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BLOW UP FOR THE WAVE EQUATION WITH A FRACTIONAL DAMPING

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Abstract. We consider the wave equation with a fractional damping of order between 0 and 1 and a polynomial source. Introducing a new functional and using an argument due to Georgiev and Todorova [1] together with some appropriate estimates, it is proved that some solutions blow up in finite time.

1. Introduction

We are concerned by the following integro-differential problem

$$\begin{cases} u_{tt} + \partial_t^{1+\alpha} u = \Delta u + a\,|u|^{p-1}\,u, & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given. The constants a, p and α are such that $a > 0$, $p > 1$ and $-1 < \alpha < 1$. The notation $\partial_t^{1+\alpha}$ stands for the Caputo's fractional derivative of order $1+\alpha$ with respect to the time variable (see [6]). It is defined as follows

$$\partial_t^{1+\alpha} w(t) := I^{-\alpha} \frac{d}{dt} w(t) \quad \text{for } -1 < \alpha < 0$$

and

$$\partial_t^{1+\alpha} w(t) := I^{1-\alpha} \frac{d^2}{dt^2} w(t) \quad \text{for } 0 < \alpha < 1,$$

where I^β, $\beta > 0$ is the fractional integral

$$I^\beta w(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s) ds.$$

See also [5] and [7] for more on fractional integrals and derivatives.

Fractional integrals and derivatives are used to describe memory and hereditary properties of various materials and processes. They have wide applications in physics, chemistry, biology, ecology, ... etc., see [5]–[7] and references therein.

The present problem (with $a = 0$) has been studied by Matignon *et al.* in [3]. The authors proved some results on well posedness and asymptotic stability. It should be noted here that the direct application of the existing methods for this case poses some difficulties due to the dependence of the solution on the whole past history and the nature of the kernel in the time convolution term. This kernel is not only singular but also non-integrable on $(0, +\infty)$. To get rid of this nonlocal term, the authors managed to transform the problem from a hereditary problem to a non-hereditary one and then used the standard methods available in the literature.

This paper is a continuation of earlier works by the second author (see [2], [8], [9]). In [2], we proved (with M. Kirane) an exponential growth result (for any power of the polynomial source) provided that the initial data are *sufficiently large*, in fact, for *very large* initial data. Then this result has been improved, using a new argument, to a wide set of initial data in [8]. In [9], using an argument involving Fourier transforms and the Hardy-Littlewood-Sobolev inequality, we proved a finite time blow up of solutions. Namely, the result was that, for any fixed time $T > 0$, there exist $0 < T^* \leq T$ and *sufficiently large* initial data (depending on T) for which u blows up at T^*. One of the main difficulties encountered is that the "energy" functional is not necessarily decreasing (in time). Indeed, the derivative of the "energy" associated to the problem is of an undefined sign. Here we shall prove finite time blow up without the *dependence* of the initial data on the time variable T. For this goal, we introduce a new functional which "controls" some "undesirable" terms that appear while using the Georgiev and Todorova [1] argument (see also Messaoudi [4]). We mention here that, combining ideas of Matignon *et al.* [3] and Georgiev and Todorova [1], we obtain the following local existence result

Theorem. *Suppose that* $p > 1$ *if* $N = 1$ *or* 2 *and* $p \leq N/(N-2)$ *if* $N \geq 3$. *For every initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there is* $T > 0$ *and a unique weak solution* $(u(t), v(t))$ *of* (1) *such that* $u \in C\left([0,T); H_0^1(\Omega)\right) \cap C^1\left([0,T); L^2(\Omega)\right)$ *and* $v \equiv u_t \in L^2\left((0,T) \times \Omega\right)$.

The paper is organized as follows:

In the next section, we prepare some material that we shall need to prove our result. Section 3 contains the statement and proof of our result.

2. Preliminaries

In this section we present some definitions and introduce some functionals needed to prove our theorem. In addition, we prove a crucial proposition for our result. Without loss of generality, we will take $a = 1$ in the whole paper. Furthermore we will consider only the case $-1 < \alpha < 0$. The classical energy functional associated to problem (1) is

$$E(t) := \int_\Omega \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right\} dx. \tag{2}$$

Clearly,

$$\frac{dE(t)}{dt} = -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\, ds\, dx.$$

Note that this derivative is of an undetermined sign. Let us define the "modified" energy functional by

$$E_\varepsilon(t) := E(t) - \varepsilon \int_\Omega u_t u\, dx \tag{3}$$

for some $0 < \varepsilon < 1$ to be determined later. Then a differentiation of (3) gives

$$E_\varepsilon'(t) = -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s)\, ds\, dx - \varepsilon \int_\Omega u_t^2\, dx + \varepsilon \int_\Omega |\nabla u|^2\, dx$$

$$+\frac{\varepsilon}{\Gamma(-\alpha)}\int_{\Omega} u \int_{0}^{t} (t-s)^{-(\alpha+1)} u_s(s)dsdx - \varepsilon \int_{\Omega} |u|^{p+1} dx. \tag{4}$$

Note here (see the theorem in the introduction) that the solutions are weak, that is

$$\frac{d}{dt}(v(t), w)_2 + \left(\partial_t^{\alpha+1} u(t), w\right)_2 + (\nabla u(t), \nabla w)_2 = \left(|u|^{p-1} u(t), w\right)_2$$

a.e. in $(0, T)$ and every $w \in H_0^1(\Omega)$. Therefore, since we are dealing with the energy of the system and the multiplier technique, formula (4) and all the computation below are justified. In particular, as $u_t \in L^2((0,T) \times \Omega)$, the scalar product of $\partial_t^{\alpha+1} u(t)$ with any $w \in L^2(\Omega)$ is well defined. Next, we introduce the functional

$$H(t) := -\left(e^{-\sigma\varepsilon t} E_\varepsilon(t) + \mu F(t) + d\right) \tag{5}$$

where

$$F(t) := \int_{0}^{t} \int_{\Omega} G(t-s) e^{-\sigma\varepsilon s} u_s^2 dxds \tag{6}$$

with

$$G(t) := e^{\beta t} \int_{t}^{+\infty} e^{-\beta s} s^{-(\alpha+1)} ds. \tag{7}$$

Here, $\sigma = (p+1)/2$ and d, μ, β are positive constants that will be precised below.

Proposition 1. *If $E_\varepsilon(0) < 0$ and p is sufficiently large, then $H(t) > 0$ and $H'(t) > 0$.*

Proof. A differentiation of (4) with respect to t yields

$$H'(t) = \sigma\varepsilon e^{-\sigma\varepsilon t} E_\varepsilon(t) - e^{-\sigma\varepsilon t} E_\varepsilon'(t) - \mu F'(t).$$

Taking into account the definitions (2) and (3), the relation

$$\begin{aligned} F'(t) =& \beta^\alpha \Gamma(-\alpha) e^{-\sigma\varepsilon t} \int_{\Omega} u_t^2 dx + \beta F(t) \\ & - \int_{0}^{t} \int_{\Omega} (t-s)^{-(\alpha+1)} e^{-\sigma\varepsilon s} u_s^2 dxds \end{aligned}$$

and (4), we get

$$
\begin{aligned}
H'(t) &= \left[\frac{\sigma\varepsilon}{2} + \varepsilon - \mu\beta^{\alpha}\Gamma(-\alpha)\right] e^{-\sigma\varepsilon t} \int_{\Omega} u_t^2 dx - \sigma\varepsilon^2 e^{-\sigma\varepsilon t} \int_{\Omega} u_t u dx \\
&+ \left(\frac{\sigma\varepsilon}{2} - \varepsilon\right) e^{-\sigma\varepsilon t} \int_{\Omega} |\nabla u|^2 dx + \left(\varepsilon - \frac{\sigma\varepsilon}{p+1}\right) e^{-\sigma\varepsilon t} \int_{\Omega} |u|^{p+1} dx \\
&+ \frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)} \int_{\Omega} u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx \\
&- \frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)} \int_{\Omega} u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx \\
&+ \mu \int_0^t \int_{\Omega} (t-s)^{-(\alpha+1)} e^{-\sigma\varepsilon s} u_s^2 dx ds - \mu\beta F(t). \qquad (8)
\end{aligned}
$$

By the Young inequality, we see that

$$
\begin{aligned}
&\int_{\Omega} u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx \\
&\leq \delta_1 \int_{\Omega} u_t^2 dx + \frac{1}{4\delta_1} \int_{\Omega} \left(\int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds \right)^2 dx, \quad \delta_1 > 0.
\end{aligned}
$$

Writing $-(\alpha+1) = -(\alpha+1)/2 - (\alpha+1)/2$ and using the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
&e^{-\sigma\varepsilon t} \int_{\Omega} u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx \qquad (9) \\
&\leq \delta_1 e^{-\sigma\varepsilon t} \int_{\Omega} u_t^2 dx + \frac{(\sigma\varepsilon)^{\alpha}\Gamma(-\alpha)}{4\delta_1} \int_{\Omega} \int_0^t (t-s)^{-(\alpha+1)} e^{-\sigma\varepsilon s} u_s^2 ds dx.
\end{aligned}
$$

Similarly, with the help of the Poincaré inequality, we obtain

$$
e^{-\sigma\varepsilon t} \int_{\Omega} u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx \leq \delta_2 C_p e^{-\sigma\varepsilon t} \int_{\Omega} |\nabla u|^2 dx
$$

$$+\frac{(\sigma\varepsilon)^{\alpha}\Gamma(-\alpha)}{4\delta_2}\int_{\Omega}\int_0^t (t-s)^{-(\alpha+1)}e^{-\sigma\varepsilon s}u_s^2 ds dx,\ \delta_2>0 \tag{10}$$

where C_p is the Poincaré constant. Now from (9), (10) and

$$\int_{\Omega} u_t u dx \leq \delta_3 C_p \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\delta_3}\int_{\Omega} u_t^2 dx\ \delta_3>0 \tag{11}$$

we infer from (8) that

$$\begin{aligned} H^{'}(t) \geq & \left[\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{\alpha}\Gamma(-\alpha)-\frac{\sigma\varepsilon^2}{4\delta_3}-\frac{\delta_1}{\Gamma(-\alpha)}\right]e^{-\sigma\varepsilon t}\int_{\Omega} u_t^2 dx \\ & +\left[\frac{\sigma\varepsilon}{2}-\varepsilon-\sigma\varepsilon^2\delta_3 C_p-\frac{\varepsilon\delta_2 C_p}{\Gamma(-\alpha)}\right]e^{-\sigma\varepsilon t}\int_{\Omega} |\nabla u|^2 dx \\ & +\varepsilon\left(1-\frac{\sigma}{p+1}\right)e^{-\sigma\varepsilon t}\int_{\Omega}|u|^{p+1}dx-\mu\beta F(t) \\ & +\left[\mu-\frac{(\sigma\varepsilon)^{\alpha}}{4\delta_1}-\frac{\varepsilon(\sigma\varepsilon)^{\alpha}}{4\delta_2}\right]\int_0^t\int_{\Omega}(t-s)^{-(\alpha+1)}e^{-\sigma\varepsilon s}u_s^2 dx ds. \end{aligned} \tag{12}$$

Adding and substracting $C_1H(t)$ to the right hand side of (12) we obtain

$$\begin{aligned} H^{'}(t)\geq & C_1H(t)+\left[\frac{C_1}{2}+\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{\alpha}\Gamma(-\alpha)-\frac{\sigma\varepsilon^2}{4\delta_3}-\frac{\delta_1}{\Gamma(-\alpha)}\right] \\ & \times e^{-\sigma\varepsilon t}\int_{\Omega} u_t^2 dx+\left[\frac{C_1}{2}+\frac{\sigma\varepsilon}{2}-\varepsilon-\sigma\varepsilon^2\delta_3 C_p-\frac{\varepsilon\delta_2 C_p}{\Gamma(-\alpha)}\right]e^{-\sigma\varepsilon t}\int_{\Omega}|\nabla u|^2 dx \\ & -C_1\varepsilon e^{-\sigma\varepsilon t}\int_{\Omega} u_t u dx+\left[\varepsilon\left(1-\frac{\sigma}{p+1}\right)-\frac{C_1}{p+1}\right]e^{-\sigma\varepsilon t}\int_{\Omega}|u|^{p+1}dx+C_1 d \\ & +(C_1-\beta)\mu F(t)+\left[\mu-\frac{(\sigma\varepsilon)^{\alpha}}{4}\left(\frac{1}{\delta_1}+\frac{\varepsilon}{\delta_2}\right)\right]\int_0^t\int_{\Omega}(t-s)^{-(\alpha+1)}e^{-\sigma\varepsilon s}u_s^2 dx ds. \end{aligned} \tag{13}$$

We apply inequality (11) to the fourth term in the right hand side of (13) and choose $C_1=(p+1)/2\varepsilon$, $\delta_1=\delta_2=\Gamma(-\alpha)\varepsilon/2$ and $\delta_3=1/2$ to get

$$\begin{aligned} H^{'}(t)\geq & \frac{p+1}{2}\varepsilon H(t)+\left[\frac{\varepsilon}{2}-\mu\beta^{\alpha}\Gamma(-\alpha)\right]e^{-\sigma\varepsilon t}\int_{\Omega}u_t^2 dx \\ & +\frac{\varepsilon}{2}\left[p-1-(p+2)C_p\varepsilon\right]e^{-\sigma\varepsilon t}\int_{\Omega}|\nabla u|^2 dx+\left(\frac{p+1}{2}\varepsilon-\beta\right)\mu F(t) \end{aligned}$$

$$+\left[\mu-\frac{(p+1)^{\alpha}\varepsilon^{\alpha}}{2^{1+\alpha}}\left(\frac{1}{\varepsilon}+1\right)\right]\int_{0}^{t}\int_{\Omega}(t-s)^{-(\alpha+1)}e^{-\sigma\varepsilon s}u_s^2dxds.$$

Selecting

$$\varepsilon<\varepsilon_1:=\min\left(1,\frac{p-1}{2(p+2)C_p}\right)$$

we see that the coefficient of the third term in the right hand side of the previous relation is greater than $(p-1)\varepsilon/4$. Observe that if $C_p<1/2$ and $p\geq\frac{1+4C_p}{1-2C_p}$, then $(p-1)/[2(p+2)C_p]\geq 1$ and this condition reduces to $\varepsilon<1$ which is already in the definition (3). Next, putting $\beta=1$ and assuming that $p+1\geq(2/\varepsilon)^{1-2/\alpha}$, it appears that the fourth coefficient is nonnegative. If $C_p\geq 1/2$, then $(p-1)[2(p+2)C_p]<1$ and we may easily check that both assumptions are satisfied for sufficiently larges values of p. Moreover, we can choose μ so that the second coefficient is nonnegative and the last coefficient is greater than $(p+1)^{\alpha}/\left[2^{1+\alpha}\Gamma(-\alpha)\varepsilon^{1-\alpha}\right]$. Consequently, we find

$$H^{'}(t)\geq\frac{p+1}{2}\varepsilon H(t)+\frac{p-1}{4}\varepsilon e^{-\sigma\varepsilon t}\int_{\Omega}|\nabla u|^2\,dx \tag{14}$$

$$+\frac{(p+1)^{\alpha}}{2^{1+\alpha}\Gamma(-\alpha)\varepsilon^{1-\alpha}}\int_{0}^{t}\int_{\Omega}(t-s)^{-(\alpha+1)}e^{-\sigma\varepsilon s}u_s^2dxds.$$

This relation will be of great help in the proof of the result in the next section. If we select $d<-E_{\varepsilon}(0)$, then $H(0)>0$. Observe then, as a consequence of (14), that $H(t)>0$ and $H^{'}(t)>0$. □

3. The theorem

In this section, we prove our theorem which ensures blow up of solutions provided that the initial energy is negative.

Theorem 1. *Assume that $-1<\alpha<0$, $E(0)<0$ and $\int_{\Omega}u_1u_0dx\geq 0$. Then the solutions of* (1) *blow up in finite time for sufficiently large values of p.*

Proof. Let us define

$$Q(t):=H^{1-\gamma}(t)+be^{-\sigma\varepsilon t}\int_{\Omega}u_tudx$$

where $\gamma = p - 1/[2(p+1)]$ and b is a positive constant to be determined. A differentiation of $Q(t)$ with respect to t yields

$$Q^{'}(t) = (1-\gamma)H^{-\gamma}(t)H'(t) - b\sigma\varepsilon e^{-\sigma\varepsilon t}\int_\Omega u_t u dx + be^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx$$

$$+be^{-\sigma\varepsilon t}\left\{-\int_\Omega |\nabla u|^2 dx - \frac{1}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_s(s)dsdx + \int_\Omega |u|^{p+1} dx\right\}.$$

Using (11) and (10) (without the Poincaré inequality for the first term) with the constants $\delta_4 > 0$ and $\delta_5 > 0$ respectively, we find

$$\begin{aligned} Q^{'}(t) &\geq (1-\gamma)H^{-\gamma}(t)H'(t) - b\left(1+\delta_4 C_p\sigma\varepsilon\right)e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ &+ b\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx + be^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1} dx - \frac{b\delta_5}{\Gamma(-\alpha)}e^{-\sigma\varepsilon t}\int_\Omega u^2 dx \\ &- \frac{b2^{\alpha-1}\sigma^\alpha\Gamma(-\alpha)\varepsilon}{\delta_5(p+1)^\alpha}H'(t) + \frac{b\sigma^\alpha 2^{\alpha-2}\Gamma(-\alpha)\varepsilon^2}{\delta_5(p+1)^{\alpha-1}}H(t) \\ &+ \frac{b2^{\alpha-3}\sigma^\alpha\Gamma(-\alpha)(p-1)\varepsilon^2}{\delta_5(p+1)^\alpha}e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{15}$$

In the last relation (15) we also made use of the inequality (14). Then

$$\begin{aligned} Q^{'}(t) &\geq \left[(1-\gamma)H^{-\gamma}(t) - \frac{b2^{\alpha-1}\sigma^\alpha\Gamma(-\alpha)\varepsilon}{\delta_5(p+1)^\alpha}\right]H'(t) + \frac{b\sigma^\alpha 2^{\alpha-2}\Gamma(-\alpha)\varepsilon^2}{\delta_5(p+1)^{\alpha-1}}H(t) \\ &- b\left[1+\delta_4 C_p\sigma\varepsilon - \frac{2^{\alpha-3}\sigma^\alpha\Gamma(-\alpha)(p-1)\varepsilon^2}{\delta_5(p+1)^\alpha}\right]e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ &+ b\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)e^{-\sigma\varepsilon t}\int_\Omega u_t^2 dx \\ &+ be^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1} dx - \frac{b\delta_5}{\Gamma(-\alpha)}e^{-\sigma\varepsilon t}\int_\Omega u^2 dx. \end{aligned} \tag{16}$$

Next we pick $\delta_5 = MH^\gamma(t)$ and add and substract the same term $H(t)$ to both sides of (16) to obtain

$$\begin{aligned} Q^{'}(t) &\geq \left[(1-\gamma) - \frac{b2^{\alpha-1}\sigma^\alpha\Gamma(-\alpha)\varepsilon}{M(p+1)^\alpha}\right]H^{-\gamma}(t)H'(t) \\ &+ \left[1+\frac{b\sigma^\alpha 2^{\alpha-2}\Gamma(-\alpha)\varepsilon^2}{\delta_5(p+1)^{\alpha-1}}\right]H(t) \end{aligned}$$

$$+\left[\frac{1}{2}+\frac{b2^{\alpha-3}\sigma^{\alpha}\Gamma(-\alpha)(p-1)\varepsilon^{2}}{M(p+1)^{\alpha}}H^{-\gamma}(t)-b(1+\delta_4 C_p\sigma\varepsilon)\right]e^{-\sigma\varepsilon t}\int\limits_{\Omega}|\nabla u|^{2}\,dx$$
$$+\mu F(t)+\left[\frac{1}{2}+b\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)\right]e^{-\sigma\varepsilon t}\int\limits_{\Omega}u_t^2dx-\varepsilon e^{-\sigma\varepsilon t}\int\limits_{\Omega}u_t u dx+d$$
$$+\left(b-\frac{1}{p+1}\right)e^{-\sigma\varepsilon t}\int\limits_{\Omega}|u|^{p+1}\,dx-\frac{bM}{\Gamma(-\alpha)}e^{-\sigma\varepsilon t}H^{\gamma}(t)\int\limits_{\Omega}u^2dx. \tag{17}$$

The term $\int_{\Omega}u_t u dx$ is estimated by the Young inequality as in (11). As for the term $H^{\gamma}(t)\int_{\Omega}|u|^2\,dx$ we adopt the estimation,

$$H^{\gamma}(t)\int\limits_{\Omega}|u|^{2}\,dxds\le\frac{C_2}{(p+1)^{\gamma}}\left(1+\int\limits_{\Omega}|u|^{p+1}\,dx\right).$$

This follows from the definition (5) of $H(t)$ and the fact that

$$\varepsilon<\varepsilon_1:=\min\left(1,\frac{p-1}{2(p+2)C_p}\right).$$

Indeed, from (5) we have

$$H(t)\le\frac{1}{p+1}\int\limits_{\Omega}|u|^{p+1}\,dx+\left(\varepsilon\int\limits_{\Omega}u_t u dx-\frac{1}{2}\int\limits_{\Omega}|u_t|^{2}\,dx-\frac{1}{2}\int\limits_{\Omega}|\nabla u|^{2}\,dx\right).$$

As $\varepsilon<\min(1,(p-1)/[2(p+2)C_p])$ (in fact, here we need $\varepsilon<\min(1,1/C_p)$) and using inequality (11) with $\delta_3=1/2$ we find $H(t)\le 1/(p+1)\int_{\Omega}|u|^{p+1}\,dx$. By Hölder's inequality we see that

$$H^{\gamma}(t)\int\limits_{\Omega}|u|^{2}\,dxds\le\frac{C_2}{(p+1)^{\gamma}}\left(\int\limits_{\Omega}|u|^{p+1}\,dx\right)^{\gamma}\left(\int\limits_{\Omega}|u|^{p+1}\,dx\right)^{2/(p+1)}$$

or

$$H^{\gamma}(t)\int\limits_{\Omega}|u|^{2}\,dxds\le\frac{C_2}{(p+1)^{\gamma}}\left(\int\limits_{\Omega}|u|^{p+1}\,dx\right)^{\gamma+2/(p+1)},$$

for some positive constant C_2. As $\gamma+2/(p+1)=(p+3)/[2(p+1)]<1$, it is clear that

$$H^{\gamma}(t)\int\limits_{\Omega}|u|^{2}\,dxds\le\frac{C_2}{(p+1)^{\gamma}}\left(1+\int\limits_{\Omega}|u|^{p+1}\,dx\right). \tag{18}$$

Taking into account (11) (with $\delta_6 > 0$) and (18) in (17) we may write

$$
\begin{aligned}
&Q'(t) \geq \left[(1-\gamma) - \frac{b2^{\alpha-1}\sigma^{\alpha}\Gamma(-\alpha)\varepsilon}{M(p+1)^{\alpha}} \right] H^{-\gamma}(t) H'(t) \\
&+ \left[1 + \frac{b\sigma^{\alpha}2^{\alpha-2}\Gamma(-\alpha)\varepsilon^2}{\delta_5 (p+1)^{\alpha-1}} \right] H(t) \\
&+ \left[\frac{1}{2} + \frac{b2^{\alpha-3}\sigma^{\alpha}\Gamma(-\alpha)(p-1)\varepsilon^2}{M(p+1)^{\alpha}} H^{-\gamma}(t) - b(1+\delta_4 C_p \sigma\varepsilon) - \delta_6 C_p \varepsilon \right] \\
&\times e^{-\sigma\varepsilon t} \int_{\Omega} |\nabla u|^2 \, dx + \left[\frac{1}{2} + b\left(1 - \frac{\sigma\varepsilon}{4\delta_4}\right) - \frac{\varepsilon}{4\delta_6} \right] e^{-\sigma\varepsilon t} \int_{\Omega} u_t^2 dx \\
&+ d - \frac{bMC_2}{\Gamma(-\alpha)(p+1)^{\gamma}} + \mu F(t) \\
&+ \left[b - \frac{1}{p+1} - \frac{bMC_2}{\Gamma(-\alpha)(p+1)^{\gamma}} \right] e^{-\sigma\varepsilon t} \int_{\Omega} |u|^{p+1} \, dx.
\end{aligned}
$$

The first coefficient is nonnegative as soon as ε is chosen small enough, namely

$$
\varepsilon \leq \varepsilon_2 := \frac{M(1-\gamma)(p+1)^{\gamma}}{b2^{\alpha-1}\sigma^{\alpha}\Gamma(-\alpha)}.
$$

We pick $b = (p+3)/[4(p+1)]$, $\delta_4 = \delta_6 = 1/2$ and $\varepsilon \leq \varepsilon_3 := 4(p-1)/[(p+11)^2 C_p]$. It follows that the coefficient of $\int_{\Omega} |\nabla u|^2 \, dx$ is also nonnegative. If we further impose $\varepsilon < \varepsilon_4 := 4(p+3)/[(p+11)^2]$, then the coefficient of $\int_{\Omega} u_t^2 dx$ is positive. In fact, it is greater than $1/2$. Next, assuming that

$$
M < \frac{\Gamma(-\alpha)(p+1)^{\gamma}}{(p+3)C_2]} \min\{4d(p+1), \frac{p-1}{2}\},
$$

we infer that the coefficient of $\int_{\Omega} |u|^{p+1} \, dx$ is bigger than $p - 1/[8(p+1)]$ and the term $d - bMC_2[\Gamma(-\alpha)(p+1)^{\gamma}]$ is nonnegative. Consequently, for $\varepsilon < \min\{\varepsilon_i : i = 1, 2, 3, 4\}$ we get

$$
Q'(t) \geq H(t) + \frac{1}{2} \int_{\Omega} u_t^2 dx + \frac{p-1}{8(p+1)} \int_{\Omega} |u|^{p+1} \, dx. \tag{19}
$$

On the other hand, from the definition of $Q(t)$, we have

$$
Q(t)^{\frac{1}{1-\gamma}} \leq 2^{\frac{1}{1-\gamma}} \left\{ H(t) + b^{\frac{1}{1-\gamma}} \left(\int_{\Omega} |uu_t| \, dx \right)^{1/(1-\gamma)} \right\}. \tag{20}
$$

Furthermore, by the Cauchy-Schwarz inequality and Hölder's inequality

$$\begin{aligned}\left(\int_\Omega |uu_t|\,dx\right)^{1/(1-\gamma)} &\le \left[\left(\int_\Omega u^2dx\right)^{1/2}\cdot\left(\int_\Omega u_t^2dx\right)^{1/2}\right]^{1/(1-\gamma)}\\ &\le C(|\Omega|,p)\left(\int_\Omega |u|^{p+1}\,dx\right)^{1/[(p+1)(1-\gamma)]}\cdot\left(\int_\Omega u_t^2dx\right)^{1/[2(1-\gamma)]}.\end{aligned}$$

By our choice of γ, we have $2(1-\gamma)>1$. Therefore, we can apply Young's inequality and once again Hölder's inequality to get

$$\left(\int_\Omega |uu_t|\,dx\right)^{1/(1-\gamma)} \le B\left\{\int_\Omega u_t^2dx+\left(\int_\Omega |u|^{p+1}\,dx\right)^{2/[(p+1)(1-2\gamma)]}\right\}$$

for some $B=B(|\Omega|,p,\gamma)>0$. But from the value of γ, we see that the exponent $2/[(p+1)(1-2\gamma)]=1$. Hence, taking into account this estimate in (20), we entail

$$Q(t)^{1/(1-\gamma)} \le 2^{1/(1-\gamma)}\left\{H(t)+b^{1/(1-\gamma)}B\left(\int_\Omega u_t^2dx+\int_\Omega |u|^{p+1}\,dx\right)\right\}.$$

We will arrive at

$$Q(t)^{1/(1-\gamma)} \le K\left\{H(t)+\frac{1}{2}\int_\Omega u_t^2dx+\frac{p-1}{8(p+1)}\int_\Omega |u|^{p+1}\,dx\right\}$$

for some $K>0$, which implies by (19) that

$$Q(t)^{1/(1-\gamma)} \le KQ'(t) \tag{21}$$

if K is chosen large enough so that

$$\begin{cases}2^{1/(1-\gamma)}\le K,\\ 2^{1/(1-\gamma)}b^{1/(1-\gamma)}B\le \dfrac{K}{2},\\ 2^{1/(1-\gamma)}b^{1/(1-\gamma)}B\le \dfrac{p-1}{8(p+1)}K.\end{cases}$$

That is K has to be selected so that

$$K\ge 2^{1/(1-\gamma)}\max\left\{1,\frac{8(p+1)}{p-1}b^{1/(1-\gamma)}B\right\}.$$

From (19) it is clear that $Q'(t)\ge 0$. Hence, by the definition of $Q(t)$ and the hypotheses on the initial data, we have $Q(t)\ge Q(0)>b\int_\Omega u_1u_0dx\ge 0$.

Thus $Q(t) > 0$. Integrating (21) over $(0, t)$, we find

$$Q(t)^{\gamma/(1-\gamma)} \geq \frac{1}{Q(0)^{-\gamma(1-\gamma)} - \dfrac{\gamma}{K(1-\gamma)}t}.$$

Consequently, $Q(t)$ blows up at some time

$$T^* \leq \frac{K(1-\gamma)Q(0)^{-\gamma/(1-\gamma)}}{\gamma}.$$

$\square$

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