, k$, and $v_R = tr(c)$, where v_R is an optimal value for (38)-(40). We can also assume that $rank(a_i) \leq \varphi_d^{-1}(k)$ for all i. Here

$$\varphi_d(x) = x + \frac{dx(x-1)}{2}.$$

The last remark easily follows from Lemma 2 and Theorem 3 in [F1].

Lemma 4 *If*

$$A = \{x \in \bar{\Omega}_l : \langle a_i, x \rangle \geq \chi\gamma, i = 1, \ldots k, \langle < c, x \rangle \leq \chi\mu tr(c)\}$$

is not empty for some $\gamma > 0$ *and some* $\mu > 0$, *then*

$$\frac{\mu}{\gamma} v_R \geq v_{min} \geq v_R, \tag{41}$$

where v_{min} *is the optimal value of (35)-(37).*

The proof of Lemma 4 is quite similar to the proof of Lemma 3.

Theorem 3 *For problem (35)-(37) and its relaxation (38)-(40) the estimate (41) holds with*

$$\mu = \frac{1}{0.99 - (\frac{e}{5})^{1/2}}, \gamma = \frac{1}{25\chi k^{1/\chi}}.$$

Remark 6 *Theorem 3 generalizes the results of [Zhang2] (Theorems 1,2), where the cases* $l = 1, d = 1, 2$ *were considered.*

Lemma 5 *For any integers* $d \geq 1, k \geq 1$ *we have:*

$$\varphi_d^{-1}(k) \leq 3\sqrt{k}.$$

Proof of Lemma 5

An easy computation shows that:

$$\varphi_d^{-1}(k) = (\frac{1}{2} - \frac{1}{d}) + \sqrt{(\frac{1}{2} - \frac{1}{d})^2 + \frac{2k}{d}}.$$

Hence,

$$\varphi_d^{-1}(k) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2k} = \frac{1 + \sqrt{1 + 8k}}{2} \leq \sqrt{1 + 8k} \leq 3\sqrt{k}.$$

Lemma 6 *For* $x > 0, \gamma > 0, c > \frac{1}{e\gamma}$ *we have:*

$$\ln x \leq cx^{\gamma}.$$

Proof of Lemma 6 Consider the function $\varphi(x) = cx^\gamma - \ln x, x > 0$. Notice that $\varphi(x) \to +\infty$, when $x \to 0$ or $x \to +\infty$. Furthermore, an easy computation shows that $\varphi^{'}(x) = 0$ if and only if

$$x = \frac{1}{(\gamma c)^{1/\gamma}}.$$

Hence, $\varphi(x)$ has a global minimum at this point. An easy computation shows that this minimum is equal to

$$\frac{1 + \ln(\gamma c)}{\gamma},$$

which is nonnegative precisely when

$$c \geq \frac{1}{e\gamma}.$$

Proof of Theorem 3 Let

$$A_i = \{x \in \bar{\Omega}_l : \langle a_i, x\rangle < \chi\gamma\}, i = 1, \dots k,$$
$$C = \{x \in \bar{\Omega}_l : \langle c, x\rangle > \chi\mu tr(c)\}.$$

Then one can easily see that

$$\mu_l(A) \geq 1 - \sum_{i=1}^{k} \mu_l(A_i) - \mu_l(C),$$

where A is defined as in Lemma 4. Notice that

$$A_i \subset \{x \in \bar{\Omega}_l : \langle a_i, x\rangle \leq \chi\gamma tr(a_i)\}, i = 1, \dots k,$$

since $tr(a_i) \geq 1$. Hence, by Proposition 6

$$\mu_l(A_i) \leq (\frac{5e\gamma}{2})^\chi$$

provided $e\gamma \ln(rank(a_i)) \leq 1/5$. Due to our assumptions, it suffices to verify that

$$e\gamma \ln \varphi_d^{-1}(k) \leq 1/5.$$

One can easily see that $\varphi_d(1) = 1$ for any d. Hence, it suffices to consider the case $k \geq 2$. But then by Lemma 5:

$$\ln \varphi_d^{-1}(k) \leq \frac{5}{2} \ln k.$$

Using Lemma 6, we have:

$$\ln k \leq \frac{\chi}{e} k^{\frac{1}{\chi}}.$$

Combining this with our choice of γ, we obtain:

$$e\gamma \ln \varphi_d^{-1}(k) \leq 1/5.$$

Hence, taking into account our choice of γ, we obtain:

$$\mu_l(A_i) \leq (\frac{e}{10\chi})^\chi \frac{1}{k} \leq (\frac{e}{5})^{1/2} \frac{1}{k}, i = 1, \dots k,$$

since $\chi \geq 1/2$. Hence,

$$\sum_{i=1}^{k} \mu_l(A_i \leq (\frac{e}{5})^{1/2}.$$

Notice, further, that by Markov inequality:

$$\mu_l(C) \leq \frac{\chi tr(c)}{\chi\mu tr(c)} = \frac{1}{\mu}.$$

Now, with our choice of μ:

$$\sum_{i=1}^{k} \mu_l(A_i) + \mu_l(C) \leq 0.99 < 1$$

and the result follows.

5 Concluding remarks

In present paper we introduced a general randomization technique in the context of Euclidean Jordan algebras. It enables us to generalize various results related to quality of symmetric relaxations of difficult optimization problems. In particular, all major results of [Ye] (including Theorem 1.1) can be generalized to the case of an arbitrary irreducible symmetric cone. Almost all results in [Zhang1] can be generalized to the case of an arbitrary symmetric cone and arbitrary rank (in [Zhang1] only cones of real symmetric and complex Hermitian matrices and rank one constraints are considered).

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