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INSTANTANEOUS KINEMATO-STATIC MODEL OF PLANAR COMPLIANT PARALLEL MECHANISMS

Cyril Quennouelle*
Laboratoire de robotique,
Département de génie mécanique,
Université Laval - Québec, Qc, Canada - G1V 0A6
cyril.quennouelle.1@ulaval.ca

Clément Gosselin
Laboratoire de robotique,
Département de génie mécanique,
Université Laval - Québec, Qc, Canada - G1V 0A6
gosselin@ulaval.ca

ABSTRACT

In this paper, the number of degrees of freedom, the kinematic constraints, the pose of the end-effector and the static constraints that lead to the Kinemato-Static Model of a Compliant Mechanism are introduced. A formulation is then provided for the Instantaneous Kinemato-Static Model. This new model enables to calculate the variation of the pose as a linear function of the motion of the actuators and the variation of the external loads through two new matrices: the compliant Jacobian matrix *and the* matrix of compliance *that give a simple and meaningful formulation of the model of the mechanism.*
Finally, a simple application to a 4-bar mechanism is presented to illustrate the use of this model and the new possibilities that it opens, notably the study of the kinematics for any range of applied load.

1 INTRODUCTION

A manipulator is a mechanism designed to displace objects in space or in a plane. Therefore, a high precision in positioning and orienting of the end-effector and a good repeatability of motion are desirable properties of a good manipulator. Parallel mechanisms are composed of several kinematic chains of rigid bodies linking a fixed base to a mobile platform. These mechanisms offer some advantages when used as manipulators. Since in this kind of architecture, the number of joints is higher than the number of degrees of freedom of the mechanism, some joints can be actuated while the others are passive, i.e., the motion of the latter is determined by that of the actuated joints. Generally, the joints selected to be actuated are close to the base in order to reduce the inertia of the moving parts of the mechanism. This relatively low inertia of the moving parts compared to the power of the actuators is a major advantage of parallel mechanisms in terms of precision of motion. on the other hand, the existence of passive joints increases the cost of fabrication and the mechanical clearance in the mechanism.

Compliant Mechanisms The use of compliant joints as passive joints is a promising subject that aims at reducing the costs of fabrication and the mechanical clearance of parallel manipulators or, in other words, to *reduce the cost of precision*. A mechanism built with compliant joints is called *elastically articulated rigid-body system* [1] or *compliant mechanism*.
In practice, the main advantage of a compliant joint is that it is made of a unique piece of material, therefore a compliant mechanism requires fewer assembly stages and the cost of fabrication is reduced. Moreover, from theoretical perspective, the absence of mechanical clearance allows an accurate modelling of the compliant joints. The main limitations of compliant joints and compliant mechanisms found in the literature are their small range of motion and their kinematic behaviour that is different from the conventional mechanisms notably because of the axis drift in the compliant joints.
Another drawback pointed out in [2] is the fact that a compliant joint cannot always be rigorously considered as a 1-DoF joint

*Address all correspondence to this author.

since it can have some small deformations along directions other than the one desired when submitted to a wrench. Therefore, a compliant mechanism is more sensitive to external wrenches than a conventional mechanism.
For the aforementioned reasons, when compliant joints are used to design an accurate manipulator, considering kinematics without statics or statics without kinematics is meaningless and the determination of a model that can take into account the motion of the actuator as well as the external and internal wrench becomes a topic of the highest interest.

Kinemato-Static Model A model that enables to calculate the configuration of the compliant mechanism taking into account the kinematic constraints and the static constraints is presented in [3]. It can notably take into account the stiffness of the passive joints. This model leads to a reduction of the *cost of precision* since it is of a great accuracy when the stiffness of the joints and the external wrenches are known. However, this model — which does not consider the dynamical effects and the gravity—, is highly non-linear and its computation can be expensive, in term of computational cost. Therefore, in this paper the "*Instantaneous Kinemato-Static Model*" (IKSM) that corresponds to the time derivative of the Kinemato-Static Model (KSM) is presented. This new model gives simple and meaningful relations between the motion of the actuators, the variations of the external wrenches and the motion of the end-effector.

Outline The first section of this paper presents the kinematic and static constraints that a compliant mechanism must satisfy. In the next section, the Instantaneous Kinemato-Static Model is introduced. Then, its contributions are discussed and finally, a concrete application is presented to illustrate the interest of this model. The derivation of the Cartesian Stiffness Matrix is briefly recalled in the appendix.

2 Kinemato-Static Model (KSM) of a Compliant Parallel Mechanism (CPM)

2.1 Kinematics of a Compliant Parallel Mechanism

2.1.1 Kinematic Degrees of Freedom The Chebychev-Grüber-Kutzbach formula, used to calculate the number of degrees of freedom l of a planar mechanism, is written as

$$l = 3(b-p-1) + \sum_{i=1}^{p} g_i, \tag{1}$$

where b is the number of rigid bodies, p is the number of joints and g_i is the number of degrees of freedom of the i^{th} joint. The number of kinematic constraints to be satisfied in a planar mechanism is noted c and is equal to 3 times the number of kinematic loops. In the Chebychev-Grüber-Kutzbachformula, c is calculated as

$$c = 3(p+1-b). \tag{2}$$

Hence, as explained in [3] and [4], a planar compliant joint is usually modelled as a 1-DoF linear or torsional spring but can also be modelled as 2 or 3 springs. Actually, the chosen number of springs depends on the motions taken into account, the ratio between the directional stiffnesses or the desired precision of the model. Therefore in a planar compliant joint, g_i can be equal to 1, 2 or 3 such that the total number of DoF in a compliant mechanism can be higher than in the equivalent mechanism built with non compliant joints.
In the rest of this paper, each k-DoF joint will be considered as k 1-DoF additional virtual joints [4] and $m = \sum g_i$ will be considered as the total number of joints in the mechanism. The joint vector containing the m joint coordinates θ_i in the mechanism is noted $\vec{\theta}$. In eq.(1), the relation between m, c and l is written as $l = -c + m$.

2.1.2 Kinematic Constraints Due to the kinematic loops existing in any parallel mechanism, c kinematic constraints must be satisfied by the joint coordinates. This is expressed as

$$\vec{\mathcal{K}}(\vec{\theta}) = \vec{0}_c \tag{3}$$

where $\vec{0}_c$ stands for the zero vector of dimension c. Any set of kinematic constraints can be chosen as long as it forms a system of c independent equations. For example, in [3] the chosen constraints are 6 equalities between the components of the pose of the platform considered as the end-effector of each of the 3 legs while in [5] the lengths of the edges of the platform are used as 2 of the 6 constraints instead of the equalities between the orientations of the platform.
Among the components of $\vec{\theta}$, c correspond to kinematically constrained joints, they are noted λ_i and the l others joints, noted χ_i, correspond to unconstrained joints. Vector $\vec{\chi}$ is the vector of the generalized coordinates of the mechanism. The relation between these vectors is expressed as $\vec{\theta} = \begin{bmatrix}\vec{\chi}^T & \vec{\lambda}^T\end{bmatrix}^T$ and the kinematic constraints are written as

$$\vec{\mathcal{K}}(\vec{\lambda}, \vec{\chi}) = \vec{0}_c, \tag{4}$$

such that $\vec{\lambda}$, the vector of kinematically constrained joint coordinates, and $\vec{\theta}$ can be calculated as functions of the generalized coordinates:

$$\vec{\lambda} = \vec{\mathcal{C}}(\vec{\chi}) \text{ and } \vec{\theta} = \vec{\mathcal{G}}(\vec{\chi}). \tag{5}$$

2.1.3 Actuated Mechanism An actuated joint is modeled as a spring with a variable free length. This free length is equal to the commanded value of the actuator noted α_i. All α_i are assembled in a vector noted $\vec{\alpha}$ and the vector of the coordinates corresponding to the actuated joint are noted $\vec{\theta}_\alpha$ ($\vec{\chi}_\alpha$ for the generalized coordinates). The actual value of the actuated joint with its compliance taken into account remains equal to θ_i such that

$$\theta_i = \chi_i = \alpha_i + \eta_i, \text{ for any actuated joint,} \tag{6}$$

where η_i represents the deformation of the actuator position due to its compliance. Since in the kinematic constraints, these free lengths $\vec{\alpha}$ do not appear, they are not directly components of $\vec{\chi}$. Indeed, the influence of the actuators is taken into account through the static constraints.
In any conventional mechanism, all unconstrained joints have to be actuated in order to ensure a control of the mechanism. Hence, the actuated joints are not compliant and hence the vector of the actuator coordinates stands for the vector of generalized coordinates ($\vec{\chi} = \vec{\chi}_\alpha = \vec{\alpha}$).
On the contrary, in compliant mechanisms some joints can be neither kinematically constrained nor actuated. The position of these joints is then ruled by the static constraints (Sec.(2.2)).

2.1.4 Control of the Pose of the Platform In any robotic manipulator, the determination of the Cartesian pose of the end-effector is of the utmost importance. In a planar mechanism, 3 parameters are used to characterized the pose, namely x, y and ϕ, which are assembled in the pose vector $\vec{x}$. This vector can be integrated in the kinematic model by adding 3 more kinematic constraints, written as

$$\vec{\mathcal{P}}(\vec{\theta}) - \vec{x} = \vec{0}_3. \tag{7}$$

The kinematic model becomes $\vec{\mathcal{K}}(\vec{\theta},\vec{x}) = \vec{\mathcal{K}}(\vec{\lambda},\vec{\chi},\vec{x}) = \vec{0}_{c+3}$. Therefore, after eliminating $\vec{\lambda}$ with the use of equation (5), the kinematic model of a parallel mechanism with l DoFs is written as

$$\vec{\mathcal{K}}'(\vec{\chi},\vec{x}) = \vec{\mathcal{P}}'(\vec{\chi}) - \vec{x} = \vec{0}_3. \tag{8}$$

2.2 Statics of a Compliant Parallel Mechanism

2.2.1 Static Constraints Since the static constraints $\vec{\mathcal{S}}$ result from the elastic potential energy and the derivative of the potentials associated to the external forces with respect to the generalized coordinates [5], $\vec{\mathcal{S}}$ comprises l independent functions. The relation between the dimension of $\vec{\mathcal{K}}$, the dimension of $\vec{\mathcal{S}}$ and m is another consequence of the duality between kinematics and statics and is written as

$$\dim(\vec{\mathcal{K}}) + \dim(\vec{\mathcal{S}}) = c + l = m = \dim(\vec{\theta}). \tag{9}$$

When the external parameters are known and all generalized coordinates appear in the equations of the static equilibrium (*i.e., if all generalized joints have a stiffness*), this latter equation implies that the system of equations has a finite number of solutions. For a specific branch of solutions, the configuration $\vec{\chi}$ of the mechanism is completely determined by the kinematic and the static constraints.
For example, a system strictly composed of springs will always remain in the same configuration if there is no *external perturbation* applied on it. The perturbation can be a new actuator position or a new wrench applied on the end-effector, on an actuator or anywhere else on the mechanism, represented by vector $\vec{\beta}$. Thus, the static constraints of a mechanism can be written as functions of $\vec{\chi}$ and $\vec{\beta}$ only. Therefore, the generalized coordinates —that are not kinematically constrained— are indeed functions of the external parameters, i.e., they are statically constrained.

$$\vec{\mathcal{S}}(\vec{\chi},\vec{\beta}) = \vec{0} \text{ s.t. } \vec{\chi} = \mathcal{F}(\vec{\beta}) \tag{10}$$

2.2.2 Parameters of the Mechanism In general, 6 external parameters are used to represent the configuration of the mechanism: the 3 components of the external wrench applied on the end-effector and the 3 coordinates of the actuated joints if controlled in position. The vector of the external wrench applied on the end-effector is noted $\vec{f}$.
Thus, the static constraints of a mechanism are written as

$$\vec{\mathcal{S}}(\vec{\chi},\vec{\alpha},\vec{f}) = \vec{0}. \tag{11}$$

Therefore, the configuration of the mechanism is a function of these 6 parameters, namely $\vec{\chi} = \mathcal{F}(\vec{\alpha},\vec{f})$.

2.3 Kinemato-Static Model of a Compliant Parallel Mechanism

The following Cartesian Kinemato-static Model (KSM) of a compliant parallel mechanism can be written, that satisfies the kinematic and static constraints:

$$\vec{x} = \vec{\mathcal{M}}(\vec{\alpha},\vec{f}) = \vec{\mathcal{P}}\left[\vec{\mathcal{G}}\left\{\vec{\mathcal{F}}\left(\vec{\alpha},\vec{f}\right)\right\}\right]. \tag{12}$$

3 Instantaneous Kinemato-Static Model of a CPM

For the most general case, the time derivative of equation (12) is written as

$$\dot{\vec{x}} = \frac{d\vec{\mathcal{M}}(\vec{\alpha},\vec{f})}{d\vec{\alpha}}\dot{\vec{\alpha}} + \frac{d\vec{\mathcal{M}}(\vec{\alpha},\vec{f})}{d\vec{f}}\dot{\vec{f}}. \tag{13}$$

If any analytical formulation of $\vec{\mathcal{M}}$ is known, equation (13) can be calculated by differentiating the kinematic and static constraints.

3.1 Instantaneous Kinematic Constraints

The differentiation of the kinematic constraints given in equation (3) leads to

$$\vec{\mathcal{V}}(\dot{\vec{\theta}}) = \frac{d\vec{\mathcal{K}}(\vec{\theta})}{d\vec{\theta}}\dot{\vec{\theta}} = \mathbf{S}\dot{\vec{\theta}} = \vec{0}_c, \tag{14}$$

where $\mathbf{S}$ stands for $d\vec{\mathcal{K}}/d\vec{\theta}$. Partitioning $\vec{\theta}$ in constrained and unconstrained coordinates, the derivation of the kinematic constraints are written as

$$\vec{\mathcal{V}}(\dot{\vec{\chi}},\dot{\vec{\lambda}}) = \mathbf{S}_\chi\dot{\vec{\chi}} + \mathbf{S}_\lambda\dot{\vec{\lambda}} = \vec{0}_c, \tag{15}$$

where $\mathbf{S}_\chi$ stands for $d\vec{\mathcal{K}}/d\vec{\chi}$ and $\mathbf{S}_\lambda$ stands for $d\vec{\mathcal{K}}/d\vec{\lambda}$, such that $\mathbf{S} = \left[\mathbf{S}_\chi\ \mathbf{S}_\lambda\right]$.The relation between $\dot{\vec{\chi}}$ and $\dot{\vec{\lambda}}$ is linear, and since the c coordinates of $\vec{\lambda}$ are the solutions of the c kinematic constraints, $\mathbf{S}_\lambda = d\vec{\mathcal{K}}/d\vec{\lambda}$ is a matrix of full rank, i.e., always invertible. Thus, $\dot{\vec{\lambda}}$ is calculated as

$$\dot{\vec{\lambda}} = -\mathbf{S}_\lambda^{-1}\mathbf{S}_\chi\dot{\vec{\chi}} = \mathbf{G}\dot{\vec{\chi}}, \tag{16}$$

where $\mathbf{G} = -\mathbf{S}_\lambda^{-1}\mathbf{S}_\chi$ is introduced to simplify the equations.
The time derivative of $\vec{\theta}$ is a function of the time derivative of the generalized coordinates:

$$\dot{\vec{\theta}} = \mathbf{R}\dot{\vec{\chi}}, \text{ with } \mathbf{R} = d\vec{\theta}/d\vec{\chi} = \begin{bmatrix}\mathbf{1} & \mathbf{G}^T\end{bmatrix}^T. \tag{17}$$

3.2 Cartesian Velocity

The Cartesian velocity is given by the time derivative of the pose of the mechanism and is obtained by differentiating eq.(7) or eq.(8):

$$\dot{\vec{x}} = \frac{\vec{\mathcal{P}}(\vec{\theta})}{d\vec{\theta}}\dot{\vec{\theta}} = \mathbf{J}_\theta\dot{\vec{\theta}} = \frac{\vec{\mathcal{P}}'(\vec{\chi})}{d\vec{\chi}}\dot{\vec{\chi}} = \mathbf{J}\dot{\vec{\chi}}, \tag{18}$$

where $\mathbf{J}_\theta$ and $\mathbf{J}$ are the Jacobian matrices of the mechanism with respect to $\vec{\theta}$ and $\vec{\chi}$. Then, using the definition of matrix $\mathbf{R}$ in eq.(17), the relation between both Jacobian matrices is written as

$$\mathbf{J} = \mathbf{J}_\theta\mathbf{R}. \tag{19}$$

3.3 Variational Constraints

The differentiation of the static constraints, given in equation (11), leads to

$$\begin{aligned}\vec{\mathcal{W}}(\dot{\vec{\chi}},\dot{\vec{\alpha}},\dot{\vec{f}}) &= \frac{d\vec{\mathcal{S}}(\vec{\chi})}{d\vec{\chi}}\dot{\vec{\chi}} + \frac{d\vec{\mathcal{S}}(\vec{\alpha})}{d\vec{\alpha}}\dot{\vec{\alpha}} + \frac{d\vec{\mathcal{S}}(\vec{f})}{d\vec{f}}\dot{\vec{f}} = \vec{0}_l. \\ &= \mathbf{V}_\chi\dot{\vec{\chi}} + \mathbf{V}_\alpha\dot{\vec{\alpha}} + \mathbf{V}_f\dot{\vec{f}} = \vec{0}_l,\end{aligned} \tag{20}$$

where $\mathbf{V}_\chi$ stands for $d\vec{\mathcal{S}}/d\vec{\chi}$, $\mathbf{V}_\alpha$ stands for $d\vec{\mathcal{S}}/d\vec{\theta}$ and $\mathbf{V}_f$ stands for $d\vec{\mathcal{S}}/d\vec{f}$. For a solution $\vec{\chi}$ of the static constraints, matrix $\mathbf{V}_\chi$ is a matrix of full rank and thus is invertible. Except in a static singular configuration, $\dot{\vec{\chi}}$ can be calculated as

$$\dot{\vec{\chi}} = -\mathbf{V}_\chi^{-1}\mathbf{V}_\alpha\dot{\vec{\alpha}} - \mathbf{V}_\chi^{-1}\mathbf{V}_f\dot{\vec{f}}. \tag{21}$$

3.4 Stiffness Matrices

The static equilibrium of CPMs is written as

$$\vec{\mathcal{S}}(\vec{\theta},\vec{\alpha},\vec{f}) = \vec{\tau}_\chi + \mathbf{G}^T\vec{\tau}_\lambda - \mathbf{J}^T\vec{f} = \vec{0}_l. \tag{22}$$

The vectors of torques/forces $\vec{\tau}_\chi$ and $\vec{\tau}_\lambda$ provided by the deformed compliant joints are calculated as

$$\vec{\tau}_\chi = \int_{\vec{\chi}_0}^{\vec{\chi}}\mathbf{K}_\chi d\vec{\chi} \text{ and } \vec{\tau}_\lambda = \int_{\vec{\lambda}_0}^{\vec{\lambda}}\mathbf{K}_\lambda d\vec{\lambda} = \int_{\vec{\chi}_0}^{\vec{\chi}}\mathbf{K}_\lambda\mathbf{G}d\vec{\chi} \tag{23}$$

where vectors $\vec{\chi}_0$ and $\vec{\lambda}_0$ are the vectors of the free lengths of all compliant joints and matrices $\mathbf{K}_\chi$ and $\mathbf{K}_\lambda$ represent the diagonal matrices containing the local stiffness coefficients of the joints. These matrices $\mathbf{K}_\chi$ and $\mathbf{K}_\lambda$ can be known as parameters of the mechanism or as functions of the geometry of the compliant joints and the properties of their material [6, 7].
When the compliant joints are modeled using the *pseudo-rigid body model* (PRBM) [6], the articular stiffness matrices are constant and the following formulations of the torques/forces vectors $\vec{\tau}_\chi$ and $\vec{\tau}_\lambda$ are used. These formulations are particular cases of eq.(23) and are written as

$$\vec{\tau}_\chi = \mathbf{K}_\chi(\vec{\chi} - \vec{\chi}_0) \text{ and } \vec{\tau}_\lambda = \mathbf{K}_\lambda(\vec{\lambda} - \vec{\lambda}_0) = \mathbf{K}_\lambda(\vec{\mathcal{C}}(\vec{\chi}) - \vec{\lambda}_0). \tag{24}$$

Matrix $\mathbf{V}_\alpha$ All the free lengths are constant except the ones corresponding to the actuated joints. Thus, $\dot{\chi}_{0i}$ and $\dot{\lambda}_{0i}$ is equal to 0 for the passive (unactuated) joints and $\dot{\chi}_{0i}$ equal $\dot{\alpha}_i$ for the actuated joints.
The differentiation of vector $\vec{\tau}_\chi$ with respect to $\vec{\alpha}$ yields a $l \times 3$ matrix composed of some **0** sub-matrices and a sub-matrix corresponding to the derivative of the torque/force applied by the actuators noted $\vec{\tau}_\alpha$.

$$\frac{d\vec{\tau}_\chi}{d\vec{\alpha}} = \mathbf{I}_{l\times 3}\frac{d\vec{\tau}_\alpha}{d\vec{\alpha}}, \tag{25}$$

where matrix $\mathbf{I}_{l\times 3} = d\vec{\chi}/d\vec{\chi}_\alpha$ is the identity matrix if l equals 3 (all generalized coordinates are actuated). Matrix $\mathbf{I}_{l\times 3}$ is a rectangular matrix composed of "0"s and 3 "1"s corresponding to $d\chi_{\alpha i}/d\chi_{\alpha i}$ when $l > 3$.
A diagonal matrix $\mathbf{K}_\alpha$ composed of the stiffness coefficients of the actuators, is defined and differentiated with respect to $\vec{\tau}_\alpha$ as

$$\frac{d\vec{\tau}_\alpha}{d\vec{\alpha}} = \frac{d}{d\vec{\alpha}}\left(\int_{\vec{\alpha}}^{\vec{\chi}_\alpha} \mathbf{K}_\alpha d\vec{\chi}_\alpha\right) = -\mathbf{K}_\alpha. \tag{26}$$

Hence, neither $\mathbf{G}$, $\vec{\tau}_\lambda$, $\mathbf{J}$ nor $\vec{f}$ are functions of $\vec{\alpha}$, and the differentiation of the static equilibrium (eq.(22)) with respect to the actuators coordinates is formulated as

$$\frac{d\vec{\mathcal{S}}}{d\vec{\alpha}} = \frac{d\vec{\tau}_\chi}{d\vec{\alpha}} = \mathbf{V}_\alpha = -\mathbf{I}_{l\times 3}\mathbf{K}_\alpha = -\mathbf{K}'_\alpha, \tag{27}$$

where $\mathbf{K}'_\alpha$, a $l \times 3$ matrix composed of the stiffness coefficients $k_{\alpha i}$, is used to simplify the equations.

Matrix $\mathbf{V}_\chi$ Taking the derivative of the static equilibrium (eq.(22)) with respect to the generalized coordinates gives the generalized stiffness matrix of a compliant mechanism noted $\mathbf{K}_M$, which is formulated as

$$\frac{d\vec{\mathcal{S}}}{d\vec{\chi}} = \mathbf{V}_\chi = \mathbf{K}_M = \mathbf{K}_\chi + \mathbf{K}_I + \mathbf{K}_E. \tag{28}$$

In this formulation, matrix $\mathbf{K}_E$ captures the effect of the changes of geometry on the impact of the external wrenches on the generalized coordinates. Matrix $\mathbf{K}_I$ captures the effect of the changes of the geometry of the internal static constraints on the generalized coordinates. The derivative of the static equilibrium and these stiffness matrices are presented in [5] and briefly recalled in appendix (6).

Matrix $\mathbf{V}_f$: The derivative of the static equilibrium (eq.(22)) with respect to the external wrench $\vec{f}$ is calculated as

$$\frac{d\vec{\mathcal{S}}}{d\vec{f}} = \mathbf{V}_f = -\mathbf{J}^T. \tag{29}$$

3.5 Instantaneous Kinemato-Static Model

With the instantaneous kinematic and static constraints, given in eqs.(17), (18) and (21), the Instantaneous Kinemato-Static Model (IKSM) of a compliant mechanism can be expressed as

$$\dot{\vec{x}} = \mathbf{J}\left(-\mathbf{V}_\chi^{-1}\mathbf{V}_\alpha\dot{\vec{\alpha}} - \mathbf{V}_\chi^{-1}\mathbf{V}_f\dot{\vec{f}}\right). \tag{30}$$

With the derivative of the static equilibrium given by eqs.(27), (28) and (29), the latter equation can be written in a very meaningful form, namely

$$\dot{\vec{x}} = \mathbf{J}\mathbf{K}_M^{-1}\mathbf{K}'_\alpha\dot{\vec{\alpha}} + \mathbf{J}\mathbf{K}_M^{-1}\mathbf{J}^T\dot{\vec{f}}. \tag{31}$$

In this equation, two important matrices appear:

- The *Jacobian matrix of the compliant mechanism* that gives the relation between a variation of the position of the actuators and the variation of the pose of the end-effector.

$$\mathbf{J}_C = \frac{d\vec{x}}{d\vec{\alpha}} = \mathbf{J}\mathbf{K}_M^{-1}\mathbf{K}'_\alpha \tag{32}$$

- The *matrix of compliance of the compliant mechanism* that gives the relation between a variation of the external wrench applied on the end-effector and the variation of its pose.

$$\mathbf{C}_C = \frac{d\vec{x}}{d\vec{f}} = \mathbf{J}\mathbf{K}_M^{-1}\mathbf{J}^T \tag{33}$$

Therefore, a simple and clear expression of the IKSM of a compliant mechanism in which the actuators are controlled in position is obtained as

$$\dot{\vec{x}} = \mathbf{J}_C\dot{\vec{\alpha}} + \mathbf{C}_C\dot{\vec{f}}. \tag{34}$$

4 Contributions of the Kinemato-Static Model

4.1 Formulation

Recalling the definition of the KSM given in eq.(12), the IKSM is defined as

$$\dot{\vec{x}} = \dot{\vec{\mathcal{M}}}(\vec{\alpha}, \vec{f}) = \frac{d\vec{\mathcal{P}}}{d\vec{\theta}}\frac{d\vec{\mathcal{G}}}{d\vec{\chi}}\left(\frac{d\vec{\mathcal{F}}}{d\vec{\alpha}}\dot{\vec{\alpha}} + \frac{d\vec{\mathcal{F}}}{d\vec{f}}\dot{\vec{f}}\right). \tag{35}$$

Matrix $d\vec{\mathcal{P}}/d\vec{\theta}$ is well known and studied in robotics and is equal to $\mathbf{J}_\theta$ (eq.(18)). Matrix $d\vec{\mathcal{G}}/d\vec{\chi}$ is also quite common in the field of parallel robots and is equal to $\mathbf{R}$ (eq.(17)) (in serial mechanisms, $\mathbf{R}=\mathbf{1}$).
But the third matrix $d\vec{\mathcal{F}}/d\vec{\alpha}$ is new in robotics and actually could not have been discovered without the correct formulation of the stiffness matrix ([8] for serial mechanisms and [5] for parallel mechanisms). The last matrix $d\vec{\mathcal{F}}/d\vec{f}$ simply corresponds to a wrench transfer matrix.
Therefore, the IKSM can be written as

$$\dot{\vec{x}} = \mathbf{J}_\theta \mathbf{R}\left(\frac{d\vec{\mathcal{F}}}{d\vec{\alpha}}\dot{\vec{\alpha}} + \frac{d\vec{\mathcal{F}}}{d\vec{f}}\dot{\vec{f}}\right) = \mathbf{J}_\theta \mathbf{R}\left(\frac{d\vec{\chi}}{d\vec{\alpha}}\dot{\vec{\alpha}} + \frac{d\vec{\chi}}{d\vec{f}}\dot{\vec{f}}\right). \quad (36)$$

4.1.1 Compliant Jacobian matrix

Matrix $d\vec{\chi}/d\vec{\alpha}$ is noted $\mathbf{T}$ such that the Compliant Jacobian matrix $\mathbf{J}_C$ becomes

$$\mathbf{J}_C = \mathbf{J}_\theta \mathbf{R}\mathbf{T} = \mathbf{J}\mathbf{T}, \text{ with } \mathbf{T} = \frac{d\vec{\mathcal{F}}}{d\vec{\alpha}} = \frac{d\vec{\chi}}{d\vec{\alpha}}. \quad (37)$$

This matrix $\mathbf{T}$ represents the matrix of the transmission ratio between the motion of the actuators and the effective motion of the actuated compliant joint. More details on matrix $\mathbf{T}$ are given in section (4.2).
The notion of "*Compliant Jacobian Matrix*" is new in robotics since matrix $\mathbf{T}$ does not appear if a correct formulation of the stiffness matrix is not used. In fact, with the stiffness matrix proposed by Salisbury [9], $\mathbf{T}$ is always equal to the identity matrix, whatever the external load applied on the mechanism.
The Compliant Jacobian matrix is equal to the classically defined Jacobian matrix $\mathbf{J}$ when $\mathbf{T}$ is equal to $\mathbf{1}$.

4.1.2 Cartesian Compliance Matrix

Matrix $d\vec{\chi}/d\vec{f}$ is equal to $\mathbf{K}_M^{-1}\mathbf{J}^T$. This matrix relates $\dot{\vec{f}}$ —the variation of the external wrench expressed in the Cartesian frame— with $\dot{\vec{\chi}}$ —the motion of the generalized coordinates expressed in the local frame—. Its inverse —written $d\vec{f}/d\vec{\chi}$— is asymmetric and equal to $\mathbf{J}^{-T}\mathbf{K}_M$. It corresponds to a stiffness matrix multiplied by a change of frame, thus it is not a stiffness matrix (a properly defined stiffness matrix cannot be asymmetric [1]).
On the contrary, the Cartesian compliance matrix of the mechanism is symmetric and written as

$$\mathbf{C}_C = \mathbf{J}_\theta \mathbf{R}\mathbf{K}_M^{-1}\mathbf{J}^T = \mathbf{J}\mathbf{K}_M^{-1}\mathbf{J}^T. \quad (38)$$

4.2 Transmission Ratio

4.2.1 Definition of the Transmission Ratio

Matrix $\mathbf{T}$ of the transmission ratios between the motions of the actuators and the effective motions of the actuated joint is defined as $=$ $d\vec{\chi}/d\vec{\alpha}$ and is calculated as

$$\mathbf{T} = \mathbf{K}_M^{-1}\mathbf{K}'_\alpha. \quad (39)$$

The transmission ratios characterize how a change in the wrench of the actuators ($\dot{\vec{\tau}}_\alpha = \mathbf{K}'_\alpha\dot{\vec{\alpha}}$) is distributed to the generalized coordinates, yielding small displacement ($\dot{\vec{\chi}} = \mathbf{K}_M^{-1}\dot{\vec{\tau}}_\alpha$).
With the definition of the generalized stiffness matrix given in appendix (6), the matrix of transmission ratios is expressed as

$$\mathbf{T} = \left(\mathbf{K}_\chi + \mathbf{K}_I + \mathbf{K}_E\right)^{-1}\mathbf{K}'_\alpha. \quad (40)$$

When all the kinematically unconstrained joints are actuated ($\vec{\chi} = \vec{\chi}_\alpha$), the matrix of transmission ratios can be written as

$$\mathbf{T} = \left[\mathbf{1} + \mathbf{K}_\alpha^{-1}\mathbf{K}_I + \mathbf{K}_\alpha^{-1}\mathbf{K}_E\right]^{-1}. \quad (41)$$

Furthermore, it is important to keep in mind that because this transmission ratio has been calculated without considering the dynamical effects, the presented formulation is valid under quasi-static conditions only.

4.2.2 Distribution of the Motion

The variation of the position of the actuators has simultaneously several effects. The wrench provided by the actuator is used to:

- preserve the static equilibrium that is modified by a change of matrix $\mathbf{J}$. This is accomplished through matrix $\mathbf{K}_E$, that is proportional to the external wrench $\vec{f}$,
- preserve the static equilibrium that is modified by a change of the kinematic constraints (matrix $\mathbf{G}$). This is accomplished through matrix $\mathbf{K}_I$, that is due to the stiffness of the passive joints $\mathbf{K}_\lambda$,
- modify the configuration of the mechanism through matrix $\mathbf{K}_\chi$.

Therefore, the matrix of transmission ratios generally differs from $\mathbf{1}$ and thus has an influence on the kinematics of the mechanism. However, even if the transmission ratios can be larger than of smaller than 1 during a given trajectory, because all the considered torques/forces are conservative, one has $\oint \mathbf{T}d\vec{\alpha} = \mathbf{0}$. This means that in the IKSM no motion of the actuator is lost, it can actually just be *stored* or *returned* by the effects of the compliances.
Finally, $\mathbf{T}$ tends to $\mathbf{1}$ when

$$\mathbf{T} \rightarrow \mathbf{1} \Leftrightarrow \mathbf{K}_\alpha^{-1}\left(\mathbf{K}_I + \mathbf{K}_E\right) \rightarrow \mathbf{0} \quad (42)$$

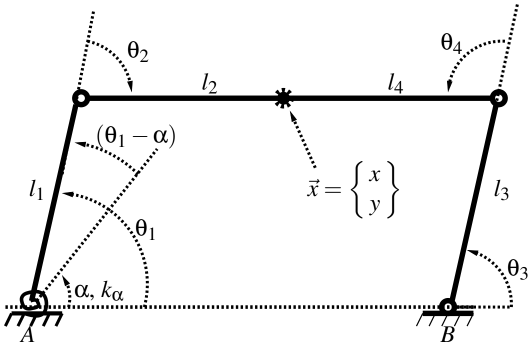

Figure 1. Four-bar mechanism with an compliant actuator at joint α.

which is equivalent to $\mathbf{K}_I + \mathbf{K}_E \lll \mathbf{K}_\alpha$. In other words, the matrix of transmission ratios is close to $\mathbf{1}$ when the stiffness of the passive joints and the effects of the external loads are negligible compared to the stiffness of the actuators.

5 Application : Four-Bar Mechanism

A simple parallel mechanism that can be used to briefly illustrate the contribution of the above model is the planar four-bar linkage with only one spring k_α representing the compliance of the actuator (Fig. 1). The geometric parameters of this mechanism are defined such that $\vec{x} = [0\ 0]^T$ when $\theta_1 = \alpha = \pi/2$. The parameters are the lengths of the links $l_1 = l_2 = l_3 = l_4 = 1m$ and the base points of each leg A and B, respectively defined by vectors $\vec{a} = (-1,-1)$ and $\vec{b} = (1,-1)$.

5.1 Kinemato-Static Model

5.1.1 Degrees of Freedom: In this mechanism, there are 4 links and 4 joints, among which the actuated one has 2 DoFs, thus this compliant four-bar mechanism has 2 degrees of freedom. One is located at the end-effector that is constrained to a circular trajectory and the other one is located at the actuated joint where actually angles α and θ_1 can evolved independently.

5.1.2 Geometric Constraints and Generalized Coordinates: Coordinates θ_1 and α are chosen as generalized coordinates, they enable a complete description of the configuration of the mechanism. The 3 kinematically constrained joints λ_i are θ_2, θ_3 and θ_4. Even if there are several solutions to the kinematic constraints, one is obvious and chosen in this application.

$$\vec{\lambda} = \vec{\mathcal{C}}(\theta_1) \Leftrightarrow \begin{cases} \theta_2 = -\theta_1 \\ \theta_3 = \ \ \theta_1 \\ \theta_4 = -\theta_1 + \pi \end{cases} \tag{43}$$

5.1.3 Pose of the End-Effector The geometrical constraints corresponding to the pose of the end-effector are written as

$$\vec{x} = \vec{\mathcal{P}}(\theta_1) = [l_1 \cos\theta_1,\ \ l_1(\sin\theta_1 - 1)]^T . \tag{44}$$

5.1.4 Constrained Joint Matrix Matrix $\mathbf{G}$ defined in equation (16) and matrix $\mathbf{R}$ are written as

$$\mathbf{G} = \frac{d\vec{\lambda}}{d\theta_1} = [-1\ \ 1\ -1]^T \text{ and } \mathbf{R} = [1\ -1\ \ 1\ -1]^T \tag{45}$$

5.1.5 Jacobian Matrix $\mathbf{J}$ is calculated as

$$\mathbf{J} = \frac{d\vec{\mathcal{P}}}{d\theta_1} = [-l_1 \sin\theta_1 \quad l_1 \cos\theta_1]^T . \tag{46}$$

5.1.6 Torque/Force Vectors The vectors τ_{θ_1}, $\vec{\tau}_\lambda$ and $\vec{f}$ are defined as

$$\tau_{\theta_1} = k_\alpha(\theta_1 - \alpha), \vec{\tau}_\lambda = [0\ \ 0\ \ 0]^T \text{ and } \vec{f} = [f_x \quad f_y]^T . \tag{47}$$

It can be noticed that because the mechanism has only two DoFs, only two independent parameters can have an effect on it. Thus, if one chosen external parameter is the position α of the actuator, the effect of all other external parameters can be reduced to one unique parameter. Indeed, the action of the 2 dimensional wrench $\vec{f}$ applied on the end-effector is equivalent to a unique torque t applied on the actuated joint. This torque t is calculated as

$$t = \mathbf{J}^T \vec{f} = -l_1 \sin\theta_1 f_x + l_1 \cos\theta_1 f_y \tag{48}$$

This equation means that all the work done by the 2-dimensional wrench $\vec{f}$ is provided by its single component orthogonal to the link l_1. The other component does not produce any work.

5.1.7 Static Equilibrium Therefore, the formula of the static equilibrium given in equation (22) is written as

$$\begin{aligned} \vec{\mathcal{S}}(\theta_1,\alpha,\vec{f}) &= \tau_{\theta_1} + \mathbf{G}^T\vec{\tau}_\lambda - \mathbf{J}^T\vec{f} = \tau_{\theta_1} - \mathbf{J}^T\vec{f} \\ &= k_\alpha(\theta_1 - \alpha) + l_1 \sin\theta_1 f_x - l_1 \cos\theta_1 f_y = 0 \end{aligned} \tag{49}$$

5.1.8 Statically Constrained Joint From equation (49), the angle θ_1 can be calculated as

$$\theta_1 = \alpha - \frac{l_1 \sin\theta_1 f_x}{k_\alpha} + \frac{l_1 \cos\theta_1 f_y}{k_\alpha} \tag{50}$$

If $\vec{f}$ is taken as an external parameter, θ_1 appears on both sides of this latter equation, $\cos\theta_1$ and $\sin\theta_1$ are replaced using eq.(44), such that θ_1 becomes a function of the external parameters α, $\vec{f}$ and $\vec{x}$. One obtains

$$\theta_1 = \alpha - (y + l_1)\frac{f_x}{k_\alpha} + x\frac{f_y}{k_\alpha}. \tag{51}$$

5.1.9 Kinemato-Static Model of the 4-bar Mechanism With eqs.(44) and (51), the KSM of the above four-bar mechanism is expressed as

$$\vec{\mathcal{M}}(\vec{x},\alpha,\vec{f}) = \vec{0} \Leftrightarrow \begin{cases} x - l_1 \cos\left(\alpha - (y+l_1)\frac{f_x}{k_\alpha} + x\frac{f_y}{k_\alpha}\right) & = 0 \\ y - l_1 \sin\left(\alpha - (y+l_1)\frac{f_x}{k_\alpha} + x\frac{f_y}{k_\alpha}\right) - l_1 & = 0 \end{cases} \tag{52}$$

5.2 Instantaneous Kinemato-Static Model

5.2.1 Pose of the End-Effector The time derivative of eq.(44) is written as

$$\dot{\vec{x}} = \dot{\vec{\mathcal{P}}}(\theta_1,\dot{\theta}_1) = \begin{bmatrix} -l_1 \sin\theta_1\dot{\theta}_1, & l_1\cos\theta_1\dot{\theta}_1 \end{bmatrix} \tag{53}$$

5.2.2 Static Equilibrium The time derivative of eq.(49) is written as

$$\begin{aligned} &\dot{\vec{\mathcal{S}}}(\theta_1,\dot{\theta}_1,\alpha,\dot{\alpha},\vec{f},\dot{\vec{f}}) = \dot{\tau}_{\theta_1} - \dot{\mathbf{J}}^T\vec{f} - \mathbf{J}^T\dot{\vec{f}} \\ &\dot{\vec{\mathcal{S}}} = k_\alpha(\dot{\theta}_1 - \dot{\alpha}) + k_I\dot{\theta}_1 - l_1(-s_1\dot{f}_x + c_1\dot{f}_y) = 0, \end{aligned} \tag{54}$$

where $k_I = -(d\mathbf{J}^T/d\theta_1)\vec{f} = -l_1\cos\theta_1 f_x - l_1\sin\theta_1 f_y$.
Thus, the relation between $\dot{\theta_1}$ and the variation of the external parameters $\dot{\alpha}$ and $\dot{\vec{f}}$ is calculated as

$$\dot{\theta_1} = \frac{k_\alpha}{k_\alpha + k_I}\dot{\alpha} + \frac{1}{k_\alpha + k_I}(-l_1\sin\theta_1\dot{f}_x + l_1\cos\theta_1\dot{f}_y). \tag{55}$$

5.2.3 Instantaneous Kinemato-Static Model Using the time derivative of the kinematic and static constraints, the variation of the pose of the mechanism is calculated for a variation of the external parameters, as

$$\dot{x} = \mathbf{J}\frac{k_\alpha}{k_\alpha + k_I}\dot{\alpha} + \mathbf{J}\frac{1}{k_\alpha + k_I}\mathbf{J}^T\dot{\vec{f}}. \tag{56}$$

Thus, the IKSM of the compliant four-link mechanism is expressed as

$$\begin{cases} \dot{x} = -\sin\theta_1\frac{k_\alpha}{k_\alpha+k_I}\dot{\alpha} + \frac{-\sin\theta_1}{k_\alpha+k_I}(-l_1\sin\theta_1\dot{f}_x + l_1\cos\theta_1\dot{f}_y) \\ \dot{y} = \quad \cos\theta_1\frac{k_\alpha}{k_\alpha+k_I}\dot{\alpha} + \frac{\cos\theta_1}{k_\alpha+k_I}(-l_1\sin\theta_1\dot{f}_x + l_1\cos\theta_1\dot{f}_y) \end{cases} \tag{57}$$

5.2.4 Derivation of the Kinemato-Static Model Since an analytic formulation of the KSM is known for this simple mechanism, the IKSM can as well be obtained by differentiating eq.(52). The details of the calculations are not reported in this paper due to space limitations.

5.3 Study of the Transmission Ratio T

The ratio between the angular velocities $\dot{\alpha}$ and $\dot{\theta_1}$ corresponds to the transmission ratio noted $\mathbf{T}$ in the other sections of this paper. In this application, since $\mathbf{T}$ is a scalar, it is noted r. It is indeed equal to the derivative of θ_1 with respect to α.

$$r = \dot{\theta_1}/\dot{\alpha} = d\theta_1/d\alpha. \tag{58}$$

This ratio is a function of the mechanism configuration and of the external wrench. It can be calculated from eq.(55) as

$$r = \mathcal{R}(\theta_1,\vec{f}) = \frac{1}{1 + \cos(\theta_1)l_1 f_x/k_\alpha + \sin(\theta_1)l_1 f_y/k_\alpha}. \tag{59}$$

5.3.1 Angle θ_1 Since no analytical solution of equation $\theta_1 = f(\alpha,\vec{f})$ can be obtained, graphs (fig.(2)) have been plotted with the numerical solutions of the equivalent equation (50). To reduce the number of parameters in this equation, two parameters b_x and b_y have been introduced. They are defined as

$$\left.\begin{aligned} b_x = l_1 f_x/k_\alpha \\ b_y = l_1 f_y/k_\alpha \end{aligned}\right\} \text{s.t. } \theta_1 = \alpha - b_x\sin\theta_1 + b_y\cos\theta_1 \tag{60}$$

These parameters are used to normalize the external wrenches with respect to the dimension of the mechanism and the stiffness of the actuator. Moreover, since the external wrench works only

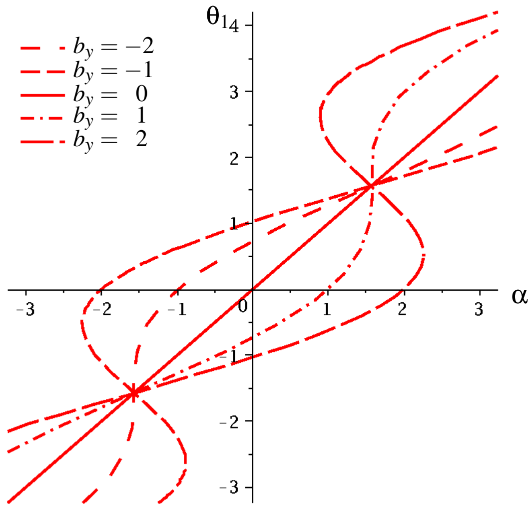

Figure 2. Mechanism configurations (α, θ_1) satisfying the static equilibrium for several external loads b_y.

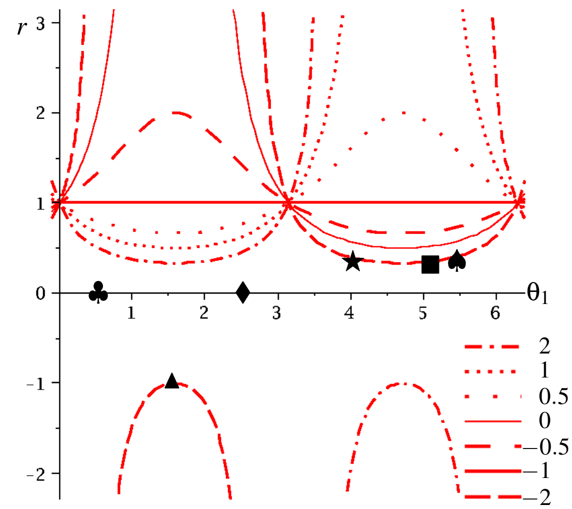

Figure 3. Transmission ratio $r = \mathcal{F}(\theta_1)$ for several values of b_y.

through its torque t (eq.(48)), only one parameter is required to represent the external wrench and to study all the possibilities. Therefore, in fig.(2) and in the other graphic, $\vec{f}$ is arbitrary chosen equal to $\left[0, \frac{k_\alpha}{l1} b_y, 0\right]^T$, i.e., $b_x = 0$ while b_y is chosen as the varying parameter.

The function has been calculated for $b_y \in \{-2, -1, 0, 1, 2\}$. The graphs (fig.(2)) reveal that θ_1 is not a *mathematical function* of α and $\vec{f}$ because for a fixed pair $(\alpha, \vec{f})$, θ_1 can take several values. However when $|b_y| < 1$, θ_1 and its instantaneous variation $\dot{\theta}_1$ are functions of α and $\vec{f}$, and respectively of α, $\dot{\alpha}$, $\vec{f}$ and $\dot{\vec{f}}$. So is r.

5.3.2 Transmission Ratio r

With parameter $b_x = 0$, r is expressed as

$$r = \mathcal{R}'(\theta_1, b_y) = 1/(1 + \sin\theta_1 b_y). \tag{61}$$

In fig.(3), the transmission ratio r is plotted as a function of α for several values of b_y. The graphs has been plotted on the whole trajectory $0 \le \alpha \le 2\pi$, but due to the symmetry according to the vertical axis, it is obvious that $\mathcal{R}'(\alpha, b_y) = \mathcal{R}'(\alpha \pm \pi, -b_y)$.

Unloaded Mechanism When the load is negligible relatively to the stiffness of the actuator ($f_y \lll k_\alpha/l_1$), parameter b_y is close to 0 and r remains almost constant and equal to 1. However the graphs (fig.(3)) shows that the assumption commonly made in robotics -namely "*external loads do not modify the kinematics of a manipulator*"- is valid only for a small range of external loads. For example, if the desired precision of the mechanism is $\pm 1\mu$m, the maximal value of b_y is $\sqrt{2}/2.10^{-3}$, which means that for $l_1 = 1$m and $k_\alpha = 1000$Nm.rad^{-1}, f_y must be smaller that 0.7N. Depending on the application, this limit can be easily reached.

Nevertheless, since the above KSM enables to quantify the impact of the loads on kinematics, there is no reason to keep a distinction between a loaded or an unloaded mechanism. Neglecting the loads would just lead to a loss of accuracy.

Loaded Mechanism When the external loads are sizeable relatively to the stiffness of the actuator, parameter r becomes an indispensable data required to compute the kinematics of a mechanism with accuracy.

Critical Load When $b_y \cos\theta_1 = -1$, i.e $k_I = -k_\alpha$, the ratio r is not defined, which means that the configuration of the mechanism becomes unstable and the control is locally lost. For example, the infinitesimal motion of the mechanism $\dot{\theta}_1$ can be different from 0 even if the actuator is set ($\dot{\alpha} = 0$), despite eq.(55) that is written as $\dot{\theta}_1 = r\dot{\alpha}$.

Beyond Critical Load To illustrate the behaviour of the overloaded mechanism, two graphics have been plotted for $\vec{f} = [0, -2k_\alpha/l_1, 0]^T$ (Twice the critical load). Fig.(4) shows the values of θ_1 that satisfy the static equilibrium for $\alpha \in [-\pi, \pi]$. It can be noticed that in a certain range of α, θ_1 can take different values. These values depend on the followed path (phenomenon of hysteresis). Fig.(5) represents the mechanisms in 6 particularly interesting configurations (α is represented by the dashed vector):

- ♣ For this value of θ_1, the stiffness $(k_\alpha + k_I)$ equals 0, the configuration is unstable and the ratio r becomes undefined. Then $\dot{\theta}_1$ can be *infinite* and θ_1 can instantly *jump* to ★.

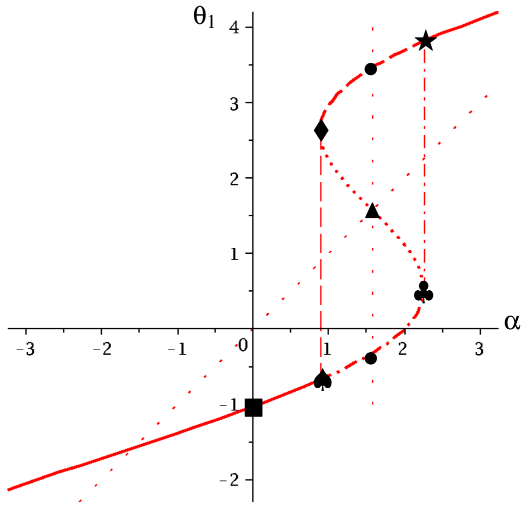

Figure 4. Trajectory of θ_1 as a function of α for an overloaded mechanism.

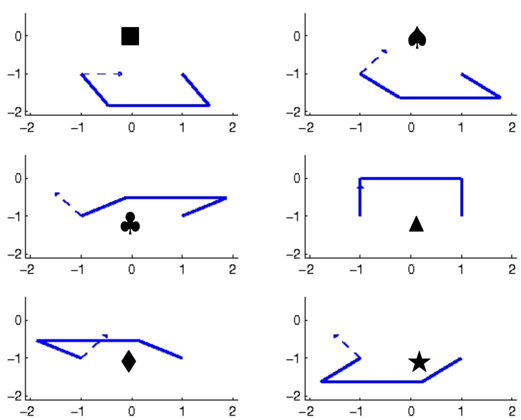

Figure 5. Some configurations of the overloaded mechanism.

- ♦ When the motion goes in the other direction (α decreasing), the unstable configuration is reached at this point and the mechanism switches to ♠.
- ▲ This configuration is theoretical, because it is unstable ($k_\alpha - k_I < 0$). In the case where some *artificial* event would have placed the mechanism in this configuration, it will move by itself to reach one of the two corresponding stable positions marked with a ●.

5.4 Contribution of this Application

In this application, the KSM and the IKSM have been implemented and used. First of all, these models enable to calculate the trajectory of the mechanism accurately whatever the applied external load. Hence, the models enable to study some interesting situations that can happen for a real 4-bar mechanism when it is subjected to a large load relatively to the power of its actuator. These models can predict the unstable configurations and provide information on what will happen in these configurations. However it is important to keep in mind the assumptions on which the models are based, namely the dynamical effects are neglected. Therefore, when a mechanism switches between to stable positions, the conditions of application of these models are no longer valid.

6 Conclusion

In this paper, a model is presented that can simultaneously consider the kinematics and the statics of parallel compliant mechanisms. This model is very general, since it works for serial and parallel mechanisms; for compliant passive joints and compliant actuators; and for any range of external or internal wrenches.

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Appendix : Stiffness matrix

In this appendix, the demonstration of the stiffness matrix of compliant parallel mechanism presented in [5] is summarized.

Potential energy of a mechanism: *A stiffness matrix is defined as the Hessian matrix of the potential energy of a mechanism.* With all the matrices and vectors introduced in this paper and noting $\vec{f}_m$ the wrench provided by the mechanism in reaction to the external wrench ($\vec{f}_m = -\vec{f}$), the potential energy of a mechanism is constant and is written as

$$\xi = \int \vec{\tau}_\chi^T d\vec{\chi} + \int \vec{\tau}_\lambda^T d\vec{\lambda} + \int \vec{f}_m^T d\vec{x} = C. \quad (62)$$

With equations (5) and (8), ξ can be expressed as

$$\xi = \int \vec{\tau}_\chi^T d\vec{\chi} + \int \vec{\tau}_\lambda^T \mathbf{G} d\vec{\chi} + \int (-\vec{f})^T \mathbf{J} d\vec{\chi} = C. \quad (63)$$

Static equilibrium: The derivative of the latter equation with respect to the generalized coordinates is written as

$$\frac{d\xi}{d\vec{\chi}} = \vec{\tau}_\chi + \mathbf{G}^T \vec{\tau}_\lambda - \mathbf{J}^T \vec{f} = \vec{0}. \quad (64)$$

Hessian matrix: The derivative of eq.(64) with respect to $\vec{\chi}$ gives 5 matrices:

$$\frac{d^2\xi}{d\vec{\chi}^2} = \mathbf{K}_\chi + \frac{d\mathbf{G}}{d\vec{\chi}}^T \vec{\tau}_\lambda + \mathbf{G}^T \mathbf{K}_\lambda \mathbf{G} - \frac{d\mathbf{J}}{d\vec{\chi}}^T \vec{f} - \mathbf{J}^T \mathbf{K}_C \mathbf{J} = \mathbf{0}. \quad (65)$$

Stiffness matrices: In the latter equation, the following rearrangements can be made:

$$\mathbf{K}_\chi + \left(\frac{d\mathbf{G}}{d\vec{\chi}}^T \vec{\tau}_\lambda + \mathbf{G}^T \mathbf{K}_\lambda \mathbf{G} \right) - \frac{d\mathbf{J}}{d\vec{\chi}}^T \vec{f} = \mathbf{J}^T \mathbf{K}_C \mathbf{J}. \quad (66)$$

Then, the Cartesian stiffness matrix can be isolated from the stiffness matrix of the generalized coordinates noted $\mathbf{K}_\chi$, from the matrices related to the constrained joints assembled in one matrix noted $\mathbf{K}_I$ and defined as

$$\mathbf{K}_I = (d\mathbf{G}^T / d\vec{\chi}) \vec{\tau}_\lambda + \mathbf{G}^T \mathbf{K}_\lambda \mathbf{G}, \quad (67)$$

and from the matrix related to the external wrench noted $\mathbf{K}_E$ and defined as

$$\mathbf{K}_E = -(d\mathbf{J}^T / d\vec{\chi}) \vec{f}. \quad (68)$$

Therefore, the generalized stiffness matrix of a mechanism noted $\mathbf{K}_M$ is calculated as

$$\mathbf{K}_M = \mathbf{K}_\chi + \mathbf{K}_I + \mathbf{K}_E = \mathbf{J}^T \mathbf{K}_C \mathbf{J} \quad (69)$$

And the Cartesian stiffness matrix is calculated as

$$\mathbf{K}_C = \mathbf{J}^{-T} \mathbf{K}_M \mathbf{J}^{-1} \quad (70)$$