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Numerical solutions of the algebraic matrix Riccati equation

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Abstract

The linear-quadratic control model is one of the most widely used control models in both empirical and theoretical economic modeling. In order to obtain the equilibrium solution of this control model, the so-called algebraic matrix Riccati equation has to be solved. In this note we present a numerical solution method for solving this equation. Our method solves the Riccati equation as a multidimensional fixed-point problem. By establishing the analytical derivative of the Riccati equation we have been able to construct a very efficient Newton-type solution method with quadratic convergence properties. Our method is an extension for the Newton–Raphson method described in Kwakernaak and Sivan (1972) and does not require any special conditions on the transition matrix as in the nonrecursive method of Vaughan (1970).

Key words: Algebraic matrix Riccati equation; Newton numerical solution method; Equilibrium solution
JEL classification: C61; C63

1. Introduction

One of the most widely used control models in economics is the Linear-Quadratic control model. Recent examples are found in McGratten (1994), Amman and Kendrick (1995), and many more. Generally, the solution of this control model for finite-horizon problems is obtained by solving the so-called

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Riccati matrix equation backward in time from the terminal date to an initial date. Solving the Riccati equation recursively in time is a simple operation which generally does not pose any difficulties. The procedure is different, however, for infinite-horizon problems. In that case the algebraic matrix Riccati equation has to be solved. Unfortunately, there seems to be no analytical solution to this equation.

Bertsekas (1976) pointed out that under certain restrictions the algebraic matrix Riccati equation could be solved by using successive substitutions and that it has a unique solution. Successive substitutions are quite inefficient, however, and it may take many iteration steps before a solution is reached. In this note we will present a fast method for solving the algebraic Riccati equation numerically, using a Newton-type solution method. The advantage of such a method, as generally known, is that it has quadratic convergence properties ensuring a rapid solution. Our method deviates from the newer approaches in the engineering literature, see Lainiotis et al. (1994) and Stoorvogel and Weeren (1994).[1] These algorithms are of a recursive nature and do not exploit derivative information.

A nonrecursive method for solving the algebraic Riccati equation, popular in economics, is the method of Vaughan (1970). Unfortunately this method is only applicable to a limited set of problems as it requires that the transition matrix should be nonsingular. For most applications in economics this condition does not hold and an alternative method should be used. In the next paragraph we present such a method that does not require any additional assumptions on the matrices in the linear-quadratic control model. Our method is an extension of the Newton–Raphson method described in Kwakernaak and Sivan (1972) in the sense that we use derivative information for solving the Riccati equation. Also, it does not require any special conditions on the transition matrix as in the nonrecursive method of Vaughan (1970).

2. The Riccati equation

Following Kendrick (1981), a commonly used form of the linear-quadratic control model is to minimize

$$J = \tfrac{1}{2}[x_T - \hat{x}_T]'W_T[x_T - \hat{x}_T] + \tfrac{1}{2}\sum_{t=0}^{T-1}\{[x_t - \hat{x}_t]'W_t[x_t - \hat{x}_t]$$

$$+[u_t - \hat{u}_t]'R_t[u_t - \hat{u}_t] + [x_t - \hat{x}_t]'F_t[u_t - \hat{u}_t]\}, \qquad (1)$$

[1] We would like to thank one of the anonymous referees for bringing to our attention a number of recent references in this area.

subject to the constraints

$$x_{t+1} = A_t x_t + B_t u_t + c_t$$
$$x_0 \quad \text{given,} \tag{2}$$

with $t \in \{0, 1, .., T\}$, $x_t \in \Re^n$ as the state vector, $u_t \in \Re^m$ as the control vector, $c_t \in \Re^n$ the vector of exogenous variables, $\hat{x}_t \in \Re^n$ and $\hat{u}_t \in \Re^n$ as target vectors, $A_t \in \Re^{(n\times n)}$ and $B_t \in \Re^{(n\times m)}$ as system matrices, and $W_t \in \Re^{(n\times n)}$, $R_t \in \Re^{(m\times m)}$, $F_t \in \Re^{(n\times m)}$ as penalty matrices. Normally W_t and R_t are assumed to be positive semi-definite and positive definite, respectively. In this general form of the linear-quadratic control model, the Riccati equation has the form

$$X_t = W_{t+1} + A_t' X_{t+1} A_t - (A_t' X_{t+1} B_t + F_t)(R_t + B_t' X_{t+1} B_t)^{-1} \times (B_t' X_{t+1} A_t + F_t'), \tag{3}$$

with the fixed-end condition

$$X_T = W_T. \tag{4}$$

Following Sargent (1987) or Lancaster and Rodman (1995), the infinite-horizon equilibrium solution of the Riccati equation, if the matrices are time-invariant, leads to the algebraic matrix Riccati equation

$$X = W + A'XA - (A'XB + F)(R + B'XB)^{-1}(B'XA + F'). \tag{5}$$

Let us now define the following function:

$$G(X) = X - W - A'XA + (A'XB + F)(R + B'XB)^{-1}(B'XA + F'). \tag{6}$$

This leads to the first derivative

$$\partial g/\partial z' = D^+[I - \{(A'XB + F)(R + B'XB)^{-1}B' - A'\} \otimes \{(A'XB + F)(R + B'XB)^{-1}B' - A'\}]D, \tag{7}$$

with $g = D^+ \text{vec}\, G$ and $z = D^+ \text{vec}\, X$. The second derivatives have the form

$$\begin{aligned} H_k &= \partial^2 g_k / \partial z \partial z' \\ &= D'[A\tilde{E}_{ij}A' \otimes B(R + B'XB)^{-1}B']D \\ &\quad - D'[A\tilde{E}_{ij}(A'XB + F)(R + B'XB)^{-1}B' \otimes B(R + B'XB)^{-1}B']D \\ &\quad - D'[B(R + B'XB)^{-1}(B'XA + F')\tilde{E}_{ij}A' \otimes B(R + B'XB)^{-1}B']D \\ &\quad + D'[B(R + B'XB)^{-1}(B'XA + F')\tilde{E}_{ij}(A'XB + F) \\ &\quad \times (R + B'XB)^{-1}B' \otimes B(R + B'XB)^{-1}B']D, \end{aligned} \tag{8}$$

where the matrix I is the identity matrix, $\otimes$ denotes the Kronecker product and $\tilde{E}_{ij} = E_{ij} + E_{ji}$, $i, j \in \{1, .., n\}$. The matrices D and D^+ are the duplication matrix and inverse duplication matrix (cf. Magnus and Neudecker, 1994). These matrices 'remove and restore' redundant upper triangular parts of the (symmetric) positive definite Riccati matrix X and the function $G(X)$. As a result the matrix X is transformed to a $p = \frac{1}{2}n(n+1)$ vector z. Consequently, H_k is a $(p \times p)$ matrix of second derivatives of the kth element g_k of the vector $g(z)$ with respect to the vector z, k being $(j-1)n+i$.

The problem here is to solve the multidimensional fixed-point problem $g(z) = 0$. There are several types of numerical methods to solve such a problem. With the help of Eq. (7) it is simple to construct an iterative scheme to solve our fixed-point problem, *viz.*

$$z^{(j+1)} = z^{(j)} - (\partial g/\partial z')^{-1} g, \tag{9}$$

with $z^{(0)} = D^+ \mathrm{vec}\, W$ as an initial condition. This updating scheme is quite robust and will generally lead to a rapid solution up to some machine precision level. However, this method will in some particular cases not converge. This can be shown by transforming our problem to a nonlinear least-squares problem of the form (see Press et al., 1986)

$$\min: \quad \psi(z) = \tfrac{1}{2} g(z)' g(z), \tag{10}$$

which has the first derivative

$$\partial \psi / \partial z' = g' (\partial g / \partial z') \tag{11}$$

and the second derivative

$$\partial^2 \psi / \partial z \partial z' = (\partial g/\partial z')'(\partial g/\partial z') + \sum_{j=1}^{n} \sum_{i=j}^{n} g_k H_k, \qquad k = (j-1)n + i, \tag{12}$$

which is the Hessian matrix of $\psi(z)$ with respect to z, k, g_k, and H_k as previously defined. Clearly, the Hessian matrix is positive (semi)definite in the neighbourhood of the optimal solution $g(z) = 0$. As such Eq. (12) serves as a diagnostic on convergence of our multidimensional fixed-point algorithm. Furthermore, if the initial value of $g(z^{(0)})$ deviates substantially from zero, nonconvergence may occur. This convergence issue can be fixed to a large extent by adding a relaxation factor λ to equation (9), which leads to

$$z^{(j+1)} = z^{(j)} - \lambda (\partial g/\partial z')^{-1} g. \tag{13}$$

The relaxation factor λ is chosen such that Eq. (10) is globally optimized. There are several strategies described in Press et al. (1986) for choosing λ efficiently.

3. Two examples

Let us assume we have the following matrices that belong to a simple control model with $n = 5$ and $m = 1$ (see Amman, 1995):[2]

$$A = \begin{pmatrix} 0.8 & 0 & 0 & 0 & -0.8 \\ 0.8 & 0 & 0 & 0 & -0.8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad F = 0, \quad W = I, \quad R = 1,$$

which leads to the solution of Eq. (5):

$$X = \begin{pmatrix} 2.2069 & 0 & 0 & 0 & -1.1976 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1.1976 & 0 & 0 & 0 & 2.3115 \end{pmatrix}.$$

Clearly, this example could not be solved by using Vaughan's (1970) nonrecursive method due to the fact that the A matrix is singular.

Table 1[3] gives the norm $\| g(z^{(j)}) \| = \sqrt{g(z^{(j)})' g(z^{(j)})}$ at each iteration step for the above model when we use the Newton method of Eq. (9). As expected the Newton method converges in a small number of iterations.

Table 1

j	$\| g(z^{(j)}) \|$
1	1.1921E-01
2	2.7930E-05
3	5.3938E-13
4	7.6919E-16
∞	2.2404E-16

[2] A GAUSS implementation of the algorithms can be obtained through email from the corresponding author.

[3] Calculations were done in GAUSS on a 486 computer.

However, if the matrix A has the form[4]

$$A = \begin{pmatrix} -0.5208 & 0.5999 & -0.4380 & -0.3014 & -0.0562 \\ 0.9405 & -1.7373 & -1.5401 & -2.1367 & -1.2417 \\ 0.1110 & -0.8929 & -0.5187 & -1.8992 & -2.1634 \\ 0.0058 & 0.9553 & -0.8661 & -0.7301 & 1.7773 \\ 1.0474 & 0.5714 & 0.3946 & 1.1376 & -0.4130 \end{pmatrix},$$

the Newton method of Eq. (9) will not converge. The eigenvalues of the Hessian matrix in (12) for the initial value $X = W$ are in descending order {430.343, 104.7753, 36.3720, 15.4705, 8.8303, 7.8615, 4.7562, 1.3750, 1.1323, 0.1360, −0.1124, −1.2031, −5 877, −24.2595, −384.9590}. Evidently, the Hessian matrix is not positive definite, which causes the nonconvergence. In this case we will need to apply the relaxation method of Eqs. (10)–(13) to get the solution

$$X = \begin{pmatrix} -3.0208 & -0.9591 & 2.1478 & 2.2488 & 0.9963 \\ -0.9591 & -0.8385 & 1.0214 & 0.6773 & 0.0265 \\ 2.1478 & 1.0214 & -2.4771 & -2.0446 & -1.1554 \\ 2.2488 & 0.6673 & -2.0446 & -2.3064 & -1.3623 \\ 0.9963 & 0.0265 & -1.1554 & -1.3623 & -1.5591 \end{pmatrix}.$$

4. Summary

In this note we have presented a fast numerical method for solving the algebraic Riccati matrix equation. In the example we investigated, the Newton method converged within a maximum of five iteration steps whatever the initial conditions. The merit of our method lies in the fact that it does not require additional assumptions for the underlying linear-quadratic control model and therefore is capable of solving a wide range of applications. Furthermore, the derivative information used in the method will generally ensure fast convergence.

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[4] This matrix was randomly selected.

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