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The Solution of Three-Dimensional, Composite Media Heat Conduction Problems by Synthesis Methods[1]

A method, known as synthesis, is applied to the task of obtaining approximate solutions to the static heat conduction equation for three-dimensional, composite media problems with mixed boundary conditions. The method is based upon an expansion in terms of known two-dimensional solutions of the problem of interest. These known two-dimensional solutions (trial functions) are blended over the remaining dimension by unknown mixing coefficients which are defined by means of variational techniques. A modified canonical variational principle is derived which permits the use of discontinuous trial functions, which expands the class of problems to which the synthesis method can be applied. The equations defining the mixing coefficients are derived in some detail, and the results of several test problems display the potential of this method for analyzing realistic heat conducting systems.

Introduction

The problem of solving the equilibrium heat conduction equation in three dimensions for composite media is indeed very difficult. Such problems usually arise in designing any heat producing mechanism, and when accurately solved permit the designer to describe the removal of heat from the device in an optimum fashion. In addition an accurate temperature distribution yields a detailed description of thermal stress sources in the device. Hence it is very desirable to know as much as possible about the spatial temperature distribution when designing realistic heat producing devices.

For three-dimensional composite media problems, such classical solution techniques as expansions in terms of orthogonal functions are not, in general, feasible; and since exact analytical solutions are certainly not attainable, one must resort to some approximation method. Quite often a designer obtains some knowledge of the three-dimensional temperature distribution by solving more tractable equivalent two-dimensional problems. However, this is not often possible or desirable if accurate tem-

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Nomenclature

T = temperature
k = thermal conductivity
S = heat source
$\mathcal{R}$ = volume of interest
$\mathcal{S}$ = external surface of $\mathcal{R}$
$\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ = connected segments of $\mathcal{S}$
H = trial function
Z = mixing coefficient
$\hat{n}$ = unit normal surface vector
T_s = surface temperature
∇ = gradient operator
h = convective surface coefficient
q_n = normal surface heat flow
N = number of trial functions
δ = variation operator
V_1, V_2 = variational functionals
V_1', V_2' = reduced functionals
$\mathcal{S}_i$ = internal surfaces on which k is discontinuous
Σ = summation symbol
$(+), (-)$ = denotes one-sided evaluation of a discontinuous function
$\langle , \rangle$ = volume integration symbol
$\{ , \}$ = surface integration symbol
a = x-y cross sectional area
c = contour of a
$\bar{z}$ = denotes axial dimension of system
$\mathfrak{z}_i$ = axial regions where k is continuous
L = axial size of test problem
R = radial size of test problem
deg F = degrees Fahrenheit
$\mathcal{L}$ = Lagrangian density
$\mathcal{U}$ = integral of surface term in functional
$\mathbf{p}$ = vector valued generalized momenta
p_x, p_y, p_z = components of $\mathbf{p}$
$\mathcal{H}$ = Hamiltonian density
$\bar{\mathcal{S}}_j$ = internal surfaces on which trial functions are discontinuous
$\mathcal{R}_j$ = volume regions in which trial functions are continuous
z_i = axial mesh points
Δ_i = axial mesh spacing
s_i, t_i = unit step functions
I = number of axial mesh points
M = number of discontinuous trial functions used
P^x, P^y, P^z = heat flow mixing coefficients
$(\tilde{\ })_i$ = denotes quantity averaged about mesh point z_1
$\delta_{i,0}; \delta_{i,I}$ = Kronecker delta function
B_i, C_i = terms in synthesis equations (see equation (40))

Subscripts

n = index of summation in synthesis expansion
i = axial mesh point indicator; internal surface and volume indicator
z = 0, $\bar{z}$ = evaluation on surface $z = 0$ on $z = \bar{z}$
j, k = components of matrix
$m(i)$ = index of summation about mesh point z_i

Superscripts

T = transpose of a matrix
-1 = inverse of a matrix
x, y, z = x, y, or z component

perature shapes are required. An obvious remedy is to superimpose a finite mesh structure on the domain of interest and obtain numerical solutions. This may be feasible if only a crude approximation is desired, but if an accurate temperature distribution is required, then a detailed mesh must be used. This can be a prohibitively expensive task even with today's fast digital computers.

This paper describes a method of constructing accurate numerical solutions for equilibrium composite media heat conduction problems in three dimensions. This method, which constructs three-dimensional solutions from known lesser dimension solutions of the problem of interest, is known as the synthesis approximation [1].[2] Synthesis techniques have been applied with much success to the solution of the neutron group diffusion problem and other problems arising in reactor theory [2].

The Synthesis Concept

The synthesis method is based on a "bracket-blend" idea which we now proceed to describe. Consider, for example, the problem of solving the heat conduction equation,

$$-\nabla \cdot k \nabla T = S \quad \text{in } R$$
$$T = 0 \quad \text{on } \mathcal{S} \tag{1}$$

for the heat conducting device shown in Fig. 1. In equation (1) $T = T(x, y, z)$ is the temperature, $k = k(x, y, z)$ is the thermal conductivity, and $S = S(x, y, z)$ is the heat source. R is the volume enclosing the device which is bounded by the surface $\mathcal{S}$. Let $H_1(x, y)$ and $H_2(x, y)$ be two-dimensional temperature distributions obtained for x-y slices through axial compositions 1 and 2, respectively. Now it is reasonable to assume that H_1 and H_2 are good approximations to the true x-y temperature shape in the central portion of axial compositions 1 and 2, respectively. In addition a linear combination of H_1 and H_2 should yield a good approximation to the x-y temperature shape near the interface $z = z'$. Then for this problem, we see that the trial functions H_1 and H_2 in some sense "bracket" the true x-y shape of the temperature. The three-dimensional temperature distribution is then synthesized as

$$T(x, y, z) \cong H_1(x, y)Z_1(z) + H_2(x, y)Z_2(z) \tag{2}$$

where the unknown mixing coefficients "blend" the trial functions H_1 and H_2. The basic difference between the synthesis approximation and usual expansion type approximations is that the synthesis trial functions are found directly from the problem of interest. Since these trial functions are so tailored to the specific problem to be solved, an accurate solution can be obtained by using only a few such functions. In general, the procedure to be followed is to subdivide the geometry of interest into "axial compositions," within which k and S are either independent of, or only slowly varying functions of z. Then we obtain an x-y temperature trial function for each different axial composition by solving the two-dimensional heat conduction equation for slices through each of these axial compositions. If there are N such axial compositions in the problem of interest we use the approximate form

$$T(x, y, z) \cong \sum_{n=1}^{N} H_n(x, y)Z_n(z) \tag{3}$$

where $H_n(x, y)$ is the known trial function found from axial composition n. For problems in which there is a drastic change in material in going from one axial composition to the next, one can include additional trial functions in the form (3) which may help in approximating the temperature shape near this interface. Such functions will be referred to as "transition trial functions." The unknown mixing coefficients, $Z_n(z)$, can be obtained by either of two methods. The first is the method of weighted residuals [3], for which the equations defining the mixing coefficients are found by substituting the approximate form (3) into the heat conduction equation (1), multiplying the resulting equation in turn by each of the trial functions and then integrating these N equations over the x-y cross section of the heat conducting system. This results in N coupled ordinary second order differential equations which define the mixing functions, $Z_n(z)$. The second way of finding the functions $Z_n(z)$ is to use the form (3) with an appropriate variational principle. The variational formulation will be used in this presentation since it is the most systematic method and because it will lead to certain generalizations which will extend the usefulness of the synthesis procedure. We shall refer to the approximation based on the expansion (3) as "continuous synthesis" since both the trial functions $H_n(x, y)$ and the mixing coefficients $Z_n(z)$ are required to be spatially continuous functions throughout the domain of interest.

[2] Numbers in brackets designate References at end of paper.

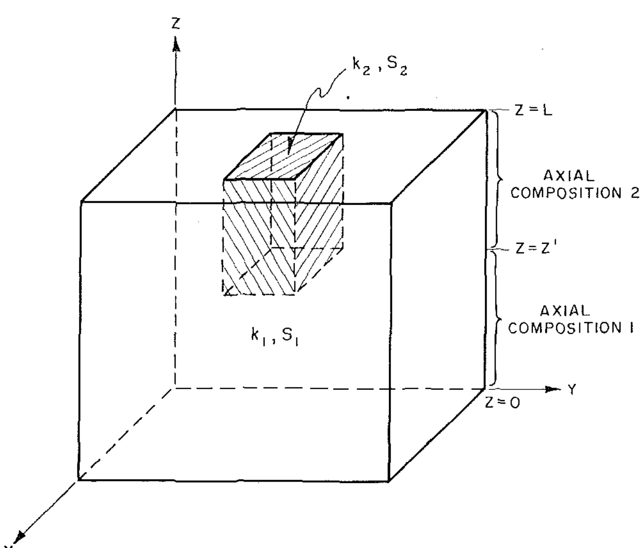

Fig. 1 Heat conducting geometry to illustrate the synthesis method

The Continuous Synthesis Method

We now focus our attention on obtaining approximate solutions to the heat conduction problem

$$\nabla \cdot k \nabla T + S = 0 \quad \text{in } \mathcal{R} \tag{4}$$

subject to the boundary conditions

$$k \nabla T \cdot \hat{n} = 0 \quad \text{on } \mathcal{S}_1, \tag{5a}$$

$$T = T_S \quad \text{on } \mathcal{S}_2, \tag{5b}$$

$$k \nabla T \cdot \hat{n} + h(T - T_S) = 0 \quad \text{on } \mathcal{S}_3, \tag{5c}$$

$$k \nabla T \cdot \hat{n} = q_n \quad \text{on } \mathcal{S}_4 \tag{5d}$$

where $\mathcal{R}$ is the volume enclosing the heat conducting system of interest, which is bounded by the external surface $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4$.

Within the framework of the continuous synthesis approximation, the temperature is approximated as

$$T(x, y, z) \cong \sum_{n=1}^{N} H_n(x, y)Z_n(z) \tag{6}$$

where the $H_n(x, y)$ are the predetermined trial functions which bracket the true solution in the sense that they are two-dimensional solutions representative of x-y slices through the heat conducting medium of interest at several z elevations where k and S are independent of, or only slowly varying functions of z. The $Z_n(z)$ are the unknown mixing coefficients which blend the trial functions in such a way as to obtain the best (in a variational sense) solution of the form (6). The variational principle to be used is

$$\delta V_1[T] = 0 \tag{7}$$

$$V_1[T] = \iiint_{\mathcal{R}} (\nabla T \cdot k \nabla T - 2TS) d\mathbf{r} + \oint\!\!\oint_{\mathcal{S}} [h(T - T_S)^2 - 2q_n T] d\mathcal{S}. \tag{8}$$

This principle admits functions $T(\mathbf{r})$, which are continuous with at least piecewise continuous first derivatives in $\mathcal{R}$, and has as its stationary conditions equations (4) and (5).

In order to simplify the algebra we define the matrix notation

$$Z(z) \equiv \begin{bmatrix} Z_1(z) \\ \cdot \\ \cdot \\ \cdot \\ Z_N(z) \end{bmatrix} \quad \text{and} \quad H(x, y) \equiv \begin{bmatrix} H_1(x, y) \\ \cdot \\ \cdot \\ \cdot \\ H_N(x, y) \end{bmatrix}.$$

Then if superscript T denotes a transpose of a matrix we may replace the synthesis expansion (6) by

$$T(x, y, z) \cong Z^T(z) H(x, y). \tag{9}$$

In addition we define the following integration notation

$$\langle A, OB \rangle \equiv \iint_a AOBdxdy$$

$$\langle OB \rangle \equiv \iint_a OBdxdy$$

where a is the x-y cross-sectional area of the system of interest, and

$$\{A, OB\} = \oint_c AOBdc$$

where c is the closed line contour bounding the x-y area a. The z dimension of the system of interest is defined by $0 \leq z \leq \bar{z}$.

We now substitute the synthesis form (9) into the functional (8) and perform the x-y integrations. This results in a reduced functional of Z, which when its first variation is required to vanish yields the stationary conditions

$$-\frac{d}{dz}\langle H, kH^T\rangle \frac{d}{dz} Z + [\langle \nabla H, k\nabla H^T\rangle + \{H, hH^T\}]Z = \langle SH\rangle + \{(hT_S + q_n)H\} \qquad \text{for } z \text{ in } \mathfrak{z}_i \tag{10}$$

subject to the boundary conditions

$$\begin{aligned} -\langle H, kH^T\rangle_{z=0} \frac{d}{dz} Z(0) + \langle H, hH^T\rangle_{z=0} Z(0) &= \langle (hT_S + q_n), H\rangle_{z=0} \\ \langle H, kH^T\rangle_{z=\bar{z}} \frac{d}{dz} Z(\bar{z}) + \langle H, hH^T\rangle_{z=\bar{z}} Z(\bar{z}) &= \langle (hT_S + q_n), H\rangle_{z=\bar{z}} \end{aligned} \tag{11}$$

and the continuity conditions

$$\left.\begin{aligned} \langle H, k(-)H^T\rangle \frac{d}{dz} Z(-) &= \langle H, k(+)H^T\rangle \frac{d}{dz} Z(+) \\ Z(-) &= Z(+) \end{aligned}\right\} \quad \text{at} \quad z = z_1, \tag{12}$$

where the z_i are those axial positions where k is discontinuous, and the $\mathfrak{z}_i$ denote the axial regions wherein k varies continuously with z.

Equation (10) is a set of $N \times N$ coupled differential equations, subject to the boundary conditions (11) and continuity conditions (12), which completely define the synthesis mixing coefficients, $Z_n(z)$.

In equation (10), $Z(z)$ is the column matrix of unknown mixing coefficients

$$Z(z) = \begin{bmatrix} Z_1(z) \\ Z_2(z) \\ \cdot \\ \cdot \\ \cdot \\ Z_N(z) \end{bmatrix}, \tag{13}$$

$\langle H, kH^T\rangle$ is a $N \times N$ matrix whose j, kth element is

$$\langle H, kH^T\rangle_{j,k} = \iint_a H_j(x, y) k(x, y, z) H_k(x, y) dxdy, \tag{14}$$

$\langle \nabla H, k\nabla H^T\rangle$ is a $N \times N$ matrix whose j, kth element is

$$\langle \nabla H, k\nabla H^T\rangle_{j,k} = \iint_a \nabla H_j(x, y) \cdot k(x, y, z) \nabla H_k(x, y) dxdy, \tag{15}$$

$\langle SH\rangle$ is an N element column matrix whose jth element is

$$\langle SH\rangle_j = \iint_a H_j(x, y) S(x, y, z) dxdy, \tag{16}$$

$\{H, hH^T\}$ is an $N \times N$ matrix whose j, kth element is

$$\{H, hH^T\}_{j,k} = \oint_c H_j(x, y) h(x, y, z) H_k(x, y) dc, \tag{17}$$

and $\{(hT_S + q_n)H\}$ is an N element column matrix whose jth element is

$$\{(hT_S + q_n)H\}_j = \oint_c H_j(x, y)[h(x, y, z)T_S(x, y, z) + q_n(x, y, z)]dc. \tag{18}$$

Continuous Synthesis Test Problems

In order to display the utility of the continuous synthesis method for obtaining accurate numerical solutions to multidimensional, composite media, heat conduction problems, we describe the results of two two-dimensional test problems. Both problems used the axially symmetric cylindrical heat conducting system described in Fig. 2. For these problems equations (10)–(12), in cylindrical geometry, were solved numerically, using a modified neutron diffusion theory synthesis program [4]. For purposes of comparison, an r-z finite difference solution [5] of the heat conduction equation was obtained for each test problem, utilizing the same mesh structure as was used for the solution of the synthesis equations (10)–(12). These pointwise temperatures were accurate to 0.001 deg F.

In all, six radial trial functions were considered in the synthesis solutions of test problems 1 and 2. Trial functions $H_1(r)$, $H_2(r)$ and $H_3(r)$ were one-dimensional finite difference solutions [6] of the heat conduction equation representing radial slices through axial compositions 1, 2, and 3, respectively. Trial functions $H_4(r)$ and $H_5(r)$ were transition type functions, which were one-dimensional finite difference solutions using material properties which are averages for axial compositions 1–2 and 2–3, respectively. These first five trial functions vanished at the radial boundary $r = R$. The final trial function, $H_6(r) = 1$ for $0 \leq r \leq R$.

Test Problem 1. This problem was concerned with approximating the temperature distribution for the system in Fig. 2, subject to the boundary conditions

$$\frac{\partial}{\partial z} T(r, z) = 0 \qquad \text{at} \quad z = 0, L,$$

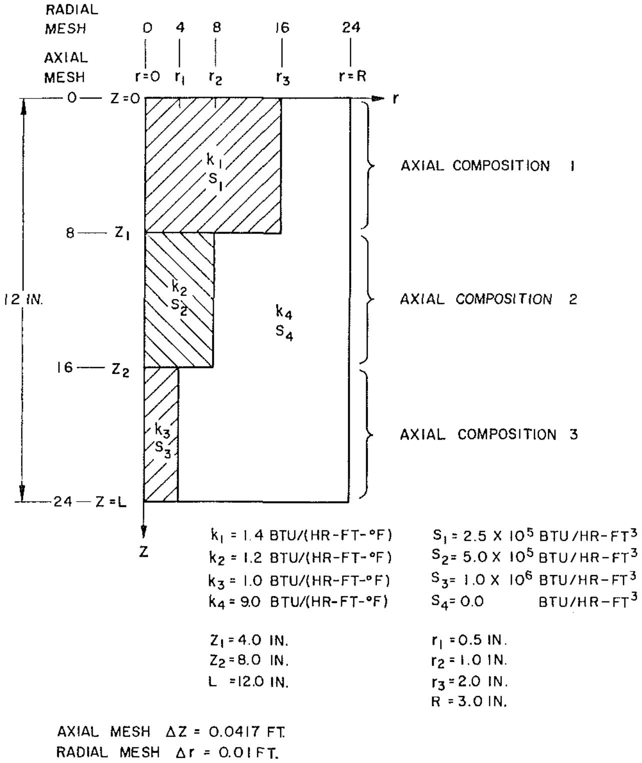

Fig. 2 Axially symmetric cylindrical conducting medium

$$T(R, z) = 80 \text{ deg F} \quad \text{for} \quad 0 < z < z_1,$$

$$T(R, z) = 100 \text{ deg F} \quad \text{for} \quad z_1 < z < z_2,$$

$$T(R, z) = 80 \text{ deg F} \quad \text{for} \quad z_2 < z < L.$$

The synthesis expansion was then constructed as

$$T(r, z) \cong \sum_{n=1}^{6} Z_n(z)H_n(r), \tag{19}$$

with $\frac{d}{dz} Z_n(z) = 0$, at $z = 0, L$. Note that the unit trial function, $H_6(r)$, was included to accommodate the nonzero boundary conditions at $r = R$. Fig. 3 presents a comparison of the synthesis solution (19) and the r-z finite difference solution for this problem.

Test Problem 2. The second test problem again considered the heat conducting system of Fig. 2. The boundary conditions for this problem included convective conditions over a portion of the external surface. Specifically the boundary conditions were

$$\frac{\partial}{\partial z} T(r, z) = 0 \qquad \text{at} \quad z = 0, L,$$

$$T(R, z) = 80 \text{ deg F} \quad \text{for} \quad 0 < z < z_1,$$

$$k \frac{\partial}{\partial r} T(R, z) + h[T(R, z) - T_S] = 0 \qquad \text{for} \quad z_1 < z < z_2,$$

where $h = T_S = 100$

$$T(R, z) = 80 \text{ deg F} \quad \text{for} \quad z_2 < z < L.$$

The synthesis solution was obtained using all six trial functions, as in (19). The resulting synthesized temperature distribution is shown in Fig. 4, along with the r-z finite difference solution of this problem at several radial points.

The preceding test problems show that the continuous synthesis method can yield very accurate approximate temperature distributions. However, this method does have some deficiencies. Consider the application of the continuous synthesis approximation to a heat conduction problem involving a material system with a great deal of complexity in the axial direction (i.e., a heat conducting system with many different axial compositions). In order to obtain an accurate solution for such a problem it may be necessary to use a large number of trial functions which, in turn, would result in a large, expensive computation effort. One would like to be able to switch sets of trial functions for problems of this type. That is, for z near the bottom of the system one would like to use only those trial functions relevant to the lower axial compositions. Then, as z increases, one would like, in stages, to discard the lower region trial functions and replace them with middle and upper region trial functions. Another way to say this is that we would like to make the trial functions $H_n(x, y)$ in equation (6) be discontinuous functions of z. Hence, if we can use trial functions which have a discontinuous z behavior, then we can synthesize heat conduction problems involving many axial material discontinuities by using only a few relevant trial functions at any specific axial position.

Another deficiency of the continuous heat conduction synthesis (which can be remedied by the use of axially discontinuous trial functions)[3] is that it cannot be applied to problems involving nonregular geometries for which the x-y boundaries change with z.

We have seen that the continuous synthesis procedure has certain deficiencies which can be circumvented if the synthesis expansion (6) is modified as

$$T(x, y, z) \cong \sum_{n=1}^{N} Z_n(z)H_n^{z}(x, y) \tag{20}$$

[3] Synthesis approximations, using x-y trial functions which are discontinuous functions of z, for a nonregular reactor geometry treated by two-group diffusion theory, have yielded very accurate results [1].

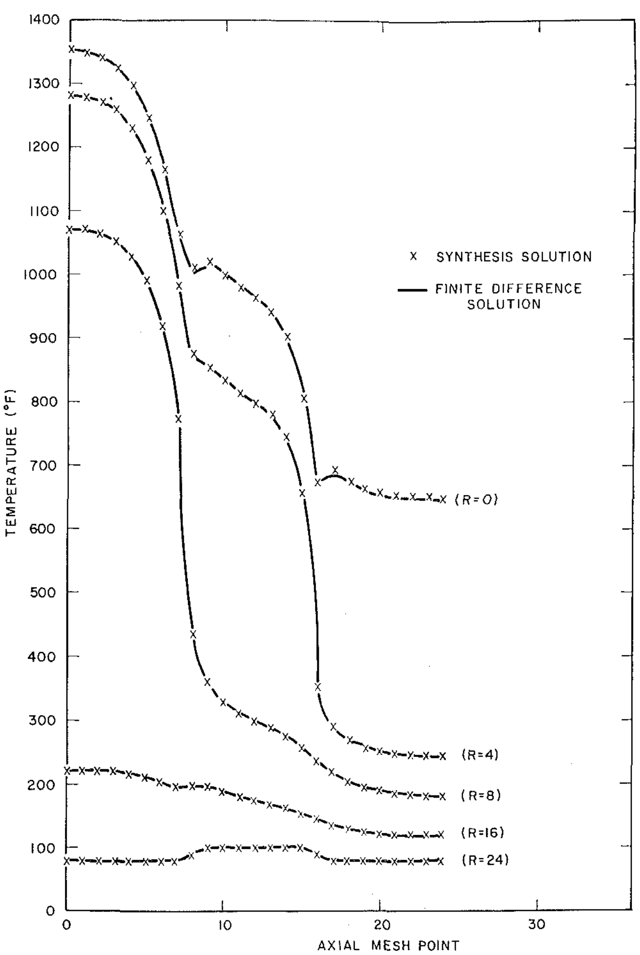

Fig. 3 Heat conduction synthesis test problem 1—axial temperature distributions

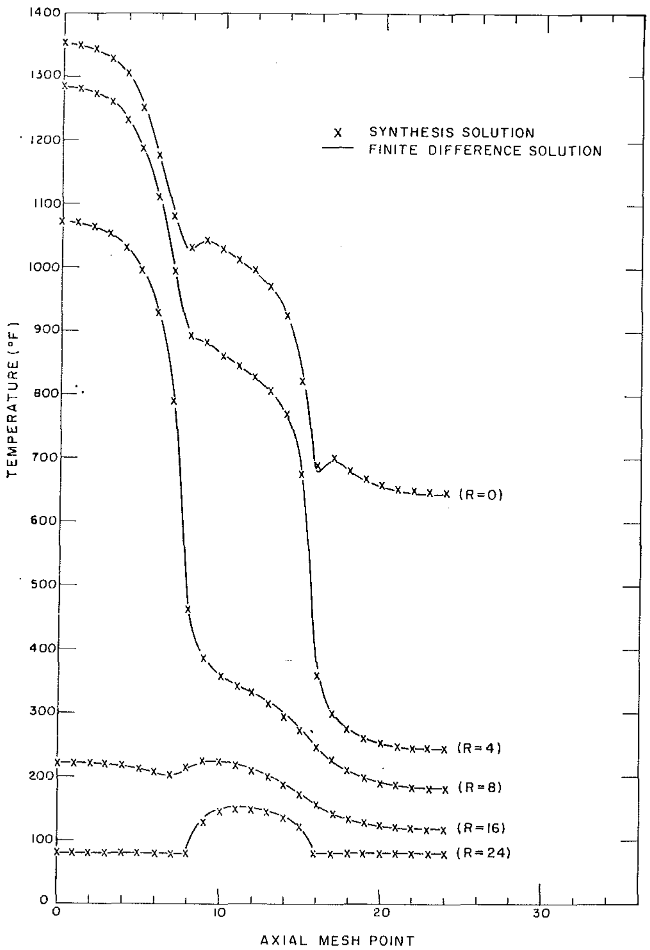

Fig. 4 Heat conduction synthesis test problem 2—axial temperature distributions

where the trial functions $H_n^z(x, y)$ are known x-y temperature distributions which apply only over specified regions of the z axis. That is, the H_n^z are to be discontinuous functions of z. As was previously mentioned, the variational principle, equations (7) and (8), admits only continuous approximating functions. Hence this principle cannot be used to develop a "discontinuous" heat conduction synthesis procedure using the axially discontinuous form (20). We now proceed to develop a discontinuous synthesis approximation for static heat conduction problems.

The Discontinuous Synthesis Method

We now concern ourselves with formulating a temperature synthesis approximation of the form

$$T(x, y, z) \cong \sum_{n=1}^{N} Z_n(z) H_n^z(x, y) \tag{21}$$

where the $H_n^z(x, y)$ are known x-y trial functions which are axially discontinuous. The discontinuous form (21) will permit one to synthesize a much larger class of static heat conduction problems than was possible within the continuous synthesis framework. However, using discontinuous trial functions complicates the problem of defining the mixing coefficients $Z_n(z)$. In order to develop a discontinuous heat conduction synthesis approximation, a variational functional must be used which admits discontinuous approximating functions. Such a functional may be obtained by transforming the functional $V_1[T]$, equation (8), into canonical form [8], and interpreting contributions due to the spatial discontinuities of the approximating functions. We now proceed to derive such a functional.

Let us write equation (8) as

$$V_1[T] = \iiint_{\mathcal{R}} \mathcal{L}(T, \nabla T, \mathbf{r}) d\mathbf{r} + \oint_{\mathcal{S}} U(T) d\mathcal{S}. \tag{22}$$

In analogy with classical mechanics we may look upon (22) as the functional of Hamilton's principle. Then we can identify $\mathcal{L}$ as the Lagrangian density

$$\mathcal{L}(T, \nabla T, \mathbf{r}) \equiv \nabla T \cdot k \nabla T - 2TS \tag{23}$$

and proceed to define generalized momenta as

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial(\nabla T)} \equiv 2k\nabla T. \tag{24}$$

Next we define a Hamiltonian density

$$\mathcal{H}(T, \mathbf{p}, \mathbf{r}) \equiv \mathbf{p} \cdot \nabla T - \mathcal{L}(T, \nabla T, \mathbf{r}) \tag{25}$$

where it is assumed that ∇T can be eliminated from (25) using (24). Such a substitution yields

$$\mathcal{H}(T, \mathbf{p}, \mathbf{r}) = \mathbf{p} \cdot \mathbf{p}/4k + 2TS. \tag{26}$$

From equations (25) and (26) we have

$$\mathcal{L} = \mathbf{p} \cdot \nabla T - \mathbf{p} \cdot \mathbf{p}/4k - 2TS \tag{27}$$

which, when used in (22), results in the canonical functional

$$V_2[\mathbf{p}, t] = \iiint_{\mathcal{R}} (\mathbf{p} \cdot \nabla T - \mathbf{p} \cdot \mathbf{p}/4k - 2TS) d\mathbf{r} + \oint_{\mathcal{S}} [h(T - T_S)^2 - 2q_n T] d\mathcal{S} \tag{28}$$

where we have set $U(T) = [h(T - T_S)^2 - 2q_n T]$. This functional does not admit the use of discontinuous functions, but it is of such a form that it can be augmented to permit $\mathbf{p}$ and T to possess a countable number of discontinuities. Such an extension requires an interpretation of the term $\iiint_{\mathcal{R}} \mathbf{p} \cdot \nabla T d\mathbf{r}$ at internal surfaces $\tilde{\mathcal{S}}_j$, on which T is discontinuous. Following the reasoning of Selengut and Wachspress [9] we interpret this term as

$$\iiint_{\mathcal{R}} \mathbf{p} \cdot \nabla T d\mathbf{r} = \sum_j \iiint_{\mathcal{R}_j} \mathbf{p} \cdot \nabla T d\mathbf{r} + \sum_j \oint_{\tilde{\mathcal{S}}_j} \left[\frac{\mathbf{p}(+) + \mathbf{p}(-)}{2}\right] [T(+) - T(-)] \cdot \hat{n} d\tilde{\mathcal{S}}_j \tag{29}$$

where the $\mathcal{R}_j$ are those regions within which T and $\mathbf{p}$ are continuous. $\hat{n}$ is a unit outward normal, and $(+)$ denotes quantities evaluated on the side of $\tilde{\mathcal{S}}_j$ toward which $\hat{n}$ points and $(-)$ denotes evaluation on the other side. Using the interpretation (29) in the functional (28) we arrive at

$$V_2[\mathbf{p}, T] = \sum_j \iiint_{\mathcal{R}_j} (\mathbf{p} \cdot \nabla T - \mathbf{p} \cdot \mathbf{p}/4k - 2TS) d\mathbf{r} + \oint_{\mathcal{S}} [h(T - T_S)^2 - 2q_n T] d\mathcal{S} + \sum_j \oint_{\tilde{\mathcal{S}}_j} \left[\frac{\mathbf{p}(+) + \mathbf{p}(-)}{2}\right] [T(+) - T(-)] \cdot \hat{n} d\tilde{\mathcal{S}}_j. \tag{30}$$

This functional now admits functions T and $\mathbf{p}$ which possess a countable number of discontinuities on the internal surfaces $\tilde{\mathcal{S}}_j$.

Those functions $T(x, y, z)$ and $\mathbf{p}(x, y, z)$ which make (30) stationary must satisfy

$$\delta V_2 \equiv V_2[\mathbf{p} + \delta\mathbf{p}, T + \delta T] - V_2[\mathbf{p}, T] = 0, \tag{31}$$

neglecting second order terms in δT and $\delta \mathbf{p}$. Using (30) and (31) we find that $V_2[\mathbf{p}, T]$ is stationary for those functions T and $\mathbf{p}$ which satisfy

$$\left.\begin{aligned} \frac{1}{2}\nabla\cdot\mathbf{p} + S &= 0 \\ \mathbf{p} &= 2k\nabla T \end{aligned}\right\} \text{ in } \mathcal{R}_j, \tag{32}$$

$$\frac{1}{2}\mathbf{p}\cdot\hat{n} + h(T - T_S) = q_n \text{ on } \mathcal{S}, \tag{33}$$

$$\left.\begin{aligned} T(+) &= T(-) \\ \mathbf{p}(+)\cdot\hat{n} &= \mathbf{p}(-)\cdot\hat{n} \end{aligned}\right\} \text{ on } \tilde{\mathcal{S}}_j. \tag{34}$$

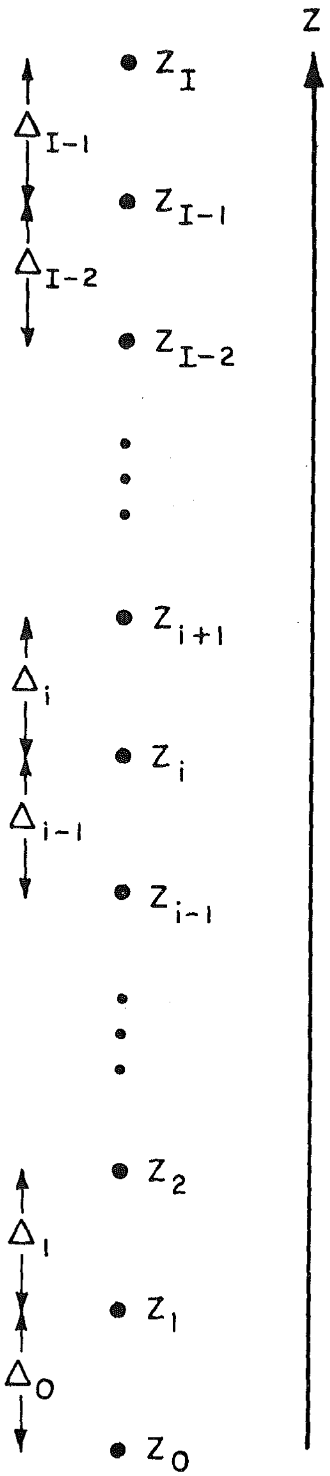

Fig. 5

Hence we see that the Euler equations (32) are just the heat conduction equation in first order form. Equation (33) is the general type boundary condition (5), and equations (34) require continuity of temperature and heat flux on all internal surfaces. Hence the functional (30) does have the desired stationary conditions to permit its use for developing a discontinuous synthesis approximation for static heat conduction problems. We now proceed to develop a synthesis approximation which permits the switching of sets of x-y trial functions at specified z locations.

We begin by superimposing a mesh structure on the z axis as is shown in Fig. 5. Material properties and surface functions are assumed constant between axial mesh points and denoted as $k(z) = k_i$, $S(z) = S_i$, $h(z) = h_i$ and $T_S(z) = T_{S_i}$, for $z_i < z < z_{i+1}$. Next we introduce unit step functions

$$s_i(z) = \begin{cases} 1, & \text{for} \quad z_i - \dfrac{\Delta_{i-1}}{2} < z < z_i + \dfrac{\Delta_i}{2} \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

where Δ_i is the mesh spacing between z_i and z_{i+1},

$$t_i(z) = \begin{cases} 1, & \text{for} \quad z_i < z < z_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

Let $H_{m(i)}(x, y)$ denote a set of continuous functions of x, y which we think may approximate the actual x-y temperature distribution in the vicinity of axial elevation z_i. Using the step functions $s_i(t)$ and $t_i(t)$, we proceed to expand the temperature and components of heat flow as

$$\begin{aligned} T(x, y, z) &\cong \sum_{i=0}^{I} \sum_{m(i)=1}^{M} s_i(z) Z_{m(i)} H_{m(i)}(x, y) \\ p_x(x, y, z) &\cong \sum_{i=0}^{I} \sum_{m(i)=1}^{M} s_i(z)[t_{i-1}(z)k_{i-1} \\ &\qquad + t_i(z)k_i] P_{m(i)}{}^x \frac{\partial}{\partial x} H_{m(i)}(x, y) \\ p_y(x, y, z) &\cong \sum_{i=0}^{I} \sum_{m(i)=1}^{M} s_i(z)[t_{i-1}(z)k_{i-1} \\ &\qquad + t_i(z)k_i] P_{m(i)}{}^y \frac{\partial}{\partial y} H_{m(i)}(x, y) \\ p_z(x, y, z) &\cong \sum_{i=0}^{I-1} \sum_{m(i)=1}^{M} t_i(z) k_i P_{m(i)}{}^z H_{m(i)}(x, y) \end{aligned} \tag{37}$$

where $t_{-1}(z) = 0$. Note here that we are free to choose different sets of trial functions, $H_{m(i)}$, to be used about any particular mesh point z_i. For simplicity we have restricted our approximation to use the same number, M, of trial functions for each mesh indicator i. In general we could specify different numbers of trial functions for different i. To simplify the algebra, we again introduce matrix notation. Let

$$\left.\begin{aligned} H_i &\equiv \begin{bmatrix} H_{1(i)}(x, y) \\ H_{2(i)}(x, y) \\ \vdots \\ H_{M(i)}(x, y) \end{bmatrix}, \quad Z_i \equiv \begin{bmatrix} Z_{1(i)} \\ Z_{2(i)} \\ \vdots \\ Z_{M(i)} \end{bmatrix}, \quad P_i{}^x \equiv \begin{bmatrix} P_{1(i)}{}^x \\ P_{2(i)}{}^x \\ \vdots \\ P_{M(i)}{}^x \end{bmatrix} \\ P_i{}^y &\equiv \begin{bmatrix} P_{1(i)}{}^y \\ P_{2(i)}{}^y \\ \vdots \\ P_{M(i)}{}^y \end{bmatrix}, \quad P_i{}^z \equiv \begin{bmatrix} P_{1(i)}{}^z \\ P_{2(i)}{}^z \\ \vdots \\ P_{M(i)}{}^z \end{bmatrix}. \end{aligned}\right\}$$

Then we may write (37) as

$$T(x, y, z) \cong \sum_{i=0}^{I} s_i Z_i{}^T H_i$$

$$p_x(x, y, z) \cong \sum_{i=0}^{I} s_i[t_{i-1}k_{i-1} + t_i k_i] P_i{}^{x^T} \frac{\partial}{\partial x} H_i \tag{38}$$

$$p_y(x, y, z) \cong \sum_{i=0}^{I} s_i[t_{i-1}k_{i-1} + t_ik_i]P_i^{y^T} \frac{\partial}{\partial y} H_i$$

$$p_z(x, y, z) \cong \sum_{i=0}^{I-1} t_ik_iP_i^{z^T}H_i$$

where superscript T denotes transpose of a matrix. We also invoke the integration notation used earlier. However, the cross-sectional area a and the contour c may now change with z if the x-y boundaries of the system of interest are functions of z. With this notation, we may substitute the approximation expansions (38) into the functional (30), observing that the $\tilde{s}_j$ consists only of horizontal surfaces midway between the mesh points z_i, and performing the x-y integrations we obtain the reduced functional $V'[Z_i, P_i^x, P_i^y, P_i^z]$. The stationary conditions for the reduced functional are found by requiring

$$\frac{\partial V_2'}{\partial Z_i} = \frac{\partial V_2'}{\partial P_i^x} = \frac{\partial V_2'}{\partial P_i^y} = \frac{\partial V_2'}{\partial P_i^z} = 0 \qquad (39)$$

for all i.

When we apply these stationary conditions, and eliminate the heat flow mixing coefficients P_i^x, P_i^y and P_i^z, we obtain the following set of three point difference equations which define the temperature mixing coefficients Z_i.

$$\langle \nabla H_i, \tilde{k}_i \nabla H_i^T \rangle Z_i - \left\langle H_i, \frac{k_{i-1}}{\Delta_{i-1}} H_{i-1}^T \right\rangle Z_{i-1}$$
$$- \left\langle H_i, \frac{k_i}{\Delta_i} H_{i+1}^T \right\rangle Z_{i+1} + \left[\left\langle H_i, \frac{k_i}{\Delta_i} H_i^T \right\rangle \right.$$
$$\left. + \left\langle H_i, \frac{k_{i-1}}{\Delta_{i-1}} H_{i-1}^T \right\rangle \langle H_{i-1}, k_{i-1}H_{i-1}^T\rangle^{-1} \langle H_{i-1}, k_{i-1}H_i^T \rangle \right] Z_i$$
$$+ B_iZ_i - C_i = \langle \tilde{S}_iH_i \rangle \quad \text{for} \quad i = 1, 2, \ldots, I \qquad (40)$$

where

$$B_i = \{H_i, \tilde{h}_iH_i^T\} + \langle H_0, h_0H_0^T\rangle\delta_{i,0} + \langle H_I, h_IH_I^T\rangle\delta_{i,I}$$

$$C_i = \{\tilde{h}T_{S_i}H_i\} + \{\tilde{q}_{n_i}H_i\} + [\langle T_{S_0}H_0\rangle + \langle q_{n_0}H_0\rangle]\delta_{i,0} + [\langle T_{S_I}H_I\rangle + \langle q_{n_I}H_I\rangle]\delta_{i,I},$$

$$\delta_{i,0} = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}; \quad \delta_{i,I} = \begin{cases} 1, & \text{if } i = I \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{k}_i = \frac{1}{2}(k_{i-1}\Delta_{i-1} + k_i\Delta_i)$$

$$\tilde{S}_i = \frac{1}{2}(S_{i-1}\Delta_{i-1} + S_i\Delta_i)$$

$$\tilde{h} = \frac{1}{2}(h_{i-1}\Delta_{i-1} + h_i\Delta_i)$$

$$\tilde{h}_iT_{S_i} = \frac{1}{2}(h_{i-1}T_{i-1}\Delta_{i-1} + h_iT_i\Delta_i).$$

It is interesting to note that if the same set of trial functions is used about mesh point z_{i-1} as is used about mesh point z_i, then the term $\langle H_{i-1}, k_{i-1}H_{i-1}^T\rangle^{-1}\langle H_{i-1}, k_{i-1}H_i^T\rangle$ is just the identity matrix, since $H_i = H_{i-1}$. In addition if the same set of trial functions is used for $0 \leq z_i \leq z_I$, then the equations (40) are just a finite difference form of the continuous synthesis equations (10)–(12). An ALGOL computer program was written to solve the synthesis equations (40). We now proceed to describe a test problem using axially discontinuous trial functions.

Discontinuous Synthesis Test Problem

The symmetric cylindrical heat conducting system considered for this problem is described in Fig. 6. Zero temperature boundary conditions were applied on all external surfaces. Four one-dimensional trial functions were considered for this problem, although only two functions were used at any z_i elevation. The computer program, WANDA [6], was used to calculate these trial functions and all necessary matrix elements. The trial functions; $H_1(r)$, $H_2(r)$, $H_3(r)$, $H_4(r)$ are one-dimensional solutions for radial slices through axial compositions 1, 2, 3, and 4, respectively, of the geometry depicted in Fig. 6. Table 1 describes the axial mesh regions over which each of these trial functions were used. We see that the trial functions used were discontinuous between axial mesh points 9–10, 24–25, and 37–38. Using these trial functions, the difference equations (40) were solved and the synthesized r-z temperature distribution was constructed. This synthesized solution is compared with an r-z finite difference solution [5] (accurate to 0.001 deg F) of this same problem in Fig. 7. From this figure we see that the synthesis solution is quite accurate, and most impressively, smoothly varying near the axial points where the trial functions are discontinuous.

Table 1 Axial dependence of discontinuous trial functions

Trial function	Used over z-Mesh points
$H_1(r)$, $H_2(r)$	$i = 0$ to $i = 9$
$H_2(r)$, $H_3(r)$	$i = 10$ to $i = 24$
$H_3(r)$, $H_4(r)$	$i = 25$ to $i = 37$
$H_4(r)$, $H_1(r)$	$i = 38$ to $i = 48$

Conclusion

In this paper variational techniques have been applied to develop continuous and discontinuous synthesis approximations for static composite media heat conduction problems. The basic idea behind these methods is that of using known two-dimensional trial functions representative of asymptotic solutions for axial regions of constant material properties. These trial functions, tailored to the problem of interest, are blended axially by means of z dependent mixing coefficients to yield the synthesized three-dimensional temperature distribution.

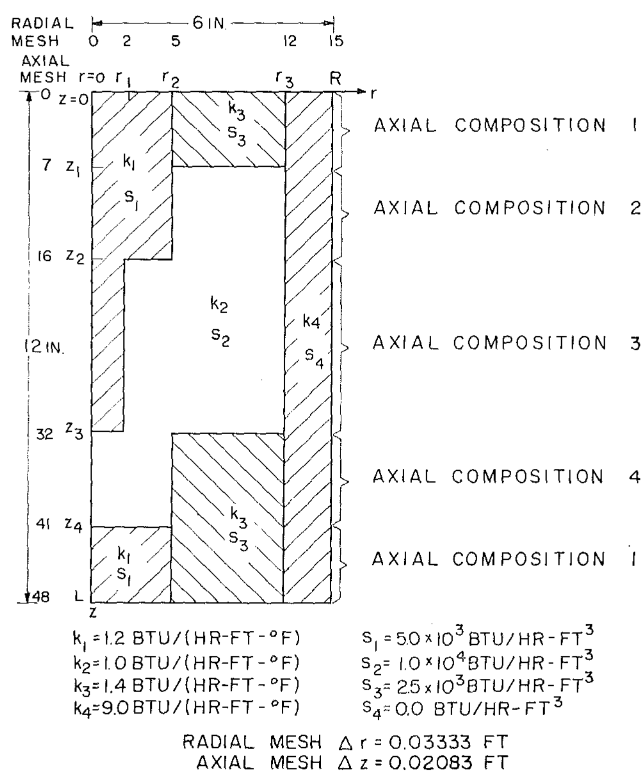

Fig. 6 Cylindrical conducting medium for discontinuous synthesis test problem

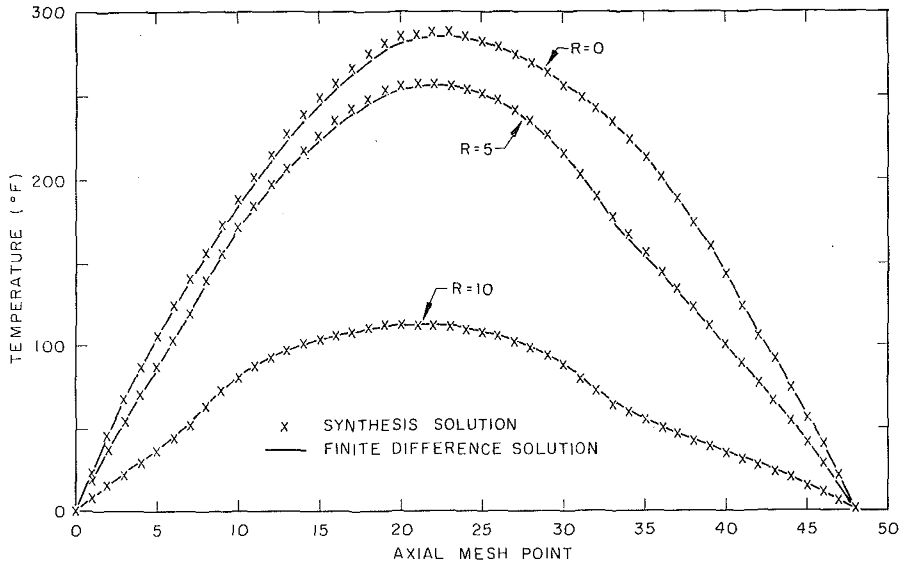

Fig. 7 Discontinuous heat conduction synthesis test problem—axial temperature distributions

The results of several two-dimensional test problems indicate the accuracy that can be obtained with the method. However, the advantage of using the synthesis methods, as opposed to finite difference methods, manifests itself when detailed three-dimensional problems must be solved. For such problems (possibly involving millions of spatial mesh points) the solution of the three-dimensional finite difference equations would be extremely expensive even on today's fast digital computers, while the synthesis method requires the solution of two-dimensional problems (to obtain the trial functions) and one one-dimensional solution (for the mixing coefficients). One- and two-dimensional numerical solutions are very inexpensive.

The ability to use axially discontinuous trial functions (switch sets of trial functions along the z axis) permits one to solve heat conduction problems involving irregular geometries. Hence synthesis methods can prove to be powerful tools for analyzing realistic heat conducting designs.

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