

# Dynamic Legislative Policy Making\*

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## Abstract

We prove existence of stationary Markov perfect equilibria in an infinite-horizon model of legislative policy making in which the policy outcome in one period determines the status quo for the next. We allow for a multidimensional policy space and arbitrary smooth stage utilities, and we assume preferences and the status quo are subject to arbitrarily small shocks. We prove that all such equilibria are essentially in pure strategies and that proposal strategies are continuous almost everywhere. We establish upper hemicontinuity of the equilibrium correspondence, and we derive conditions under which each equilibrium of our model determines a unique invariant distribution characterizing long run policy outcomes. We provide a convergence theorem giving conditions under which the invariant distributions generated by stationary equilibria must be close to the core in a canonical spatial model.

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# 1 Introduction

In this paper we study policy making within a legislative body under the assumption that an agreement to replace the status quo in the current period influences the status quo in the next period. We develop a benchmark model of legislative interaction that accounts for the endogenous determination of the status quo, for the multidimensional aspects of public policy, for a wide range of policy preferences, and for the kinds of random shocks (e.g., on preferences and the environment) to which political interaction is subjected over time. We address some of the central theoretical difficulties arising in this dynamic environment, namely, the existence of equilibrium, regularity properties of the equilibrium set, and the long-run convergence of the equilibrium policy process. The model is intentionally austere, in that we do not incorporate the rich spectrum of political institutions observed in the real world, but our approach is very general and can accommodate fine institutional detail. Furthermore, although we are motivated by the application to legislatures and democratic politics, the issues we address are fundamental and would arise in a host of dynamic bargaining contexts, such as wage negotiation in labor markets, or collusion among members of a cartel, or deliberations among a board of directors, or treaty talks among states.

The model is fully dynamic, with play of the game generating an infinite sequence of policies over time. Each period begins with a status quo policy and the random draw of a legislator, who may propose any feasible policy, which is then subject to an up or down vote. The policy outcome in that period is the proposed policy if it receives the support of a “decisive” coalition of legislators, the status quo otherwise, and the status quo in the next period is determined by the outcome that prevails in the current period. This process continues ad infinitum, and in equilibrium legislators must anticipate future policy consequences of their decisions. When voting on a proposal, a legislator must compare the distribution over policy streams generated by the proposed policy with the distribution generated by the status quo, and legislators must select their proposed policies optimally in light of the future, while factoring in whether a proposed policy will garner the support of a decisive coalition. Although we do not preclude the possibility of equilibria supported by complex, history-dependent punishments, our focus is on stationary (Markov perfect) equilibria. This strengthens our existence result, at a technical level; from a practical point of view, stationarity significantly facilitates the estimation of empirical models in applications.<sup>1</sup>

We do not impose specific assumptions about the policy space or functional forms for legislators’ utility functions. Instead, we allow the set of alternatives to be a very general subset of any finite-dimensional Euclidean space defined by arbitrary smooth feasibility constraints, and we assume smooth stage utility functions but do not impose any further restrictions on preferences. Thus, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, public good economies, and distributive models in which a fixed surplus is allocated across legislators, and we obtain even a finite policy space as a special case. In addition to these features of the model, we assume that legislators’ stage utilities are subjected to transitory, publicly observed preference shocks in each period;

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<sup>1</sup>Examples of empirical applications exploiting stationarity are [Ericson and Pakes \(1995\)](#), and [Aguirregabiria and Mira \(2007\)](#). See [Maskin and Tirole \(2001\)](#) for an elaboration and further grounds for interest in stationary equilibria.

and when the policy space is infinite, we assume the transition from policy outcomes to next period's status quo is stochastic, e.g., next period's status quo is realized as the sum of the current period's policy outcome and a stochastic shock. The second of these two assumptions captures the fact that the impact of current policies may be relatively clear in the present but impossible to predict with certainty. If legislators choose the level of a durable public good that depreciates over time, for example, then it is reasonable to expect that the rate of depreciation, and therefore the stock of the public good at the beginning of the next period, is subject to uncertainty. Both of these two types of noise can be arbitrarily small.

We deduce the existence of stationary equilibria in pure strategies. Continuation values of the legislators are differentiable, and equilibrium proposal strategies are continuous almost everywhere. Moreover, we show that *all* stationary equilibria are essentially pure and that, in fact, equilibria are strict in the sense that proposers almost always have unique optimal proposals. The proof of existence of equilibrium requires the usual compactness and continuity conditions, and we rely on the two types of noise in order to establish these conditions.<sup>2</sup> The status quo transition probability gives us compactness when the policy space is infinite (it is not needed when the policy space is finite), an approach that is not novel to our paper; similar techniques have been used in the problems of intergenerational transfers in growth economies (Bernheim and Ray (1989) and Nowak (2006)) and existence of stationary equilibria in stochastic games. The role of the preference shocks in the continuity argument, however, is novel. Specifically, these shocks allow us to establish existence of equilibrium despite the fact that in our model, the action of the proposer uniquely determines the pair of alternatives (the proposal and status quo) compared at the voting stage of the game. Such deterministic transitions have proved incongruous with general existence of equilibria in stochastic games, and extant approaches in the literature assume stronger continuity conditions on state transitions and rely either on correlation of strategies or the relaxation of stationarity.<sup>3</sup> In our approach, given arbitrary continuation values for the legislators, the best response problem of a proposer can be formulated as a constrained optimization problem in which the objective function of the proposer includes his continuation value and in which the other legislators' continuation values appear in the constraints. As these continuation values are endogenous, the optimization problem of the proposer need not be convex, and the solution set need not be upper hemicontinuous as a function of continuation values. We apply the transversality theorem, however, to show that for almost all preference shocks, the proposer's problem satisfies the linear independence constraint qualification, giving us continuity of optimal proposals for almost all shocks. We thus establish existence of stationary equilibria without recourse to correlation or departures from stationarity.

Two types of noise also appear in the industrial organization literature on dynamic models of competition between firms (e.g., Aguirregabiria and Mira (2007) and Doraszelski and Satterthwaite (2009)). Doraszelski and Satterthwaite (2009), for example, assume noise on the transition from firm decisions in the current period to firm states in the next period,

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<sup>2</sup>The existence counterexample due to Harris et al. (1995), once formulated as a stochastic game, implies that some such measure must be taken to be assured of existence.

<sup>3</sup>Nowak and Raghavan (1992) prove existence of correlated equilibria, and Chakrabarti (1999) provides results on semi-Markov stationary equilibria. Both require norm continuity of the state transition; see Nowak (2007) for a recent overview and Duggan and Kalandrakis (2007) for a detailed literature review.

as well as idiosyncratic shocks to each firm’s scrap value/setup cost. In their setting, the idiosyncratic shocks are private information and are used to purify entry/exit decisions of firms. In our setting the preference shocks are publicly observed, but similarly ensure that proposers’ best responses are strict, enhancing the computational tractability of our model; we address this in separate work (Duggan et al. (2008), Duggan and Kalandrakis (2009b)).<sup>4</sup> In contrast to our equilibrium existence argument, however, the preference shocks do not play a role in the continuity of each firm’s optimal level of investment as a function of continuation values, which these authors ensure using a separate sufficient condition.<sup>5</sup> Moreover, these authors assume a finite set of states for each firm, circumventing compactness issues. A byproduct of our existence argument is upper hemicontinuity of the equilibrium correspondence with respect to the parameters of the model. Of note, we include the policy space itself as a parameter, allowing us to consider finite approximations of an infinite policy space, and we apply this upper hemi-continuity result in Duggan and Kalandrakis (2009b) to obtain an equilibrium of the continuum model by taking the limit of equilibria generated by finite approximations of the policy space.

We also give conditions under which each equilibrium admits a unique invariant distribution with desirable ergodic properties, providing an unambiguous prediction of long run policy outcomes generated by the equilibrium. We specialize to the multidimensional spatial model in which legislative preferences are close to admitting a core policy that cannot be overturned by a decisive coalition, and we provide bounds on equilibrium policies induced by proximity to the spatial model in which a core policy exists. Our equilibrium bounds from the core hold when the stochastic shocks in our model are small and legislative preferences are close to admitting a unique core policy, the ideal point of a “core legislator,” and so we approximate a setting in which a social-choice analysis yields an unambiguous prediction on the basis of individual preferences alone (abstracting from the institutional details of the bargaining process). We show that the invariant distributions over policy outcomes generated by stationary legislative equilibria must be close, in the sense of weak convergence, to the point mass on the core policy. In the one-dimensional special case, the core is always non-empty and coincides with the ideal policy of the median legislator, and the equilibria of our model generate long run policy outcomes close to the median legislator’s ideal point with high probability. Thus, we reconcile the static prediction of the median voter theorem with the strategic incentives of farsighted agents in the context of dynamic bargaining. In the multidimensional setting, we allow the core to be empty as long as legislative preferences approximate the canonical model in which a core policy exists. We therefore generalize the results of Ferejohn et al. (1984), who show that myopic majority voting concentrates probability near the core policy of the canonical model, to a setting in which voting is farsighted and policy transitions are governed by the optimal proposals of strategic agents. Our results hold for an arbitrary, fixed discount factor—they do not rely on any assumptions regarding the patience of the legislators—distinguishing them from “core equivalence” results of Banks and Duggan (2000, 2006a). In

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<sup>4</sup>The role of pure strategies for the computational tractability of dynamic games is highlighted by Herings and Peeters (2004) and Nowak (2007) and emphasized by Doraszelski and Satterthwaite (2009) in the context of a dynamic oligopoly model.

<sup>5</sup>Their Condition 1 ensures this by imposing a concavity condition directly on the transition probability on firm states.

the framework of those papers, the game ends once an agreement is reached, so the influence of the core is mitigated unless agents are patient, in which case an alternative close to the core is proposed immediately. In the dynamic framework of this paper, however, the legislators may be impatient and policies far from the core may be reached with positive probability following any history; but with high probability they will be close to the core in the long run.

In Section 2, we give a detailed review of the literatures in political bargaining and stochastic games, as well as the related literature in dynamic industrial organization. In Section 3, we present the model formally and describe our solution concept. In Section 4, we state our existence, characterization, and robustness results. In Section 5, we study ergodic properties of equilibria. In Section 6, we analyze the long run equilibrium policies of our model as the stochastic shocks become small and the preferences of the legislators are close to admitting a core policy. We conclude in Section 7, and we collect all proofs in Appendix A.

## 2 Literature Review

Most of the existing work in political economy on bargaining considers an infinite-horizon game where in each period one agent proposes a division of surplus and that proposal is either accepted, in which case the game ends with the proposed outcome, or rejected, in which case bargaining continues for at least one more round. [Baron and Ferejohn \(1989\)](#) extend the models of [Rubinstein \(1982\)](#) and [Binmore \(1987\)](#) to cover legislative politics by allowing for an arbitrary number of legislators and requiring the support of a majority to pass a proposal. A substantial literature cutting across economics and political science has grown from these papers, but most assume that bargaining terminates once a proposal passes. While these models can be used to examine policy choices across legislative sessions by simply repeating the bargaining game each session, this is appropriate only if policies remain in place for a single session with an exogenously fixed default outcome at the beginning of the next. This is often the case in budgetary negotiations, but the model is inadequate for the analysis of the enactment of legal statutes or continuing legislation, where policy remains in place for the indefinite future and endogenously determines the status quo in subsequent periods.

A growing literature considers the effects of endogenizing the status quo. In this framework, each period begins with a status quo, then one agent makes a proposal, and that proposal is either accepted, in which case it becomes the current policy and the status quo for the next period, or rejected, in which case the current status quo remains in place until next period. In any case, the process is repeated next period, and so on. Extant studies provide constructions of stationary equilibria in special cases of the model. [Baron \(1996\)](#) analyzes the one-dimensional version of the model with single-peaked stage utilities. [Kalandrakis \(2004, 2009\)](#) establishes existence and continuity properties of equilibrium strategies in the distributive model, obtains a fully strategic version of [McKelvey's \(1979\)](#) dictatorial agenda setting result in that setting, and studies the composition of equilibrium coalitions and the effect of risk-aversion on equilibrium.<sup>6</sup> [Baron and Herron \(2003\)](#) give a numerical calculation of equilibrium in a three-legislator, finite-horizon model. [Fong \(2005\)](#) considers

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<sup>6</sup>In contrast, [Epple and Riordan \(1987\)](#) allow for history dependent strategies and derive folk theorem results in the distributive model.

a three-legislator model in which policies consist of locations in a two-dimensional space and allocations of surplus. [Cho \(2005\)](#) analyzes policy outcomes in a similar environment but with a stage game emulating aspects of parliamentary government. Similar in spirit to the above, [Battaglini and Coate \(2007b\)](#) characterize stationary equilibria in a model of public good provision and taxation with identical legislators and a stock of public goods that evolves over time. [Battaglini and Coate \(2007a\)](#) consider a dynamic model of public spending and taxation in which the state variable is the amount of public debt. All of the above analyses of stationary equilibria consist of explicitly constructing equilibrium strategies, which, given the dependence of proposals on the status quo, can be extremely complex. [Battaglini and Palfrey \(2007\)](#) study a discrete three-player distributive model and [Battaglini et al. \(2010\)](#) study a durable public good environment, finding that equilibrium predictions are roughly consistent with experimental data.<sup>7</sup> [Diermeier and Fong \(2008\)](#) characterize the pure strategy stationary equilibria in a discretized distributive model in which one player has monopoly agenda setting power and players are patient.

A number of related papers diverge in various ways from the above literature and our model. [Acemoglu et al. \(2008\)](#) prove existence and characterize pure strategy stationary equilibria in a finite model with endogenous status quo, assuming a small transition cost and sufficiently patient players but allowing the voting rule to vary with the state. [Bernheim et al. \(2006\)](#) analyze a model of a single policy choice in which the proposal on the floor is subject to change over time, and after a fixed number of rounds, the implemented policy is determined by a final up or down vote between the proposal offered in the last round and the previous proposal on the floor. The authors assume a finite policy space and strict preferences over policies for all legislators, so that backward induction yields a unique equilibrium outcome. They then extend the model to a finite number of policy choices over time, with the finite horizon again permitting backward induction. [Penn \(2009\)](#) considers a dynamic voting game with randomly generated policy proposals and probabilistic voting on these proposals. [Lagunoff \(2005a,b\)](#) investigates a class of stochastic games that incorporate a social choice solution concept and analyzes endogenous political institutions. Finally, [Gomez and Jehiel \(2005\)](#) consider a class of stochastic games and characterize efficiency properties of equilibrium when players are patient. Unlike our model, they assume a finite number of states and transferable utility.

### 3 Legislative Model

**Framework** We posit a policy space  $X \subseteq \mathfrak{R}^d$  and a finite set  $N$  of legislators,  $i = 1, \dots, n$ , who determine policy over an infinite horizon. Legislative interaction proceeds as follows in each period  $t = 1, 2, \dots$ . A status quo policy  $q \in \mathfrak{R}^d$  and a vector  $\theta = (\theta_1, \dots, \theta_n) \in \mathfrak{R}^{nm}$  of preference parameters are realized and publicly observed. A legislator  $i$  is drawn at random, with fixed probabilities  $p_1, \dots, p_n$ ,<sup>8</sup> to propose a policy  $y \in X(q)$ ,

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<sup>7</sup>The latter authors find some evidence of non-Markovian behavior, though observed long run levels of the public good match equilibrium predictions well.

<sup>8</sup>It is straightforward to extend the analysis to allow proposal probabilities to depend measurably on the status quo  $q$ . To obtain our equilibrium bounds from the core, in Section 6, we would assume, e.g., that the core legislator's recognition probability has a positive lower bound.

where  $X(q) \subseteq \mathfrak{R}^d$  represents the set of feasible policies at status quo  $q$ . The legislators vote simultaneously to accept  $y$  or reject it in favor of the status quo  $q$ .<sup>9</sup> The proposal passes if a coalition  $C \in \mathcal{D}$  of legislators vote to accept, and it fails otherwise; here,  $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$  is a collection of *decisive coalitions* satisfying only the minimal monotonicity requirement that if one coalition is decisive and we add legislators to that coalition, then the larger coalition is also decisive. Formally, we assume that if  $C \in \mathcal{D}$  and  $C \subseteq C' \subseteq N$ , then  $C' \in \mathcal{D}$ .<sup>10</sup> The policy outcome for the current period, denoted  $x$  in general, is  $y$  if the proposal passes and is  $q$  otherwise. Each legislator  $j$  receives utility  $u_j(x, \theta_j)$ , where  $\theta_j \in \mathfrak{R}^m$  is a utility shock for legislator  $j$ . Finally, the status quo  $q'$  for the next period is drawn from the density  $g(\cdot|x)$ , a new vector  $\theta' = (\theta'_1, \dots, \theta'_n)$  of preference shocks is drawn from the density  $f(\cdot)$  and publicly observed, and the above procedure is repeated in period  $t + 1$ . Payoffs in the dynamic game are given by the expected discounted sum of stage utilities, and we denote the discount factor of legislator  $j$  by  $\delta_j \in [0, 1)$ .

We impose a number of regularity conditions on the model. For future reference, we partition them into assumptions on the set of feasible policies, preference shocks, and status quo transition. **(A1)** We assume that the set of feasible policies at status quo  $q$ , denoted  $X(q)$ , is cut out by a finite number  $k$  of functions  $h_\ell: \mathfrak{R}^d \times \mathfrak{R}^d \rightarrow \mathfrak{R}$  indexed by  $K = \{n+1, \dots, n+k\}$ , where we partition  $K$  into inequality constraints,  $K^{in}$ , and equality constraints,  $K^{eq}$ . Furthermore, we assume  $X(q)$  always contains the status quo, so that

$$X(q) = \{q\} \cup \{x \in \mathfrak{R}^d : h_\ell(x, q) \geq 0, \ell \in K^{in}, h_\ell(x, q) = 0, \ell \in K^{eq}\}.$$

We further assume that  $X$  is compact, that  $X(q) \subset X$  for all status quos  $q$ , and that  $h_\ell$  is measurable in  $q$  and  $r$ -times continuously differentiable in  $x$  for all  $\ell \in K$ , where we maintain the assumption that  $r \geq \max\{2, d\}$ .<sup>11</sup> For technical reasons, we impose a weak linear independence on the gradients of binding feasibility constraints. Formally, let  $K(x, q)$  denote the subset of  $\ell \in K$ , including equality constraints, such that  $h_\ell(x, q) = 0$ ; we then assume that for all  $q$  and for all  $x \in X(q) \setminus \{q\}$ ,  $\{D_x h_\ell(x, q) : \ell \in K(x, q)\}$  is linearly independent. With assumption **(A1)**, we capture standard models with resource and consumption constraints, such as the classical spatial model of politics, public good economic environments, and distributive models in which an amount of surplus is to be allocated among the legislators' districts. Equality constraints allow us to capture quite general policy spaces, and in particular we obtain an arbitrary finite set of feasible policies as a special case.

The presence of preference shocks in the model captures uncertainty about the legislators'

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<sup>9</sup>Equilibrium policies would be unaffected if voting were sequential. In that version of the model, we could drop our stage-dominance refinement on voting strategies, below, but the formal description of voting strategies would be somewhat more complicated.

<sup>10</sup>We capture majority rule in the obvious way, by setting  $\mathcal{D} = \{C \subseteq N : |C| > \frac{n}{2}\}$ , and we obtain unanimity rule and other quota rules similarly. See [Banks and Duggan \(2000, 2006a\)](#) for examples of more complex voting rules captured by these assumptions. Our results extend to the case in which the voting rule depends on the status quo, i.e.,  $\mathcal{D}(q)$ , as long as the correspondence  $\mathcal{D}: \mathfrak{R}^d \rightrightarrows 2^N$  is measurable. To formulate our equilibrium bounds from the core, in Section 6, we require that  $\mathcal{D}$  is independent of the status quo.

<sup>11</sup>Of course, we allow  $r = \infty$ . Later, we assume stage utilities and the status quo density  $g(q|\cdot)$  are  $r$ -times continuously differentiable, and we deduce that equilibrium continuation values are  $r$ -times continuously differentiable. By allowing  $r$  to exceed  $\max\{2, d\}$ , we see that continuation values inherit the differentiability properties imposed on these primitives.

future policy preferences. **(A2)** We assume that the stage utility  $u_i: \mathfrak{R}^d \times \mathfrak{R}^m \rightarrow \mathfrak{R}$  is  $r$ -times continuously differentiable and that  $D_{\theta_i}[u_i(x, \theta_i) - u_i(x', \theta_i)] \neq 0$  for all distinct  $x, x' \in \mathfrak{R}^d$  and every legislator  $i$ . This assumption plays a key role in the derivation of Lemmas 1 to 3 in the appendix ensuring, among other things, that legislators have unique optimal proposals for all status quos  $q$  and almost all preference shocks  $\theta_i$ . An example is  $m = d$  and  $u_i(x, \theta_i) = \hat{u}_i(x) + \theta_i \cdot x$ , where  $\hat{u}_i: \mathfrak{R}^d \rightarrow \mathfrak{R}$  is  $r$ -times continuously differentiable. If  $X$  is a finite set, say  $\{x_1, \dots, x_m\}$ , then we capture the special case in which  $\theta_i = (\theta_{i,1}, \dots, \theta_{i,m}) \in \mathfrak{R}^m$  consists of additive preference shocks of the form  $u_i(x_j, \theta_i) = \hat{u}_i(x_j) + \theta_{i,j}$ . We assume that the vector of preference shocks  $\theta = (\theta_1, \dots, \theta_n)$  is drawn independently across periods from a density  $f$  with respect to Lebesgue measure, and we assume a bound  $b_f$  and an open set  $\Theta \subseteq \mathfrak{R}^{nm}$  containing the support of  $f$  such that  $|u_i(x, \theta_i)|f(\theta) \leq b_f$  for all  $i \in N$ , all  $\theta \in \Theta$ , and all  $x \in X$ .

Noise on the status quo reflects uncertainty about the way policy decisions today will be implemented in the future. **(A3)** We assume that for each  $x$  the distribution of  $q$  is absolutely continuous with respect to some Borel measure, that the density  $g(q|x)$  relative to this measure has support contained in  $X$ , that  $g: \mathfrak{R}^d \times \mathfrak{R}^d \rightarrow \mathfrak{R}$  with values  $g(q|x)$  is jointly measurable in  $(q, x)$ , and that  $g$  is bounded on  $X \times X$ . For future use, we let  $\mu_\theta$  denote Lebesgue measure on  $\mathfrak{R}^{nm}$  and  $\mu_q$  the fixed Borel measure on  $\mathfrak{R}^d$ , and we let  $\mu$  denote the product measure on  $\mathfrak{R}^d \times (\mathfrak{R}^{nm})$  with marginal  $\mu_q$  on  $\mathfrak{R}^d$  and  $\mu_\theta$  on  $\mathfrak{R}^{nm}$ , i.e.,  $\mu = \mu_q \times \mu_\theta$ . Furthermore, we assume a bound  $b_g$  such that for all  $q$ , we have:  $g(q|x)$  is  $r$ -times continuously differentiable in  $x$ ; if  $r < \infty$ , then all derivatives of order  $1, \dots, r$  are bounded in norm by  $b_g$ , and the  $r$ -th derivative of  $g(q|x)$  with respect to  $x$  is Lipschitz continuous with modulus  $b_g$ ; and if  $r = \infty$ , then derivatives of all orders  $1, 2, \dots$  are bounded in norm by  $b_g$ . Our setup allows for the possibility that  $\mu_q$  is discrete, and in the special case that the policy space  $X$  is finite, we can specify  $g$  so that the transition is deterministic, e.g., outcome  $x$  in the current period determines status quo  $q = x$  in the next period. For later reference, let  $b = (b_f, b_g)$  denote the vector of bounds described above.

Our approach to existence involves the addition of noise to policy outcomes and legislator utilities, but we emphasize that the status quo and the utility shocks at the beginning of a period  $t$  are commonly known, so that a proposer knows whether any given policy will pass or fail if proposed. Furthermore, once a vote is taken, the policy outcome is pinned down for the current period: the legislators know, contingent on the outcome of voting, what the policy in the current period will be, and a new status quo is drawn for next period only after legislators receive their current utilities. Thus, our formulation of noise in the model is consistent with the view that while legislators are completely informed in the current period, there is some uncertainty about future policy preferences and the policy environment. We view these as natural modeling assumptions. In any case, the variance of the distributions  $f$  and  $g(\cdot|x)$  may be arbitrarily small. Thus, we allow for the selection of preference shocks and the status quo to be arbitrarily close to deterministic, so that the element of noise in the model can be made negligible from a substantive standpoint.

**Strategies and Payoffs** A strategy in the game consists of two components, one giving the proposals of legislators when recognized to propose and the other giving the votes of legislators after a proposal is made. While these choices can conceivably depend on histories arbitrarily, we seek subgame perfect equilibria in which legislators use stationary Markov



strategies, which we denote  $\sigma_i = (\pi_i, \alpha_i)$ . Our main focus will be on pure strategies, which, as we show in Theorem 2, is without loss of generality. Thus, legislator  $i$ 's proposal strategy is a measurable mapping  $\pi_i: \mathfrak{R}^d \times \Theta \rightarrow \mathfrak{R}^d$ , where  $\pi_i(q, \theta)$  is the policy proposed by  $i$  given status quo  $q$  and utility shocks  $\theta$ . And legislator  $i$ 's voting strategy is a measurable mapping  $\alpha_i: \mathfrak{R}^d \times \mathfrak{R}^d \times \Theta \rightarrow \{0, 1\}$ , where  $\alpha_i(y, q, \theta) = 1$  if  $i$  accepts proposal  $y$  given status quo  $q$  and utility shocks  $\theta$  and  $\alpha_i(y, q, \theta) = 0$  if  $i$  rejects. We let  $\sigma = (\sigma_1, \dots, \sigma_n)$  denote a stationary strategy profile. We may equivalently represent voting strategies by the set of feasible proposals a legislator would vote to accept. We define this *acceptance set* for  $i$  as  $A_i(q, \theta; \sigma) = \{y \in X(q) : \alpha_i(y, q, \theta_i) = 1\}$ . Letting  $C$  denote a coalition of legislators, we define

$$A_C(q, \theta; \sigma) = \bigcap_{i \in C} A_i(q, \theta; \sigma) \quad \text{and} \quad A(q, \theta; \sigma) = \bigcup_{C \in \mathcal{D}} A_C(q, \theta; \sigma)$$

as the *coalitional acceptance set* for  $C$  and the *legislative acceptance set*, respectively. The latter consists of all policies that would receive the votes of all members of at least one decisive coalition, and would therefore pass if proposed. The strategy profile  $\sigma$  is *no-delay* if for all  $i$ , all  $q$ , and all  $\theta$ ,  $\pi_i(q, \theta) \in A(q, \theta; \sigma)$ , so that proposals are always accepted.

Given strategy profile  $\sigma$ , we define legislator  $i$ 's dynamic policy preferences by

$$U_i(x, \theta_i; \sigma) = (1 - \delta_i)u_i(x, \theta_i) + \delta_i v_i(x; \sigma),$$

where  $v_i(x; \sigma)$  is  $i$ 's continuation value at the beginning of period  $t + 1$  from policy outcome  $x$  in period  $t$ .<sup>12</sup> Our measurability assumptions on strategies imply that continuation values are also measurable, and continuation values have the form

$$v_i(x; \sigma) = \int_q \int_\theta \sum_j p_j U_i(\pi_j(q, \theta), \theta_i; \sigma) f(\theta) g(q|x) d\mu \quad (1)$$

for every no-delay strategy profile.

To extend these ideas to allow for mixing and non-deferential voting, we let  $\bar{\pi}_i: \mathfrak{R}^d \times \Theta \rightarrow \mathcal{P}(\mathfrak{R}^d)$  denote a mixed proposal strategy, where  $\mathcal{P}(\mathfrak{R}^d)$  is the set of Borel probability measures on  $\mathfrak{R}^d$ . We equip this space with the weak\* topology, and we assume  $\bar{\pi}_i$  is Borel measurable. Here,  $\bar{\pi}_i(q, \theta)$  represents the distribution of  $i$ 's policy proposal given status quo  $q$  and shocks  $\theta$ . We define voting strategies as measurable mappings  $\bar{\alpha}_i: \mathfrak{R}^d \times \mathfrak{R}^d \times \Theta \rightarrow [0, 1]$ , where now  $\bar{\alpha}_i(y, q, \theta)$  is the probability, ranging between zero and one, that  $i$  accepts proposal  $y$  given  $q$  and  $\theta$ . A mixed strategy for legislator  $i$  is then  $\bar{\sigma}_i = (\bar{\pi}_i, \bar{\alpha}_i)$ , and we let  $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_n)$  denote a mixed strategy profile. Given a profile  $\bar{\sigma}$  of mixed strategies, we define induced preferences  $U_i(y, \theta_i; \bar{\sigma})$  as above, but the legislators' continuation values now have the following more complicated form:

$$v_i(x; \bar{\sigma}) = \int_q \int_\theta \sum_j p_j \int_y [\bar{\alpha}_i(y, q, \theta; \bar{\sigma}) U_i(y, \theta_i; \bar{\sigma}) + (1 - \bar{\alpha}_i(y, q, \theta; \bar{\sigma})) U_i(q, \theta_i; \bar{\sigma})] \bar{\pi}_j(q, \theta)(dy) f(\theta) g(q|x) d\mu, \quad (2)$$

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<sup>12</sup>Note that these continuation values are “ex ante,” in the sense that they are calculated at the beginning of the period, before  $q$  and  $\theta$  are realized.

where

$$\bar{\alpha}(y, q, \theta; \bar{\sigma}) = \sum_{C \in \mathcal{D}} \left( \prod_{j \in C} \bar{\alpha}_j(y, q, \theta) \right) \left( \prod_{j \notin C} (1 - \bar{\alpha}_j(y, q, \theta)) \right)$$

is the probability that a proposal  $y$  is accepted by a decisive coalition of legislators. Note that, in (2), we now integrate over the policies proposed by each legislator  $i$ , and given a realization  $y$  from the mixed proposal strategy we now account for the possibility that  $y$  may pass with a probability intermediate between zero and one.

**Legislative Equilibrium** We focus on a class of stationary Markov perfect equilibria, a refinement that precludes more complicated forms of history-dependence. Due to their relative simplicity, such strategies minimize the difficulty of strategic calculations and may therefore possess a focal quality. Intuitively, we require that legislators always propose optimally and that they always vote in their best interest. It is well-known that the latter requirement is unrestrictive in simultaneous voting games, however, as arbitrary outcomes can be supported by Nash equilibria in which no voter is pivotal. To address this difficulty, we follow the standard approach of refining the set of Nash equilibria in voting subgames by requiring that legislators delete votes that are dominated in the stage game. Thus, we say a strategy profile  $\sigma$  is a *pure stationary legislative equilibrium* if the following conditions hold:

- for all shocks  $\theta$ , every status quo  $q$ , every proposal  $y$ , and every legislator  $i$ ,

$$\alpha_i(y, q, \theta_i) = \begin{cases} 1 & \text{if } U_i(y, \theta_i; \sigma) \geq U_i(q, \theta_i; \sigma) \\ 0 & \text{else.} \end{cases}$$

- for all shocks  $\theta$ , every status quo  $q$ , and every legislator  $i$ ,  $\pi_i(q, \theta)$  solves

$$\max_{y \in A(q, \theta; \sigma)} U_i(y, \theta; \sigma).$$

This notion will be the main equilibrium concept of our analysis. It requires that legislators use pure strategies; it precludes delay, because the optimal proposal problem of a proposer is restricted to the legislative acceptance set; and it not only imposes the requirement that legislators eliminate stage-dominated voting strategies, but it also builds in the feature that voters defer to the proposer when indifferent. Formally, we say a profile  $\sigma$  is *deferential* if for all  $i$ , all  $q$ , all  $\theta$ , and all  $y \in X$ ,  $U_i(y, \theta_i; \sigma) = U_i(q, \theta_i; \sigma)$  implies  $\alpha_i(y, q, \theta_i) = 1$ . Thus, our notion of equilibrium is relatively restrictive.

In contrast, we also define the following, conceptually less restrictive notion of equilibrium. We say a profile  $\bar{\sigma}$  of mixed strategies is a *mixed stationary legislative equilibrium* if

- for all shocks  $\theta$ , every status quo  $q$ , every proposal  $y$ , and every legislator  $i$ ,

$$\bar{\alpha}_i(y, q, \theta_i) = \begin{cases} 1 & \text{if } U_i(y, \theta_i; \bar{\sigma}) > U_i(q, \theta_i; \bar{\sigma}) \\ 0 & \text{if } U_i(y, \theta_i; \bar{\sigma}) < U_i(q, \theta_i; \bar{\sigma}). \end{cases}$$

- for all shocks  $\theta$ , every status quo  $q$ , and every legislator  $i$ ,  $\bar{\pi}_i$  puts probability one on solutions to

$$\max_{y \in X \cup \{q\}} \bar{\alpha}(y, q, \theta; \bar{\sigma}) U_i(y, \theta_i; \bar{\sigma}) + (1 - \bar{\alpha}(y, q, \theta; \bar{\sigma})) U_i(q, \theta_i; \bar{\sigma}).$$

One difference between this notion of equilibrium and that of pure stationary legislative equilibrium is that it allows a legislator, when he has multiple optimal proposals, to mix over those proposals. A second difference is that it allows a legislator to accept with arbitrary probability when indifferent between a proposed policy and the status quo. Consistent with stage-game weak dominance, however, the vote of a legislator with a strict preference is pinned down uniquely. This complicates the optimization problem of a proposer, as the utility maximizing policies in the legislative acceptance set may no longer pass with probability one.

We say that a mixed strategy profile  $\bar{\sigma}$  is *equivalent* to a strategy profile  $\sigma$  if for all  $q$  and almost all  $\theta$ , the policy outcome determined by  $(q, \theta)$  is  $\pi_i(q, \theta)$  with probability one. Formally, for all  $q$ , there exists a measure zero set  $\Theta(q) \subseteq \Theta$  such that for all  $i$  and all  $\theta \notin \Theta(q)$ , we have: (i) if  $\pi_i(q, \theta) \neq q$ , then  $i$  proposes  $\pi_i(q, \theta)$  and this passes with probability one, i.e.,  $\bar{\pi}_i(q, \theta)(\{\pi_i(q, \theta)\}) = \bar{\alpha}(\pi_i(q, \theta), q, \theta; \bar{\sigma}) = 1$ , and (ii) if  $\pi_i(q, \theta) = q$ , then no proposal other than  $\pi_i(q, \theta)$  passes with positive probability, i.e.,  $\int_{X \setminus \{q\}} \bar{\alpha}(y, q, \theta; \bar{\sigma}) \bar{\pi}_i(q, \theta)(dy) = 0$ . We consider two cases in the preceding definition because there are two, payoff equivalent ways the status quo can prevail during a given period—the status quo can be proposed and pass or a proposal can be rejected—either of which suffices for the definition of equivalence. We will see that every mixed stationary legislative equilibrium is essentially pure, in the sense just defined, so that the added conceptual flexibility afforded by mixed strategies is not realized in equilibrium.

## 4 Existence and Robustness

In this section, we take up the existence and characterization of pure and mixed stationary legislative equilibria, and we analyze the robustness of equilibria. The main result of this section is that there is a stationary legislative equilibrium satisfying a number of desirable regularity properties.

**Theorem 1** *There exists a pure stationary legislative equilibrium,  $\sigma$ , possessing the following properties.*

1. *Continuation values are smooth: for every legislator  $i$ ,  $v_i(x; \sigma)$  is  $r$ -times continuously differentiable as a function of  $x$ .*
2. *Proposals are almost always strictly best: for every status quo  $q$ , almost all shocks  $\theta$ , every legislator  $i$ , and every  $y \in A(q, \theta; \sigma)$  distinct from the proposal  $\pi_i(q, \theta)$ , we have  $U_i(\pi_i(q, \theta), \theta_i; \sigma) > U_i(y, \theta_i; \sigma)$ .*
3. *Proposal strategies are almost always continuous: for every status quo  $q$ , almost all shocks  $\theta$ , and every legislator  $i$  such that  $\pi_i(q, \theta) \neq q$ ,  $\pi_i(q, \theta)$  is continuous at  $(q, \theta)$ .*

4. *Binding voters, if any, are almost always not redundant: for every status quo  $q$ , almost all shocks  $\theta$ , and every legislator  $i$ , if  $\pi_i(q, \theta) \neq q$  and there exists  $j$  such that  $U_j(\pi_i(q, \theta), \theta_j; \sigma) = U_j(q, \theta_j; \sigma)$ , then*

$$\{\ell \in N : U_\ell(\pi_i(q, \theta), \theta_\ell; \sigma) \geq U_\ell(q, \theta_\ell; \sigma)\} \setminus \{j\} \notin \mathcal{D}.$$

Part 1 of Theorem 1 establishes that equilibrium continuation values inherit the differentiable structure of the components of the model,  $u_i$ ,  $h_\ell$ , and  $g$ . By part 2, the equilibrium exhibited in Theorem 1 is, in a sense, strict: for almost all realizations of noise on preferences and the status quo, a proposer has a unique optimal policy choice. Part 3 of the theorem establishes a potentially useful technical property of equilibrium policy proposals: although equilibrium policy strategies will generally be discontinuous, they are continuous on an open set of full measure.<sup>13</sup> Part 4 of Theorem 1 establishes conditions under which a proposer will form minimal winning coalitions. We show that for all  $q$  and almost all  $\theta$ , if the proposer is “constrained,” in the sense that the optimal policy proposal renders any legislator indifferent between the proposal and the status quo, then all legislators who are indifferent between the proposal and the status quo are necessary coalition partners: the proposal fails if we remove any such legislator’s assent. Thus, no legislator outside a decisive coalition that support an equilibrium proposal can be indifferent between the proposal and the status quo. This is reminiscent of Riker’s (1962) size principle, which maintains that winning coalitions are of minimal size necessary in order for a proposal to pass, and no larger. Part 4 can be viewed as a formalization of the size principle in a general, non-cooperative, dynamic model of policy making. Note that there is nothing in the logic of equilibrium that precludes the possibility that a larger than minimum winning coalition of legislators strictly prefer a legislator’s proposal to the status quo, though in such cases part 4 implies that there is almost always no legislator who is exactly indifferent between the proposal and the status quo.

As expected, the proof of Theorem 1 proceeds by defining a suitable mapping, establishing the existence of a fixed point, and then verifying that it corresponds to a stationary legislative equilibrium with the claimed properties. To give intuition for the key steps in the proof, we focus on the case  $r = \infty$ . Let  $C^\infty(\mathbb{R}^d, \mathbb{R}^n)$  denote the space of smooth mappings from  $\mathbb{R}^d$  to  $\mathbb{R}^n$ , endowed with the topology of  $C^\infty$ -uniform convergence on compacta.<sup>14</sup> Given a vector  $v = (v_1, \dots, v_n) \in C^\infty(\mathbb{R}^d, \mathbb{R}^n)$  of continuation value functions, define  $U_i(y, \theta; v)$  and  $A_i(q, \theta; v)$ , in the obvious way, as the induced utilities and acceptance sets when continuation values are given by  $v$ . Consider a legislator  $i$ ’s optimal proposal problem,

$$\max_{y \in A(q, \theta; v)} U_i(y, \theta_i; v), \tag{3}$$

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<sup>13</sup>In part 3 of Theorem 1, we state that equilibrium proposal strategies are almost everywhere continuous in order to conserve space. See the working paper version, [Duggan and Kalandrakis \(2007\)](#), for a more in-depth analysis where we prove almost everywhere differentiability of proposal strategies. There, we actually demonstrate that for almost all realizations of noise, the optimal proposal problem of an agent can be written as a standard optimization problem with mixed constraints; furthermore, we show that the linear independence constraint qualification holds almost always, giving us a characterization of equilibrium proposals in terms of the Kuhn-Tucker first order conditions.

<sup>14</sup>See the appendix for the precise definition of this topology.

and let  $\pi_i(q, \theta; v)$  denote a selection from the solutions to this program. This selection determines a vector of “best response” continuation values,  $\hat{v} = (\hat{v}_1, \dots, \hat{v}_n)$ , for the legislators, and we define  $\psi$  as the mapping that takes the vector  $v$  to the vector  $\hat{v}$ , i.e.,  $\psi(v) = \hat{v}$ . The existence proof consists in verifying that  $\psi$  satisfies the conditions of Glicksberg’s fixed point theorem. The noise on the status quo plays the standard role of smoothing out continuation values and allowing us to restrict the domain and range of  $\psi$  to a compact subset of  $C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ . To see how this technique is applied in our setting, note that the new continuation value  $\hat{v}_i$  of legislator  $i$  is defined by

$$\hat{v}_i(x) = \int_q \int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\mu, \quad (4)$$

and note further that the current period’s policy choice  $x$  enters this continuation value only through the density  $g(q|x)$ . Thus,  $\hat{v} = \psi(v)$  is, in essence, the convolution of the function  $\int_\theta \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) d\mu_\theta$ , which is generally discontinuous in  $q$ , with the function  $g(q|x)$ . The result is a smooth function of the policy outcome  $x$ . Furthermore, if we define  $\mathcal{V}$  as the compact space consisting of all functions  $v \in C^\infty(\mathbb{R}^d, \mathbb{R}^n)$  such that  $v$  is appropriately bounded, then it is straightforward to verify that  $\psi$  maps  $\mathcal{V}$  into itself.

As discussed in the introduction, the proof of existence must overcome a difficult continuity issue. Given continuation values  $v$ , the best response continuation values  $\hat{v}$  in (4) are determined by the solutions,  $\pi_j(q, \theta; v)$ , to the optimal proposal problems of the legislators. The objective function  $U_i(\cdot, \theta_i; v)$  in (3) incorporates the continuation value of the proposer, and best responses are further intermediated by the other legislators’ continuation values through the constraints  $A(q, \theta; v)$ . Since these quantities are endogenous, we cannot assume a priori that the optimal proposal problems are well-behaved,<sup>15</sup> and it is not in general possible to take a continuous selection from a legislator’s optimal proposal problem. The preference shocks allow us, however, to take an almost everywhere continuous selection, which is sufficient for our purposes. We begin with the observation, underlying part 2 of Theorem 1, that for any given  $v$ , for every status quo  $q$ , and almost all shocks  $\theta$ , the proposer’s maximization problem has a unique solution. Thus, the selection  $\pi_i(q, \theta; v)$  is uniquely pinned down almost everywhere. The intuition behind this uniqueness result is straightforward: if legislator  $i$  is indifferent between proposing two policies for one realization of  $\theta_i$ , then, generically, a perturbation  $\theta'_i$  of  $\theta_i$  will break that indifference. To be more precise, note that the constraint in  $i$ ’s optimal proposal problem in (3) can be reformulated to exclude the constraint requiring that  $i$  accept his own proposal, so we can write the constraint set as  $A(q, \theta_{-i}; v)$ , which is independent of  $\theta_i$ . Then a small perturbation to  $\theta'_i$  leads to a unique maximizer  $z$ . This is depicted in Figure 1, where policies  $x$  and  $y$  maximize  $U_i(\cdot, \theta_i; v)$  over  $A(q, \theta_{-i}; v)$ , the shaded region in the figure. Key here is the fact that a perturbation of  $\theta_i$  does not affect the payoffs of other legislators or, therefore, the effective constraints of  $i$ ’s maximization problem.

Having proved uniqueness of the selection  $\pi_i(q, \theta; v)$  almost everywhere, the preference shock delivers continuity of the mapping  $\psi$  as follows. Using differentiability of  $U_i(y, \theta_i; v)$ , we apply the transversality theorem to deduce that for any given  $v \in C^\infty(\mathbb{R}^d, \mathbb{R}^n)$ , for every

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<sup>15</sup>See [Duggan et al. \(2008\)](#) for examples showing that dynamic utilities do not inherit the convexity properties of the legislators’ stage utilities.

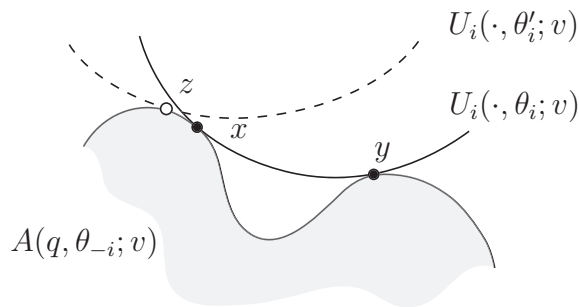


Figure 1: Genericity of unique maximizer

status quo  $q$ , almost all shocks  $\theta$ , and every policy  $y \in A(q, \theta; v)$  distinct from  $q$ , the linear independence constraint qualification (LICQ) is satisfied at  $y$ .<sup>16</sup> This is depicted in Figure 2. Here, for simplicity, we suppose the legislative acceptance set is the intersection of legislators 1's and 2's acceptance sets, which are shaded. Although the gradients of legislators 1 and 2 are linearly dependent at  $y$ , so LICQ is violated, a small shock to  $\theta_1$  will lead to a perturbation of the acceptance set of legislator 1, given by the dashed curve in the figure. We then have the generic situation, in which LICQ is satisfied over the legislative acceptance set, save possibly the status quo. This in turn implies lower hemicontinuity of the legislative acceptance set correspondence  $A(q, \theta; v)$  for all  $q$  and almost all  $\theta$ , and by the theorem of the maximum, a legislator's optimal proposal  $\pi_i(q, \theta; v)$  will be jointly continuous in  $(q, \theta; v)$  for all  $q$  and almost all  $\theta$ .<sup>17</sup> This implies that for all  $q$ , the inner integral in (4), namely,

$$\int_{\theta} \sum_{j \in N} p_j U_i(\pi_j(q, \theta; v), \theta_i; v) f(\theta) g(q|x) d\mu_{\theta},$$

is continuous as a function of  $(x, v)$ . Then continuity of  $\hat{v}_i(x) = \psi(v)_i(x)$  in  $(x, v)$  follows from Lebesgue's dominated convergence theorem. It is straightforward to apply this argument to all higher derivatives, delivering continuity of the mapping  $\psi$  in the topology of  $C^{\infty}$ -uniform convergence on compacta, thereby permitting the application of Glicksberg's theorem.

The next result justifies our focus on pure stationary equilibria. It establishes that every mixed equilibrium is equivalent to a pure one. Furthermore, because every pure equilibrium is a special case of mixed, it shows that every pure stationary legislative equilibrium satisfies the properties of Theorem 1.

**Theorem 2** *Every mixed stationary legislative equilibrium is equivalent to a pure stationary legislative equilibrium satisfying the properties in parts 1–4 of Theorem 1.*

<sup>16</sup>This means that the gradients of all binding feasibility constraints and all indifferent voters are linearly independent. See the appendix for the precise definition.

<sup>17</sup>This step is where preference shocks play their key role: allowing us to take an almost everywhere continuous selection from the optimal proposal problems. Note that they do not serve as a correlation device in the normal sense, as they are payoff relevant and for almost all shocks, the stage game (taking play in future periods as given) has a unique equilibrium.

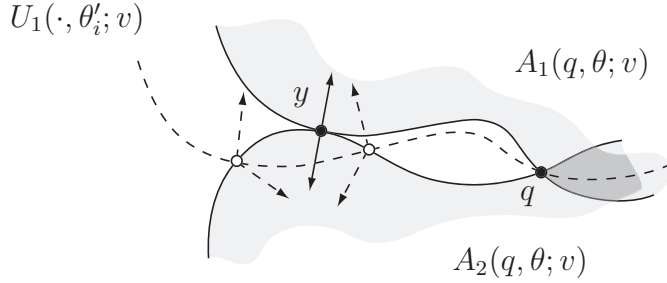


Figure 2: Genericity of LICQ

Much of the intuition for this result has already been discussed. Given a mixed stationary legislative equilibrium, with continuation value  $v$ , our earlier observation that the solution,  $\pi_i(q, \theta)$ , to a legislator's optimal proposal problem in (3) is almost always unique carries over without change. This does not immediately rule out the possibility of non-degenerate mixed strategies, however, because one or more legislators may be indifferent between  $\pi_i(q, \theta)$  and the status quo, and these legislators could conceivably vote to accept with probability less than one. But our subsequent claim that LICQ holds at every policy  $y \in A(q, \theta; v)$  distinct from  $q$  relied only on the differentiability of the equilibrium continuation values  $v$ , and inspection of (2) reveals that even in a mixed equilibrium, continuation values will inherit the differentiability assumed in the model: the current period's policy  $x$  enters the righthand side of (2) only through the function  $g(q|x)$ , which is appropriately smooth in  $x$ . An implication is that the proposer can find policies arbitrarily close to  $\pi_i(q, \theta)$  that are strictly better than the status quo for a decisive coalition of legislators. Such proposals will pass with probability one in equilibrium, and existence of an optimal proposal (a necessary condition for equilibrium) demands that  $\pi_i(q, \theta)$  will also pass with probability one. Since  $\pi_i(q, \theta)$  is the unique solution to (3), any optimal mixed proposal strategy must put probability one on that policy.

A desirable property of the equilibrium set is robustness with respect to the parameters of the model. In our framework, a model is represented by an ordered tuple  $\gamma = ((p_i^\gamma, u_i^\gamma, \delta_i^\gamma)_{i \in N}, X^\gamma, (h_\ell^\gamma)_{\ell \in K^\gamma}, f^\gamma, \Theta^\gamma, g^\gamma, \mu^\gamma)$ , where  $K^\gamma$  indexes equality and inequality constraints and for each  $q$ ,  $X^\gamma(q)$  is the corresponding set of feasible policies, which is contained in  $X^\gamma$ . Let  $\Gamma$  be a metric space of possible parameterizations satisfying the maintained assumptions (A1)–(A3) from Section 3, and moreover assume there is a fixed compact  $X \subseteq \mathfrak{R}^d$  as in (A1) independent of  $\gamma \in \Gamma$ , there are uniform bounds  $b = (b_f, b_g)$  for which (A2) and (A3) hold, and there is a single measure  $\mu_q$  fulfilling (A3) for all  $\gamma \in \Gamma$ . In addition, we assume that the parameterization is continuous: (i)  $p_i^\gamma$  and  $\delta_i^\gamma$  are continuous in  $\gamma$ , (ii)  $u_i^\gamma(x, \theta_i)$  is jointly continuous in  $(x, \theta_i, \gamma)$ , (iii) for each  $q$ ,  $X^\gamma(q)$  is continuous in  $\gamma$  with the Hausdorff metric on closed subsets of  $\mathfrak{R}^d$ , (v) for all  $\theta$ ,  $f^\gamma(\theta)$  is continuous in  $\gamma$ , and (vi) for all  $q$ ,  $g^\gamma(q|x)$  is continuous in  $(x, \gamma)$ . Note that our parameterization is especially general with respect to the set of feasible policies, for we do not assume that feasible policies are generated by a common set of parameterized constraints. Moreover, we do not assume

that the number of feasible constraints is bounded across  $\Gamma$ , and so our notion of continuity allows us to approximate a general policy space with a sequence of finite approximations cut out by an increasing number of constraints.

We define the equilibrium correspondence  $E: \Gamma \rightrightarrows C^r(\mathfrak{R}^d, \mathfrak{R}^n)$  so that  $E(\gamma)$  consists of the set of pure stationary legislative equilibrium continuation values  $v \in C^r(\mathfrak{R}^d, \mathfrak{R}^n)$ . Theorem 1 shows that  $E$  is nonempty-valued. The next result establishes that the equilibrium correspondence  $E$  is upper hemicontinuous. The proof follows, in the expected way, from a more general version of the above continuity argument, as the mapping  $\phi$  varies continuously in the parameters of the model.

**Theorem 3** *The correspondence  $E: \Gamma \rightrightarrows C^r(\mathfrak{R}^d, \mathfrak{R}^n)$  is upper hemicontinuous.*

Thus, equilibrium predictions of the model are robust in the sense that a small perturbation of the parameters of our model cannot produce new equilibria far from the original equilibrium set.<sup>18</sup> The usefulness of this result for applications may not be immediately apparent. Because we take the feasible set as a parameter, Theorem 3 allows us to compute equilibria along a sequence of finite approximations and to obtain an equilibrium of a model with a continuum of alternatives in the limit. In fact, Theorem 3 of [Duggan and Kalandrakis \(2009b\)](#) establishes that the equilibrium continuation values generated by a sequence of finite approximations must have a convergent subsequence, and, using the above result, that the limit corresponds to an equilibrium of the continuum model.

## 5 Ergodic Properties of Equilibria

A stationary legislative equilibrium, say  $\sigma^*$ , determines a stochastic process on policies, and we may then consider the equilibrium dynamics of policy outcomes in our model. Given Borel measurable  $Y \subseteq \mathfrak{R}^d$ , let  $I_Y$  denote the indicator function of  $Y$ . We define the transition probability on policy outcomes by

$$P(x, Y) = \int_q \int_\theta \sum_{i \in N} p_i I_Y(\pi_i^*(q, \theta)) f(\theta) g(q|x) d\mu,$$

which is the probability, conditional on policy outcome  $x$  this period, that next period's outcome will lie in the set  $Y$ . We define the associated Markov operator  $T$  on the space of bounded, Borel measurable functions  $\phi: X \rightarrow \mathfrak{R}$  by  $T\phi(x) = \int \phi(z)P(x, dz)$ . The adjoint  $T^*$  operates on the Borel measures on  $X$ , denoted  $\xi$ , and is defined by  $T^*\xi(Y) = \int P(x, Y)\xi(dx)$ . This describes the distribution of outcomes in the next period, given a distribution  $\xi$  of policy outcomes in the current period. The iterates of  $T^*$ , denoted  $T^{*t}$ , give the distribution of policy outcomes  $t$  periods hence and are key in describing the long run policy outcomes of the model. Say  $P$  satisfies *Doebelin's condition* if there is a finite Borel measure  $\varphi$ , a natural number  $t$ , and  $\epsilon > 0$  such that  $\varphi(Y) \leq \epsilon$  implies  $P^t(x, Y) \leq 1 - \epsilon$ , where  $P^t$  is the  $t$ -period transition defined inductively by  $P^t(x, Y) = \int P^{t-1}(y, Y)P(x, dy)$ . Intuitively, this means

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<sup>18</sup>To be clear, we do not prove lower hemicontinuity, i.e., that a small perturbation of parameters will produce equilibria close to the original equilibria. That property is not typically expected of equilibrium correspondences, and our framework is no exception.



that if a set is small according to  $\varphi$ , then  $P$  cannot assign a high probability to the set for any initial policy  $x$ . We can then define a measurable set  $Y$  to be *invariant* if for  $\varphi$ -almost every  $x \in Y$ , we have  $P(x, Y) = 1$ . Finally, we say  $P$  is *aperiodic* if there do not exist a natural number  $\beta \geq 2$  and nonempty, pairwise disjoint, measurable subsets  $C_1, \dots, C_\beta$  such that for all  $j = 1, \dots, \beta$  and all  $x \in C_j$ , we have  $P(x, C_{j+1 \bmod \beta}) = 1$ . This condition is useful in deducing strong convergence properties for the Markov chain on policies.

In the next section, we provide a sharp characterization of long run equilibrium policies when the model is close to a canonical spatial model for which there exists a core policy. Here, we consider general properties of the long run distribution of equilibrium policies while imposing minimal structure on the model. It is straightforward to show that  $T$  maps continuous functions to continuous functions and, therefore, satisfies the Feller property.<sup>19</sup> Since  $X$  is compact,  $P$  then admits at least one invariant distribution  $\xi^*$ , so that  $\xi^* = T^*\xi^*$ . Thus, each stationary legislative equilibrium determines an “ergodic Markov equilibrium,” in the sense of Duffie et al. (1994). In fact, the main result of this section establishes that  $P$  satisfies Doeblin’s condition, so that from any initial distribution  $\xi$  on  $X$ , the sequence of long run average distributions,  $\frac{1}{m} \sum_{t=1}^m T^{*t}\xi$ ,  $m = 1, 2, \dots$ , converges to an invariant distribution in the total variation norm (Doob (1953)).<sup>20</sup> While it provides a minimal characterization of long run policy outcomes, however, this result is weak in several respects: it concerns the long run average distributions, rather than the distribution of policy outcomes in each period  $t$ ; the limiting invariant distribution can depend on the initial distribution; and the rate of convergence is only known to be arithmetic. In particular, we have not precluded the possibility that there are multiple invariant distributions.

Under further restrictions on the transition probability, standard results on Markov processes can be applied to address these shortcomings. Although the equilibrium transition probability is endogenous, we can obtain the desired properties by imposing restrictions on the exogenous density  $g(q|x)$ . We first address the convergence issues raised above by assuming that every policy  $x$  lies in the support of  $g(\cdot|x)$ , so that  $x$  itself is a possible status quo in the next period. This weak assumption is sufficient for aperiodicity of  $P$  and delivers fast convergence to an invariant distribution from any initial policy. With this assumption, we obtain convergence of per period policy distributions instead of the long run average distributions, and we obtain geometric convergence in the total variation norm. We then address the possibility of multiple invariant distributions by requiring overlapping supports of the status quo densities. Specifically, we assume that for every pair of policies  $x, x' \in X$ , there is a status quo  $q$  that lies in the supports of  $g(\cdot|x)$  and  $g(\cdot|x')$ . Though this assumption is restrictive from a theoretical perspective, we allow for the status quo densities to place arbitrarily low (but positive) probability at such a status quo, consonant with applications. This delivers uniqueness of the invariant distribution corresponding to a given stationary legislative equilibrium.

**Theorem 4** *Let  $\sigma^*$  be a stationary legislative equilibrium and  $T$  be the associated Markov operator with adjoint  $T^*$ .*

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<sup>19</sup>The transition probability  $P$  satisfies the *Feller property* if for all bounded, continuous  $\phi: X \rightarrow \mathfrak{R}$ , the mapping  $T\phi: X \rightarrow \mathfrak{R}$  is also bounded and continuous.

<sup>20</sup>This result is similar in spirit to that of Hellwig (1980), who uses Doeblin’s condition to establish ergodic properties of temporary equilibria.

1. *P* satisfies Doeblin's condition, and given any initial distribution  $\xi$ , the sequence of long run average distributions,  $\frac{1}{m} \sum_{t=1}^m T^{*t} \xi$ , converges arithmetically to an invariant distribution  $\xi^*$  in the total variation norm: there is a constant  $c > 0$  such that for all  $m$ , we have  $\|\frac{1}{m} \sum_{t=1}^m T^{*t} \xi - \xi^*\| \leq \frac{c}{m}$ .
2. If for every policy  $x \in X$ , we have  $g(x|x) > 0$ , then *P* is aperiodic and given any initial distribution  $\xi$ , the sequence of per period policy distributions converges geometrically to an invariant distribution  $\xi^*$  in total variation norm: there are constants  $c$  and  $\rho$ , with  $c > 0$  and  $0 < \rho < 1$ , such that for all  $t$ , we have  $\|T^{*t} \xi - \xi^*\| \leq c\rho^t$ .
3. If for every pair of policies  $x, x' \in X$  there exists  $q \in X$  such that  $g(q|x)g(q|x') > 0$ , then *P* admits a unique invariant distribution, say  $\xi^*$ . Given any initial distribution  $\xi$ , the sequence of iterates,  $T^{*t} \xi$ , converges geometrically to  $\xi^*$  in the total variation norm: there are constants  $c$  and  $\rho$ , with  $c > 0$  and  $0 < \rho < 1$ , such that for all  $t$ , we have  $\|T^{*t} \xi - \xi^*\| \leq c\rho^t$ .

Note that part 1 holds very generally and does not exploit the details of equilibrium strategies we have established in Theorem 1. Part 2, in contrast, does rely on the equilibrium incentives of proposers: given any invariant set  $Y$  and any legislator  $i$  with positive recognition probability, there is some preference shock  $\theta_i$  such that  $i$ 's dynamic utility is maximized over the closure of  $Y$  at a unique alternative, say  $x$ . When the policy outcome is  $x$ , there is a positive probability that next period's status quo is near  $x$ ,  $i$ 's preference shock is near  $\theta_i$ , and the selected proposer is  $i$ , who then maintains a policy outcome near  $x$ . This incentive precludes cyclic subsets and implies aperiodicity. Part 3 yields an unambiguous prediction of long run policy outcomes determined by an equilibrium, allowing us to compare observed data to the distribution predicted by an equilibrium. This result is especially helpful both for purposes of estimation when moment conditions are derived from the long-run distribution over outcomes and for the conduct of counterfactual experiments when, e.g., evaluating the long run effects of institutional features. Part 3 does not exploit the structure of equilibrium strategies, and we conjecture that in more structured environments, uniqueness of the ergodic distribution may follow under even weaker conditions on the status quo density. Note that the conditions of Theorem 4 do not preclude multiple equilibria, each possibly determining a different invariant distribution. In the next section, we address this issue in models approximating a canonical spatial model.

## 6 Equilibrium Bounds from the Core

In this section, we deduce restrictions on the equilibria of our model based on proximity to a canonical spatial model. In particular, we consider models in which the stochastic shocks are small and the profile of legislators' stage utilities is close to a quadratic profile admitting a core policy, say  $\hat{x}$ . We establish that the stage utility of the legislator with ideal policy  $\hat{x}$  in the canonical model provides an arbitrarily tight lower bound on his equilibrium dynamic utility, and we show that the invariant distribution over policy outcomes generated by stationary legislative equilibria must be close, in the sense of weak convergence, to the point mass on  $\hat{x}$ . This core convergence result is ostensibly similar to results of [Banks and](#)

Duggan (2000, 2006a), but they assume legislators become arbitrarily patient to show that alternatives near the core are passed with probability one in the first period, after which the game ends. In contrast, we make no assumptions about the patience of legislators, other than discount factors are (close to) common. In addition, equilibrium policy outcomes may “drift” away from  $\hat{x}$  in our model, but they drift slowly as the model is closer to canonical, and the core legislator is able to pass policies close to his ideal point whenever recognized to propose. Finally, while these authors assume the existence of a core policy such as  $\hat{x}$  in the canonical model we posit, the policy  $\hat{x}$  does *not* belong to the core of our model for almost all realizations of the legislators’ preference shocks, even in the one-dimensional case in which a core policy (different than  $\hat{x}$ ) is guaranteed to exist for all preference shocks.

Our results bear on an earlier literature on dynamic social choice that considers the location of policies relative to the core. Ferejohn et al. (1984) posit a transition probability on the space of policies generated by myopic majority voting: given a policy  $x_t$  in period  $t$ , the distribution over policies in period  $t + 1$  is determined by first randomizing over majority coalitions and then uniformly drawing  $x_{t+1}$  from the set of policies myopically preferred to  $x_t$  by all members of the coalition. The authors establish that this transition probability admits a unique invariant distribution, and that if individual preferences are Euclidean and  $\hat{x}$  is close to satisfying the conditions for the core, then the invariant distribution generated by myopic voting piles probability close to  $\hat{x}$ . In contrast, policy dynamics in our model are governed by equilibrium behavior among farsighted players in a non-cooperative game-theoretic framework. Thus, for example, even if the policy outcome  $x_t$  is close to being a core policy relative to the stage utilities of the legislators, the equilibrium policy in the following period is determined by the legislators’ strategic preferences, which incorporate expectations of future play; and it is not clear a priori that the stability properties of  $x_t$  with respect to stage utilities are maintained when legislators condition their votes on future expectations. Furthermore, in our model the transition on policies is intermediated by stochastic shocks to the status quo and to the stage utilities of legislators. Due to the noise on the status quo, the distribution of policies will not generally converge to a point mass in the long run, and the concept of the core cannot even be defined independently of preference shocks. It is not immediately clear how a game-theoretic convergence result in the spirit of Ferejohn et al. (1984) should be formulated; our approach is to consider models in which legislator stage utilities are close to admitting a core policy  $\hat{x}$  and in which the shocks to the status quo and stage utilities are small.

In the analysis, we fix the policy space  $X \subseteq \mathfrak{R}^d$  and the measure  $\mu_q$  on  $\mathfrak{R}^d$ . We also fix a voting rule  $\mathcal{D}$  that is *proper*, i.e., for all  $C \in \mathcal{D}$ , we have  $N \setminus C \notin \mathcal{D}$ , and *strong*, i.e., for all  $C \subseteq N$ , either  $C \in \mathcal{D}$  or  $N \setminus C \in \mathcal{D}$ . A well-known example is majority rule with  $n$  odd. When  $n$  is even, it is trivial to modify majority rule by designating one legislator who breaks ties.<sup>21</sup> Because we consider models with arbitrarily small noise, we cannot maintain bounds on derivatives of densities that are uniform across models. Therefore, given a vector  $b = (b_f, b_g)$  of bounds, let  $\Gamma^b$  be the class of models fulfilling **(A1)** with  $X \subseteq \mathfrak{R}^d$ , satisfying **(A2)**–**(A3)** with respect to the vector of bounds  $b$ , and fulfilling **(A3)** with  $\mu_q$ . Let  $\Gamma^\infty = \bigcup \{\Gamma^b : b \in \mathfrak{R}_+^2\}$  be the union of these classes.

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<sup>21</sup>More generally, any voting rule  $\mathcal{D}$  can be extended to a strong rule  $\hat{\mathcal{D}}$  defined as follows:  $C \in \hat{\mathcal{D}}$  if and only if either  $C \in \mathcal{D}$  or  $v \in C$  and  $N \setminus C \notin \mathcal{D}$ , where legislator  $v$  is a designated tie-breaker.

We consider an arbitrarily fixed canonical spatial model specified by a profile  $(\hat{x}_i)_{i \in N}$  of ideal policies, a core legislator  $k$  with  $\hat{x}_k \in X$ , and a common discount factor  $\delta < 1$  such that legislators' stage utilities are quadratic and admit a core policy, the ideal point of legislator  $k$ . Formally, letting  $u_i^c$  be quadratic with ideal policy  $\hat{x}_i \in X$ , i.e.,  $u_i^c(x) = -\|\hat{x}_i - x\|^2$ , we assume that for all  $x \in \mathfrak{R}^d$ ,  $\{i \in N : u_i^c(x) > u_i^c(\hat{x}_k)\} \notin \mathcal{D}$ . Given  $\epsilon > 0$ , we say the model  $\gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, (h_\ell)_{\ell \in K}, f, \Theta, g, \mu) \in \Gamma^\infty$  is  $\epsilon$ -canonical if:

- (i)  $\max\{|u_i(x, 0) - u_i^c(x)| : i \in N, x \in X\} < \epsilon$ ,
- (ii)  $\max\{\min\{\|x - \hat{x}_k\| : x \in X(q)\} : q \in X\} < \epsilon$ ,
- (iii)  $\frac{|\delta_i - \delta|}{1 - \delta_i} < \epsilon$ ,
- (iv)  $\text{supp} f \subseteq B_\epsilon(0)$ ,
- (v) for all  $x \in X$ ,  $\text{supp} g(\cdot|x) \subseteq B_\epsilon(x)$ .

The canonical model can be interpreted as a limiting version of our model in which stage utilities are quadratic and admit a core policy, the core policy is feasible from every status quo, discount factors are common, the distribution of the preference shock  $\theta$  is degenerate at zero, and the transition from current policy,  $x$ , to next period's status quo,  $q = x$ , is deterministic; in particular, the absence of stochastic transitions places the canonical model outside of our framework.

In the appendix, Lemmas 7 and 8 establish that in an  $\epsilon$ -canonical model for  $\epsilon$  small, the core legislator is almost decisive: a proposed policy will pass only if it does not give legislator  $k$  a dynamic utility much less than his dynamic utility from the status quo; and if a policy gives  $k$  a dynamic utility slightly higher than the status quo, then it will pass if proposed. The proof relies on a result of Banks and Duggan (2006b) establishing that the core legislator is decisive over lotteries in the canonical spatial model: a decisive coalition of legislators prefers one lottery to another if and only if the core legislator does. This result must be applied with care, however, for in an  $\epsilon$ -canonical model the core may be empty, discount factors may not be common, and the legislator's preferences are subject to shocks. For this reason, we find that the core legislator is *nearly* decisive when the model is close to canonical; this implies that in equilibrium, the core legislator can implement policies that nearly maximize his dynamic utility when recognized to propose, and that policy outcomes cannot deliver dynamic utility much lower than the status quo for the core legislator when someone else is recognized. Although these lemmas are stated in terms of strategic preferences, we use them to obtain Lemma 9, which provides a lower bound for the core legislator in terms of stage utility. This lower bound holds with slack, due to the facts that the status quo is realized with noise and that the core legislator is only nearly decisive, but that slack disappears as  $\epsilon$  becomes small.

The main result of this section characterizes equilibria in models that are close to canonical: the core legislator becomes pivotal, in the sense that his stage utility provides a lower bound for his dynamic payoffs, and long run equilibrium policies are concentrated close to the core with high probability. This does not imply that the dynamics of the model become trivial: given any  $\epsilon$ -canonical model, any equilibrium of the model, and any status quo  $q$ ,

the path of play will lead away from  $q$  with positive probability, and optimal proposals and votes obey the usual dynamic incentives. Indeed, even starting from the core of the canonical model, policy outcomes will drift away with positive probability as stage utilities are perturbed, legislators other than  $k$  are recognized, and new status quos are realized. We show, however, that from any initial condition, the equilibrium path of play will gravitate toward the core with high probability as the model gets closer to the canonical model.<sup>22</sup> In what follows, we say a sequence  $\{\gamma^m\}$  of models in  $\Gamma^\infty$  *canonical* if there is a sequence  $\{\epsilon^m\}$  such that  $\gamma^m$  is  $\epsilon^m$ -canonical for all  $m$ ,  $\liminf p_k^m > 0$ , and  $\epsilon^m \rightarrow 0$ .<sup>23</sup>

**Theorem 5** *Assume that  $\mathcal{D}$  is proper and strong and that  $\{\gamma^m\}$  is a canonical sequence, and for each  $m$ , let  $\sigma^m \in E(\gamma^m)$  be a stationary legislative equilibrium.*

1. *For all sequences  $\{\theta^m\}$  and  $\{y^m\}$  satisfying  $\theta^m \in \text{supp} f^m$  and  $y^m \in X$  for all  $m$ , we have  $\liminf_{m \rightarrow \infty} U_k(y^m, \theta_k^m; \sigma^m) \geq u_k(y, 0)$ , and  $\liminf_{m \rightarrow \infty} v_k(y^m; \sigma^m) \geq u_k(y, 0)$ .*
2. *Every sequence  $\{\xi^m\}$  of equilibrium invariant distributions converges weak\* to the unit mass on  $\hat{x}_k$ .*

Part 1 follows directly from Lemma 9. The proof of part 2 proceeds by showing that along the sequence, the core legislator  $k$  proposes policies close to  $\hat{x}_k$  with probability one. Using part 1, we show that for any finite number of periods  $T$  and for large enough  $m$ , the proposals of all legislators must stay arbitrarily close to  $\hat{x}_k$  for all  $T$  periods following a proposal by the core legislator. This, with the fact that the core legislator's proposal probability has a strictly positive lower bound that is uniform across  $m$ , enables us to show that the invariant probability cannot place positive mass far from  $\hat{x}_k$  as  $m$  goes to infinity, as required.

## 7 Conclusion

We establish existence of stationary Markov perfect equilibria satisfying a number of desirable regularity properties in a general model of legislative policy making. Our analysis imposes no constraints on the dimensionality of the policy space, we do not assume convexity conditions on policy preferences, and we allow for any voting rule that can be expressed in terms of a collection of decisive coalitions. The main technical assumption we impose is differentiability, which, in combination with uncertainty about future policy preferences and noise in the implementation of future policies, allows us to bring methods of differentiable topology to bear on the existence problem. Specializing to the unidimensional model, or the multidimensional model in which legislator preferences are close to a canonical form, we find that a median voter theorem holds: as the stochastic shocks in the model become small, the long run equilibrium policies are concentrated near the median, or the core policy in higher dimensions, with high probability. For reasons of space, we have limited the scope of our analysis to a benchmark model that is institutionally austere, in the sense that we abstract away from

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<sup>22</sup>See [Duggan and Kalandrakis \(2009b\)](#) for a numerical illustration of core convergence in a two-dimensional, quadratic model with nine legislators.

<sup>23</sup>We use the obvious convention  $\gamma^m = ((p_i^m, u_i^m, \delta_i^m)_{i \in N}, X, (h_\ell^m)_{\ell \in K^m}, f^m, \Theta^m, g^m, \mu)$  and  $X^m(q)$  for the set of policies feasible at  $q$ .

much of the detail of real-world political systems. It encapsulates all of the difficult technical issues we would encounter in more complex models, while offering advantages of efficiency in presentation. But our approach to existence and related issues is quite general and extends to a much larger class of models that can capture a substantial amount of institutional detail. It is trivial to augment the model with a finite set of states that control stage utilities, discount factors, the feasible policies, the voting rule, and the identity of the proposer and that evolve according to an exogenous Markov process; this allows us to capture institutional features such as a legislative committee system and permits the analysis of electoral incentives in policy-making. Such richer versions of the model open the opportunity for the fine-tuned analysis of constitutional design issues using computational analysis (e.g., [Duggan et al. \(2008\)](#) on the effect of the presidential veto) or structural estimation (e.g., [Duggan and Kalandrakis \(2009a\)](#) on the dynamics of US presidential and congressional elections).

## A Proofs of Theorems

The appendix is organized as follows. We first prove four lemmas that establish continuity properties and necessary conditions for solutions to the optimization problem of the proposer. We proceed to define the mapping  $\psi$ , described in Section 4, and with Lemma 5 we establish that this mapping is continuous and that its domain and range can be restricted to a compact set. We then prove existence of legislative equilibrium in Theorem 1 by an application of Glicksberg’s theorem, and parts 1–4 of the theorem follow immediately from Lemmas 1–3. In Lemma 6, we show that all legislative equilibrium continuation values are fixed points  $\psi$ . The proof of Theorem 2, which reduces all mixed legislative equilibria to pure, relies mainly on Lemmas 1 and 4. Theorem 3, on upper hemicontinuity of the equilibrium correspondence, follows from Lemmas 5 and 6. Theorem 4 uses the continuity of optimal proposals along with known results on ergodicity of Markov chains. Theorem 5 follows with the help of Lemmas 7–9, which develop the near decisiveness of the core legislator.

Let  $C^r(\mathbb{R}^d, \mathbb{R}^n)$  be the  $r$ -times continuously differentiable functions from  $\mathbb{R}^d$  into  $\mathbb{R}^n$  with the topology of  $C^r$ -uniform convergence on compacta. To describe this topology, let  $s$  be a natural number and  $Y \subseteq \mathbb{R}^d$ , and define the norm  $\|\phi\|_{s,Y}$  on  $C^s(\mathbb{R}^d, \mathbb{R}^n)$  as  $\sup\{\|\partial^s \phi(x)\| : x \in Y\}$ , where  $\partial^s \phi$  is the  $s$ -th derivative of  $\phi$ .<sup>24</sup> Then a sequence  $\{\phi^m\}$  of functions converges to  $\phi$  in  $C^r(\mathbb{R}^d, \mathbb{R}^n)$  if and only if for every  $s = 0, 1, \dots, r$  and every compact set  $Y \subseteq \mathbb{R}^d$ , we have  $\|\phi^m - \phi\|_{s,Y} \rightarrow 0$ . We say  $\phi^m \rightarrow \phi$  in  $C^\infty(\mathbb{R}^d, \mathbb{R}^n)$  if and only if it converges in  $C^r(\mathbb{R}^d, \mathbb{R}^n)$  for all  $r = 0, 1, \dots$ . Given  $v = (v_1, \dots, v_n) \in C^r(\mathbb{R}^d, \mathbb{R}^n)$ , define the induced utility

$$U_i(y, \theta_i; v) = (1 - \delta_i)u_i(y, \theta_i) + \delta_i v_i(y),$$

where future payoffs are as though generated by  $v$ , and define the associated acceptance sets

$$A_i(q, \theta; v) = \{y \in X(q) : U_i(y, \theta_i; v) \geq U_i(q, \theta_i; v)\}.$$

Let  $C \subseteq N$  be any coalition and  $\mathcal{C} \subseteq 2^N$  any nonempty collection of coalitions, and, following the conventions of Section 3, define

$$A_C(q, \theta; v) = \bigcap_{i \in C} A_i(q, \theta; v) \quad \text{and} \quad A_{\mathcal{C}}(q, \theta; v) = \bigcup_{C \in \mathcal{C}} \bigcap_{i \in C} A_i(q, \theta; v).$$

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<sup>24</sup>We reserve the notation  $D\phi$  for the two-dimensional Jacobian matrix of  $\phi$ .

When  $C = \emptyset$ , we adopt the convention that  $A_C(q, \theta; v) = X(q)$ . Lastly, let

$$\max_{y \in A_{\mathcal{C}}(q, \theta; v)} U_i(y, \theta; v) \quad \mathcal{P}_i(\mathcal{C}, q, \theta; v)$$

be the optimal proposal problem of legislator  $i$ , given status quo  $q$  and preference shocks  $\theta$ , if the collection of decisive coalitions were  $\mathcal{C}$  and continuation values were  $v$ . When  $\mathcal{C}$  consists of a single coalition,  $C$ , we use the obvious shorthand  $\mathcal{P}_i(C, q, \theta; v)$ , substituting  $C$  for  $\mathcal{C}$  in the notation defined above. Henceforth, the vector of functions  $v$  will be assumed to range over  $C^r(\mathbb{R}^d, \mathbb{R}^n)$ , unless otherwise restricted.

Our first lemma establishes, among other things, that the legislators' optimal proposals are essentially unique.

**Lemma 1**

1. For all  $\mathcal{C}$ , the correspondence  $A_{\mathcal{C}}: \mathbb{R}^d \times \Theta \times C^r(\mathbb{R}^d, \mathbb{R}^n) \rightrightarrows \mathbb{R}^d$  has nonempty, compact values; and for all  $q$ ,  $A_{\mathcal{C}}(q, \theta, v)$  has closed graph in  $(\theta, v)$ ,
2. Fix  $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$ . For all  $i$  and all  $\mathcal{C}$ , there is a measurable function  $\pi_i^{\mathcal{C}}(\cdot; v): \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$  such that for all  $q$  and all  $\theta$ ,  $\pi_i^{\mathcal{C}}(q, \theta; v)$  solves  $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$ ,
3. Fix  $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$ . For all  $q$ , there is a measure zero set  $\Theta_1(q; v) \subseteq \Theta$  such that for all  $\theta \notin \Theta_1(q; v)$ , all  $i$ , and all  $\mathcal{C}$ ,  $\pi_i^{\mathcal{C}}(q, \theta; v)$  is the unique solution to  $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$ .

**Proof** We have  $A_{\mathcal{C}}(q, \theta; v) \neq \emptyset$  for all  $(q, \theta, v)$ , as the status quo  $q$  belongs to  $A_i(q, \theta; v)$  for all  $i \in N$ . Compactness of  $A_{\mathcal{C}}(q, \theta; v)$  follows since it is a closed subset of  $X \cup \{q\}$ , a compact set. By [Mas-Colell's \(1985\) Theorem K.1.2](#), the function  $U_i(y, \theta; v)$  is jointly continuous in  $(y, \theta, v)$ . It follows that for all  $q$ ,  $A_i(q, \theta; v)$  has closed graph in  $(\theta, v)$ , and  $A_{\mathcal{C}}(q, \theta; v)$  inherits this property. This completes the proof of part 1. To prove part 2, fix  $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$ , and consider any  $i$  and  $\mathcal{C}$ . We first argue that the correspondence  $X(\cdot)$  of feasible policies is weakly measurable. Let  $K^{in} = \{n+1, \dots, n+|K^{in}|\}$  index the inequality constraints and  $K^{eq} = \{n+|K^{in}|\dots, k\}$  index equality constraints. Define the mapping  $H: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^k$  by  $H(y, q) = (h_{n+1}(y, q), \dots, h_k(y, q))$ , and define the set  $F = (\mathbb{R}_+^d)^{|K^{in}|} \times (\{0\})^{|K^{eq}|}$ . Note that  $H$  is a Caratheodory function by assumption, and  $X(q) = \{x \in \mathbb{R}^d : H(x, q) \in F\}$ . Define the correspondence  $\varphi: \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  by  $\varphi(q) = \{x \in \mathbb{R}^d : H(x, q) \notin F\}$ . Then [Aliprantis and Border's \(1999\) Lemma 17.7](#) implies that  $\varphi$  is measurable, and we also have  $X(q) = \mathbb{R}^d \setminus \varphi(q)$ . To confirm weak measurability of  $X(\cdot)$ , consider any open  $G \subseteq \mathbb{R}^d$ , and note that  $\{q \in \mathbb{R}^d : X(q) \cap G \neq \emptyset\} = \{q \in \mathbb{R}^d : \varphi(q) \subseteq \mathbb{R}^d \setminus G\}$ , which is indeed measurable, as claimed. Now, since  $U_i(\cdot; v)$  is a Caratheodory function and  $X(\cdot)$  is weakly measurable, [Aliprantis and Border's \(1999\) Theorem 17.18](#) yields a measurable selection  $\pi_i^{\mathcal{C}}(\cdot; v): \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$  from the correspondence of solutions to  $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$ , as required. To prove part 3, fix  $v \in C^0(\mathbb{R}^d, \mathbb{R}^n)$ , and consider any  $q$ , any  $i$ , and any  $\mathcal{C}$ . Given preference shocks  $\theta_{-i}$ , let

$$A_{\mathcal{C}}^{-i}(q, \theta_{-i}; v) = \bigcup_{C \in \mathcal{C}} A_{C \setminus \{i\}}(q, \theta; v)$$

denote the set of policies acceptable to all members, except possibly  $i$ , of some coalition in the collection  $\mathcal{C}$ . Note that if  $y$  solves  $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$ , then it also solves

$$\begin{aligned} & \max_y U_i(y, \theta_i; v) \\ & \text{s.t. } y \in A_{\mathcal{C}}^{-i}(q, \theta_{-i}; v). \end{aligned}$$

Note that if  $y \neq y'$ , then  $D_{\theta_i}[U_i(y, \theta_i; v) - U_i(y', \theta_i; v)] = (1 - \delta_i)D_{\theta_i}[u_i(y, \theta_i) - u_i(y', \theta_i)] \neq 0$ . Thus, Mas-Colell's (1985) Theorem I.3.1 yields a measure zero set  $\Theta_1^{i, \mathcal{C}, \theta_{-i}}(q; v) \subseteq \mathbb{R}^d$  such that for all  $\theta_i \notin \Theta_1^{i, \mathcal{C}, \theta_{-i}}(q; v)$ , the program  $\mathcal{P}_i(\mathcal{C}, q, \theta; v)$  admits a unique solution. Then

$$\Theta_1^{i, \mathcal{C}}(q; v) = \bigcup_{\theta_{-i} \in \mathbb{R}^{(n-1)d}} (\Theta_1^{i, \mathcal{C}, \theta_{-i}}(q; v) \times \{\theta_{-i}\})$$

is measure zero. Finally, since  $N$  is finite,

$$\Theta_1(q; v) = \bigcup_{i \in N} \bigcup_{\mathcal{C} \subseteq 2^N} \Theta_1^{i, \mathcal{C}}(q; v)$$

is measure zero, as desired. ■

Before we state the next lemma, we develop necessary notation and recall some definitions. For the moment, fix continuation values  $v$  and status quo  $q$ . For any subsets  $C \subseteq N$  and  $L \subseteq K$ , define the functions  $U^C(\cdot; q, v): \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^{|C|}$  by  $U^C(y, \theta; q, v) = (U_j(y, \theta_j; v) - U_j(q, \theta_j; v))_{j \in C}$  and  $h^L(\cdot; q): \mathbb{R}^d \rightarrow \mathbb{R}^{|L|}$  by  $h^L(y, q) = (h_\ell(y, q))_{\ell \in L}$ . Define the mapping  $F^{C, L}: (\mathbb{R}^d \setminus \{q\}) \times \Theta \rightarrow \mathbb{R}^{|C|+|L|}$  by

$$F^{C, L}(y, \theta; q, v) = \begin{bmatrix} U^C(y, \theta; q, v) \\ h^L(y; q) \end{bmatrix},$$

where here (and whenever relevant) we view vectors as column matrices, making  $F^{C, L}(y, \theta; q, v)$  a  $(|C| + |L|) \times 1$  matrix. Derivatives are expanded via rows, e.g.,  $D_y U^C(y, \theta; q, v)$  is a  $|C| \times d$  matrix. With regard to the program  $\mathcal{P}_i(C, q, \theta; v)$ , consider  $y \in A_C(q, \theta; v)$  and let  $\overline{C} \subseteq C$  and  $\overline{K} \subseteq K$ , with  $K^{eq} \subseteq \overline{K}$ , represent the voting and feasibility constraints, respectively, that hold with equality at  $y$ . We suppress the dependence of these sets on the pair  $(q, \theta)$ . Taking the coalition  $C$  as fixed, we say that  $y$  satisfies the *linear independence constraint qualification (LICQ)* at  $(q, \theta)$  if  $D_y F^{\overline{C}, \overline{K}}(y, \theta; q, v)$  has full row rank. The next lemma establishes that for all  $q$  and almost all  $\theta$ , LICQ holds at every policy other than the status quo.

**Lemma 2** *Fix  $v$ . For all  $q$ , there exists a measure zero set  $\Theta_2(q; v) \subseteq \Theta$  such that for all  $\theta \notin \Theta_2(q; v)$ , all  $i$ , and all  $C$ , and all  $y \in A_C(q, \theta; v) \setminus \{q\}$ ,  $y$  satisfies LICQ at  $(q, \theta)$ .*

**Proof** Fix  $v$ , and consider any  $q$  and any  $\overline{C} \subseteq N$  and  $\overline{K} \subseteq K$  such that  $\overline{C} \cup \overline{K} \neq \emptyset$ . The derivative of the mapping  $F^{\overline{C}, \overline{K}}(\cdot; q, v)$  at  $(y, \theta) \in \mathbb{R}^d \setminus \{q\} \times \Theta$  is the  $(|\overline{C}| + |\overline{K}|) \times (d + |\overline{C}|m + (n - |\overline{C}|)m)$  matrix

$$DF^{\overline{C}, \overline{K}}(y, \theta; q, v) = \begin{bmatrix} D_y U^{\overline{C}}(y, \theta; q, v) & [D_{\theta_j} U^{\overline{C}}(y, \theta; q, v)]_{j \in \overline{C}} & 0 \\ D_y h^{\overline{K}}(y; q) & 0 & 0 \end{bmatrix}.$$



The  $\overline{C}$  rows of  $[D_{\theta_j} U^{\overline{C}}(y, \theta; q, v)]_{j \in \overline{C}}$  are linearly independent since  $D_{\theta_j} U^{\overline{C}}(y, \theta; q, v)$  is a  $|\overline{C}| \times m$  matrix with entry  $(1 - \delta_j) D_{\theta_j} [u_j(y, \theta_j) - u_j(q, \theta_j)] \neq 0$  in the row corresponding to  $j \in \overline{C}$  and zeros in the remaining rows. For all  $(y, \theta)$  such that  $F^{\overline{C}, \overline{K}}(y, \theta; q, v) = 0$ ,  $\overline{K}$  must be contained in the binding feasibility constraints at  $y$ , and therefore the rows of  $Dh^{\overline{K}}(y, q)$  are linearly independent by assumption. Thus,  $DF^{\overline{C}, \overline{K}}(y, \theta; q, v)$  has full row rank. We conclude that  $F^{\overline{C}, \overline{K}}$  is transversal to  $\{0\}$ . For each  $\theta$ , define  $F_{\theta}^{\overline{C}, \overline{K}}: \mathfrak{R}^d \setminus \{q\} \rightarrow \mathfrak{R}^{|\overline{C}| + |\overline{K}|}$  by  $F_{\theta}^{\overline{C}, \overline{K}}(y; q, v) = F^{\overline{C}, \overline{K}}(y, \theta; q, v)$ . Note that  $F^{\overline{C}, \overline{K}}$  is  $r$ -times continuously differentiable and  $r \geq d > \max\{0, d - (|\overline{C}| + |\overline{K}|)\}$ . Thus, it follows by the transversality theorem that for almost all  $\theta$ ,  $F_{\theta}^{\overline{C}, \overline{K}}$  is transversal to  $\{0\}$ . Let  $\Theta_2^{\overline{C}, \overline{K}}(q; v)$  be the measure zero set of  $\theta$ 's where this does not hold, and let  $\Theta_2(q; v)$  be the finite union of these sets over all  $\overline{C}$  and  $\overline{K}$  with  $\overline{C} \cup \overline{K} \neq \emptyset$ , which also has measure zero.  $\blacksquare$

The next lemma shows that for all  $q$  and almost all  $\theta$ , if any legislator is indifferent between the optimal proposal and the status quo, then all such legislators are necessary in order for the proposal to be approved by a coalition in  $\mathcal{C}$ : if one is removed, then the resulting coalition no longer belongs to  $\mathcal{C}$ . Note that the proof of the lemma does not rely on Lemma 2.

**Lemma 3** *Fix  $v$ . For all  $q$ , there is a measure zero set  $\Theta_3(q; v) \subseteq \Theta$  such that for all  $\theta \notin \Theta_3(q; v)$ , all  $i$ , and all  $\mathcal{C}$ , if  $\pi_i^{\mathcal{C}}(q, \theta; v) \neq q$  and we define*

$$C^* = \{j \in N : U_j(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_j; v) \geq U_j(q, \theta_j; v)\},$$

*then for all nonempty  $C$  with  $U_j(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$  for all  $j \in C$ , we have  $C^* \setminus C \notin \mathcal{C}$ .*

**Proof** Fix  $v$ , and consider any  $q$ . We claim there is a measure zero set  $\tilde{\Theta}_3(q; v)$  such that for all  $\theta \notin \tilde{\Theta}_3(q; v)$ , all  $i$ , all  $C$ , and all  $j \notin C \cup \{i\}$ , if  $\pi_i^C(q, \theta; v) \neq q$ , then  $U_j(\pi_i^C(q, \theta; v), \theta_j; v) \neq U_j(q, \theta_j; v)$ . Indeed, fix  $i$  arbitrarily, and consider any coalition  $C$ . Note that for all  $j \notin C \cup \{i\}$ ,  $\pi_i^C(q, \theta_j, \theta_{-j}; v)$  is independent of  $\theta_j$ . Thus, if  $\theta_{-j}$  satisfies  $\pi_i^C(q, \theta; v) \neq q$ , then the equality  $U_j(\pi_i^C(q, (\theta_{-j}, \theta'_j); v), \theta'_j; v) = U_j(q, \theta'_j; v)$  holds for a measure zero set of shocks  $\theta'_j$ . In particular, define the function  $F_j: \Theta \rightarrow \mathfrak{R}$  by  $F_j(\theta'_j) = U_j(\pi_i^C(q, (\theta_{-j}, \theta'_j); v), \theta'_j; v) - U_j(q, \theta'_j; v)$ ; since  $DF_j(\theta'_j) = (1 - \delta_j) D_{\theta'_j} [u_j(\pi_i^C(q, (\theta_{-j}, \theta'_j); v), \theta'_j) - u_j(q, \theta'_j)] \neq 0$ , it follows that zero is a regular value of  $F_j$ , and we conclude by the preimage theorem (see Mas-Colell's (1985) Theorem H.2.2) that  $F_j^{-1}(0)$  is a  $(m - 1)$ -dimensional set. We infer that there is a measure zero set  $\tilde{\Theta}_3^{i, C, j, \theta_{-j}}(q; v) \subseteq \mathfrak{R}^d$  such that for all  $\theta_j \notin \tilde{\Theta}_3^{i, C, j, \theta_{-j}}(q; v)$ , we have  $U_j(\pi_i^C(q, (\theta_{-j}, \theta_j); v), \theta_j; v) \neq U_j(q, \theta_j; v)$ . Then

$$\tilde{\Theta}_3^{i, C, j}(q; v) = \bigcup \left\{ \theta \in \Theta : \theta_j \in \tilde{\Theta}_3^{i, C, j, \theta_{-j}}(q; v), \pi_i^C(q, \theta; v) \neq q \right\}$$

is measure zero. Since  $N$  is finite,

$$\tilde{\Theta}_3(q; v) = \bigcup \left\{ \tilde{\Theta}_3^{i, C, j}(q; v) : i \in N, C \subseteq N, j \notin C \cup \{i\} \right\}$$

is also measure zero, as desired. We now define  $\Theta_3(q; v) = \Theta_1(q; v) \cup \tilde{\Theta}_3(q; v)$ . Consider any  $\theta \notin \Theta_3(q; v)$ , any  $i$ , and any  $\mathcal{C}$ , and suppose that  $\pi_i^{\mathcal{C}}(q, \theta; v) \neq q$ . Define  $C^*$  as in the statement of the lemma. Consider any nonempty  $C$  satisfying  $U_j(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$

for all  $j \in C$ . Since  $\theta \notin \Theta_1(q; v)$ , part 3 of Lemma 1 implies  $U_i(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_i; v) > U_i(q, \theta_i; v)$ , so that  $i \notin C$ . Suppose, to obtain a contradiction, that  $C' = C^* \setminus C \in \mathcal{C}$ , and take any  $j \in C$ . Note that  $\pi_i^{\mathcal{C}}(q, \theta; v)$  solves  $\mathcal{P}_i(C^*, q, \theta; v)$ , and since  $\mathcal{P}_i(C', q, \theta; v)$  removes at least one constraint, we have  $U_i(\pi_i^{C'}(q, \theta; v), \theta_i; v) \geq U_i(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_i; v)$ . Since  $C' \in \mathcal{C}$ , we have  $\pi_i^{C'}(q, \theta; v) \in A_{\mathcal{C}}(q, \theta; v)$ . Then, since  $\theta \notin \Theta_1(q; v)$ , part 3 of Lemma 1 implies  $\pi_i^{C'}(q, \theta; v) = \pi_i^{\mathcal{C}}(q, \theta; v) \neq q$ . But then  $j \notin C' \cup \{i\}$  and  $U_j(\pi_i^{C'}(q, \theta; v), \theta_j; v) = U_j(\pi_i^{\mathcal{C}}(q, \theta; v), \theta_j; v) = U_j(q, \theta_j; v)$  contradict  $\theta \notin \Theta_3(q; v)$ . We conclude that  $C^* \setminus C \notin \mathcal{C}$ . ■

The next lemma shows that, generically, any feasible policy  $x$  that is weakly preferred to the status quo by a decisive coalition of legislators can be approximated by feasible policies that are strictly preferred to the status quo by a decisive coalition.

**Lemma 4** *Fix  $v$ . For all  $q$ , there is a measure zero set  $\Theta_4(q; v) \subseteq \Theta$  such that for all  $\theta \notin \Theta_4(q; v)$ , all  $i$ , all  $\mathcal{C}$ , and all  $y \in A_{\mathcal{C}}(q, \theta; v) \setminus \{q\}$ , there exists a sequence  $\{y^m\}$  in  $A_{\mathcal{C}}(q, \theta; v)$  such that  $y^m \rightarrow y$  and for all  $j$  and all  $m$ ,  $U_j(y^m, \theta_j; v) \neq U_j(q, \theta_j; v)$ .*

**Proof** Fix  $v$ , consider any  $q$ , and define  $\Theta_4(q; v) = \Theta_2(q; v)$ . Consider any  $\theta \notin \Theta_4(q; v)$ , any  $i$ , any  $\mathcal{C}$ , and any  $y \in A_{\mathcal{C}}(q, \theta; v) \setminus \{q\}$ . Let  $C^* = \{j \in N : U_j(y, \theta_j; v) \geq U_j(q, \theta_j; v)\}$ . Since  $y \in A_{\mathcal{C}}(q, \theta; v)$ , we have  $C^* \in \mathcal{C}$ . Let  $\overline{C}$  and  $\overline{K}$  denote the voting and feasibility constraints, respectively, that bind at  $y$  in program  $\mathcal{P}_i(C^*, q, \theta; v)$ . Since  $y \neq q$  and  $\theta \notin \Theta_2(q; v)$ ,  $y$  satisfies LICQ at  $(q, \theta)$ . Define the mapping  $F: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{|\overline{C}|+|\overline{K}|}$  by

$$F(x, \epsilon) = \begin{bmatrix} (U_j(x, \theta_j; v) - \epsilon - U_j(q, \theta_j; v))_{j \in \overline{C}} \\ (h_\ell(x, q))_{\ell \in \overline{K}} \end{bmatrix},$$

and note that  $F(y, 0) = 0$ . By LICQ,  $D_x F(y, 0)$  has full row rank, and the implicit function theorem (see Loomis and Sternberg (1968)) yields open sets  $P \subseteq \mathbb{R}$  around zero and  $Y \subseteq \mathbb{R}^d$  around  $y$  and a continuous mapping  $\phi: P \rightarrow Y$  such that  $\phi(0) = y$  and for all  $\epsilon \in P$ ,  $F(\phi(\epsilon), \epsilon) = 0$ . Defining the sequence  $\{y^m\}$  by  $y^m = \phi(1/m)$ , continuity of  $\phi$  implies  $y^m \rightarrow y$ . For all  $\ell \in K^{in} \setminus \overline{K}$ , we have  $h_\ell(y, q) > 0$ , and continuity of  $h_\ell$  implies that for high enough  $m$ ,  $h_\ell(y^m, q) > 0$ . And  $F(y^m, 1/m) = 0$  implies that for all  $\ell \in \overline{K}$ , we have  $h_\ell(y^m, q) = 0$ . Thus,  $y^m \in X$  for sufficiently high  $m$ . For all  $j \in C^* \setminus \overline{C}$ , so that  $U_j(y, \theta_j; v) > U_j(q, \theta_j; v)$ , continuity of  $U_j$  implies that for sufficiently high  $m$ , we have  $U_j(y^m, \theta_j; v) > U_j(q, \theta_j; v)$ . And  $F(y^m, 1/m) = 0$  implies that for all  $j \in \overline{C}$ , we have  $U_j(y^m, \theta_j; v) - U_j(q, \theta_j; v) = 1/m > 0$ . Therefore,  $y^m \in A_{C^*}(q, \theta; v) \subseteq A_{\mathcal{C}}(q, \theta; v)$  for sufficiently high  $m$ . Furthermore, for all  $j \notin C^*$  such that  $U_j(y, \theta_j; v) < U_j(q, \theta_j; v)$ , continuity implies that for high enough  $m$ ,  $U_j(y^m, \theta_j; v) < U_j(q, \theta_j; v)$ . Thus, we have established the existence of a subsequence  $\{y^m\}$  in  $A_{\mathcal{C}}(q, \theta; v)$ , such that  $y^m \rightarrow y$  and  $U_j(y^m, \theta_j; v) \neq U_j(q, \theta_j; v)$  for all  $j$ , as required. ■

As in Section 4, we now index models by  $\gamma = ((p_i^\gamma, u_i^\gamma, \delta_i^\gamma)_{i \in N}, X^\gamma, (h_\ell^\gamma)_{\ell \in K^\gamma}, f^\gamma, \Theta^\gamma, g^\gamma, \mu^\gamma)$ ; we let  $\Gamma$  denote the metric space of parameterizations satisfying the assumptions of the legislative model, with  $X$  in (A1) fixed, uniform bounds  $b = (b_f, b_g)$  fulfilling (A2)–(A3), and the measure  $\mu_q$  in (A3) fixed; and we continue to assume that the parameterization is continuous in the sense of that section. We define the induced utility  $U_i^\gamma(y, \theta_i; v)$  in model  $\gamma$  in the obvious way, and so that  $U_i$  is jointly continuous in  $(y, \theta_i, v, \gamma)$ . Given model  $\gamma \in \Gamma$  and continuation value functions  $v$ , Lemma 1 allows us to define measurable mappings  $\pi_i^\gamma(\cdot; v): \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$  such that for all  $q$  and almost all  $\theta$ ,  $\pi_i^\gamma(q, \theta; v)$  solves  $\mathcal{P}_i^\gamma(\mathcal{D}, q, \theta; v)$ , i.e., it solves the proposer's optimization problem at  $(q, \theta)$  in model  $\gamma$  when the voting rule is given by  $\mathcal{D}$  and

continuation values are given by  $v$ . We use these optimal proposal mappings to define a best response continuation value mapping  $\psi$  as follows: define  $\psi: C^0(\mathbb{R}^d, \mathbb{R}^n) \times \Gamma \rightarrow C^0(\mathbb{R}^d, \mathbb{R}^n)$  by

$$\psi(v, \gamma)(x) = \int_q \int_\theta \sum_j p_j^\gamma U_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta_i; v) f^\gamma(\theta) g^\gamma(q|x) d\mu.$$

where  $\psi(v, \gamma) \in C^0(\mathbb{R}^d, \mathbb{R}^n)$  follows from the fact that  $\psi(v, \gamma)$  depends on  $x$  only through the  $g^\gamma(q|x)$ , which is continuous. When  $\gamma$  is fixed, we may write  $\psi^\gamma(v)$  for the value  $\psi(v, \gamma)$ .

The next lemma establishes that the domain and range of  $\psi$  can be restricted to a compact space and that  $\psi$  is continuous on this domain. Recall that  $b_f$  bounds  $|u_i^\gamma(x, \theta_i)| f^\gamma(\theta)$  on  $\Theta^\gamma$ , and  $b_g$  bounds the norms of derivatives of  $g^\gamma(q|x)$  with respect to  $x$  on  $X$ , and let  $b_h = \mu_q(X)$ . Furthermore, let  $b_g, b_h \geq 1$ . Define  $\mathcal{V}$  to consist of functions  $v \in C^r(\mathbb{R}^d, \mathbb{R}^n)$  such that (i) if  $r < \infty$ , then the derivatives of  $v$  of order  $0, 1, \dots, r$  are bounded in norm by  $\sqrt{n} b_f b_g b_h$ , and the  $r$ -th derivative of  $v$  is Lipschitz continuous with modulus  $\sqrt{n} b_f b_g b_h$ ; and (ii) if  $r = \infty$ , then the derivatives of  $v$  of all orders  $0, 1, 2, \dots$  are bounded in norm by  $\sqrt{n} b_f b_g b_h$ . Denote by  $M(\mathbb{R}^d, \mathbb{R}^n)$  the set of Borel measurable mappings from  $\mathbb{R}^d$  to  $\mathbb{R}^n$ .

**Lemma 5**

1. The space  $\mathcal{V}$  is nonempty, convex, and compact.
2. Consider  $\gamma \in \Gamma$  and  $\phi \in M(\mathbb{R}^d, \mathbb{R}^n)$  such that for all  $i$ ,  $\phi_i$  is bounded in absolute value by  $b_f$  over  $X$ . Define the mapping  $\hat{\phi} \in M(\mathbb{R}^d, \mathbb{R}^n)$  by  $\hat{\phi}(x) = \int_q \phi(q) g^\gamma(q|x) d\mu_q$  for all  $x$ . Then  $\hat{\phi} \in \mathcal{V}$ .
3. The mapping  $\psi: \mathcal{V} \times \Gamma \rightarrow \mathcal{V}$  is continuous.

**Proof** The space  $\mathcal{V}$  is obviously nonempty and convex. In case  $r < \infty$ , [Mas-Colell's \(1985\) Theorem K.2.2](#) implies that  $\mathcal{V}$  is compact as well in the topology of  $C^r$ -uniform convergence on compacta. In case  $r = \infty$ , compactness of  $\mathcal{V}$  in the topology of  $C^\infty$ -uniform convergence on compacta follows from [Mas-Colell's Theorems K.2.2.1 and K.2.2.2](#).

For part 2, consider any  $\gamma \in \Gamma$  and  $\phi \in M(\mathbb{R}^d, \mathbb{R}^n)$  such that for all  $i$ ,  $\phi_i$  is bounded in absolute value by  $b_f$ . Define  $\hat{\phi}$  as in the statement of the lemma. By [Aliprantis and Burkinshaw's \(1990\) Theorem 20.4](#), each function  $\hat{\phi}_i$  is partially differentiable. Let  $\partial^\alpha$  denote a partial derivative operator with respect to the coordinates of  $x$  of any order  $s = 1, 2, \dots, r$  with multi-index  $\alpha$ . Using [Aliprantis and Burkinshaw's result](#), we write

$$\partial^\alpha \hat{\phi}(x) = \int_q \phi(q) \partial^\alpha g^\gamma(q|x) d\mu_q. \tag{5}$$

Since this depends on  $x$  only through  $\partial^\alpha g^\gamma(q|x)$ , which is continuous, it follows that  $\partial^\alpha \hat{\phi}_i$  is continuous. Indeed, consider a sequence  $\{x^m\}$  in  $\mathbb{R}^d$  converging to  $x$ . Then the integrand in  $\partial^\alpha \hat{\phi}_i(x^m)$ , as a function of  $q$ , converges pointwise to the integrand in  $\partial^\alpha \hat{\phi}_i(x)$ . Furthermore, we assume that the  $s$ -th derivative of  $g^\gamma(q|x)$  with respect to  $x$  is bounded in norm by  $b_g$ , which implies  $|\partial^\alpha g^\gamma(q|x^m)| \leq b_g$  for all  $m$ . Since the support of  $g^\gamma(\cdot|x)$  lies in  $X$ , a compact

set, it follows that  $\partial^\alpha g^\gamma(q|x)$  is identically zero for all  $q \notin X$ . Therefore, since  $\phi$  is bounded in absolute value by  $b_f$  on  $X$ , we have  $|\phi(q)\partial^\alpha g^\gamma(q|x)| \leq b_f b_g I_X(q)$  for all  $q$ , and the claimed continuity follows from Lebesgue's dominated convergence theorem. Therefore,  $\hat{\phi}$  is  $r$ -times continuously differentiable. To prove that  $\hat{\phi} \in \mathcal{V}$ , first suppose  $r < \infty$ , and let  $\partial$  be a derivative operator of order  $s = 0, 1, \dots, r$ , where we view  $\partial\hat{\phi}(x)$  as a  $n \times d^s$  matrix and  $\partial g^\gamma(q|x)$  as a  $1 \times d^s$  row vector. Then, viewing  $\hat{\phi}(x)$  and  $\phi(q)$  as  $n \times 1$  column vectors, we have from (5) that  $\partial\hat{\phi}(x) = \int_q \phi(q)\partial g^\gamma(q|x)d\mu_q$ , and consequently,

$$\|\partial\hat{\phi}(x)\| \leq \int_q \|\phi(q)\partial g^\gamma(q|x)\|d\mu_q \leq \int_q \|\phi(q)\| \|\partial g^\gamma(q|x)\|d\mu_q,$$

where the first inequality follows from Jensen's inequality and the second follows from Aliprantis and Border's (1999) Lemma 6.6. Note that  $\|\phi(q)\| \leq \sqrt{nb_f}$ . Again,  $\partial g^\gamma(\cdot|x)$  is identically zero outside  $X$ , and we therefore have

$$\|\partial\hat{\phi}(x)\| \leq \int_X \sqrt{nb_f} \|\partial g^\gamma(q|x)\|d\mu_q \leq \int_X \sqrt{nb_f} b_g d\mu_q = \sqrt{nb_f} b_g b_h.$$

Let  $\partial$  be the  $r$ -th order derivative with respect to  $x$ , and note that for all  $x$  and  $y$ ,

$$\begin{aligned} \|\partial\hat{\phi}(x) - \partial\hat{\phi}(y)\| &= \left\| \int_q \phi(q)(\partial g^\gamma(q|x) - \partial g^\gamma(q|y))d\mu_q \right\| \\ &\leq \int_q \|\phi(q)\| \|\partial g^\gamma(q|x) - \partial g^\gamma(q|y)\|d\mu_q \\ &\leq \int_X \sqrt{nb_f} \|\partial g^\gamma(q|x) - \partial g^\gamma(q|y)\|d\mu_q \\ &\leq \sqrt{nb_f} b_g b_h \|x - y\|, \end{aligned}$$

where the last inequality follows from our assumption that the  $r$ -th derivative of  $g^\gamma(q|x)$  with respect to  $x$  is Lipschitz continuous with modulus  $b_g$  and that  $b_h = \mu(X)$ . Thus,  $\partial\hat{\phi}$  is Lipschitz continuous with modulus  $\sqrt{nb_f} b_g b_h$ , fulfilling (i). Now suppose  $r = \infty$ , and consider any  $s \geq 1$ . As above, we have  $\|\partial\hat{\phi}(x)\| \leq \sqrt{nb_f} b_g b_h$ , fulfilling (ii) and implying  $\hat{\phi} \in \mathcal{V}$ .

For part 3, first consider any  $(v, \gamma) \in \mathcal{V} \times \Gamma$ , and define the mapping  $w: \mathfrak{R}^d \rightarrow \mathfrak{R}^n$  by

$$w_i(q) = \int_\theta \sum_j p_j^\gamma U_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta_i; v) f^\gamma(\theta) d\mu_\theta \quad (6)$$

for all  $i$  and all  $q$ . Recall that  $|u_i^\gamma(x, \theta_i)| f^\gamma(\theta) \leq b_f$  for all  $i$ , all  $\theta \in \Theta^\gamma$ , and all  $x \in X$ . Since  $\pi_i^\gamma(q, \theta; v) \in X^\gamma(q) \subseteq X$ , we then have for all  $i$  and all  $q \in X$ ,

$$|w_i(q)| \leq \int_\theta \sum_j p_j^\gamma \left[ (1 - \delta_i^\gamma) |u_i^\gamma(\pi_j^\gamma(q, \theta; v), \theta_i)| + \delta_i^\gamma |v_i(\pi_j^\gamma(q, \theta; v))| \right] f^\gamma(\theta) d\mu_\theta,$$

which is bounded above by  $b_f$ . Noting that  $\psi(v, \gamma)(x) = \int_q w(q) g^\gamma(q|x) d\mu_q$ , it follows from part 2 of the lemma that  $\psi(v, \gamma) \in \mathcal{V}$ . We conclude that  $\psi: \mathcal{V} \times \Gamma \rightarrow \mathcal{V}$ , as desired.

To prove continuity of  $\psi$ , consider sequences  $\{v^m\}$  in  $\mathcal{V}$  and  $\{\gamma^m\}$  in  $\Gamma$  with  $v^m \rightarrow v^* \in C^r(\mathfrak{R}^d, \mathfrak{R}^n)$  and  $\gamma^m \rightarrow \gamma^* \in \Gamma$ . We use superscript  $m$  for variables corresponding to model  $\gamma^m$ , and we use a superscript asterisk for variables corresponding to  $\gamma^*$ . We claim that for all  $q$ , all  $i$ , and all  $\theta \notin \Theta_1^*(q; v^*) \cup \Theta_2^*(q; v^*)$ ,  $\pi_i^m(q, \theta; v^m) \rightarrow \pi_i^*(q, \theta; v^*)$ . If not, then because  $\pi_i^m(q, \theta; v^m)$  lies in the compact set  $X$  for high enough  $m$ , we may go to a subsequence, still indexed by  $m$ , such that  $\pi_i^m(q, \theta; v^m) \rightarrow x \neq \pi_i^*(q, \theta; v^*)$ . Since  $\pi_i^m(q, \theta; v^m) \in A_{\mathcal{D}}^m(q, \theta; v^m)$  for all  $m$ , part 1 of Lemma 1 implies that  $x \in A_{\mathcal{D}}^*(q, \theta; v^*)$ . And since  $\theta \notin \Theta_1^*(q; v^*)$ , part 3 of Lemma 1 implies that  $U_i^*(\pi_i^*(q, \theta; v^*), \theta_i; v^*) > U_i^*(x, \theta_i; v^*)$ . We consider two cases. First, suppose  $\pi_i^*(q, \theta; v^*) \neq q$ , so by  $\theta \notin \Theta_4^*(q; v^*)$ , Lemma 4 implies that there exists  $y \in A_{\mathcal{D}}^*(q, \theta; v^*)$  arbitrarily close to  $\pi_i^*(q, \theta; v^*)$  such that  $U_j^*(y, \theta_j; v^*) \neq U_j^*(q, \theta_j; v^*)$  for all  $j$ . Thus, there exists a decisive coalition  $C \in \mathcal{D}$  such that  $U_j^*(y, \theta_j; v^*) > U_j^*(q, \theta_j; v^*)$  for all  $j \in C$ . Furthermore, by  $U_i^*(\pi_i^*(q, \theta; v^*), \theta_i; v^*) > U_i^*(x, \theta_i; v^*)$  and continuity of  $U_i^*$ , we may suppose  $U_i^*(y, \theta_i; v^*) > U_i^*(x, \theta_i; v^*)$ . Since  $y \in X^*(q)$ , and since  $X^m(q) \rightarrow X^*(q)$  Hausdorff, there exists a sequence  $\{y^m\}$  in  $\mathfrak{R}^d$  such that  $y^m \in X^m(q)$  for all  $m$  and  $y^m \rightarrow y$ . By joint continuity, we then have for all  $j \in C$  and for high enough  $m$ ,  $U_j^m(y^m, \theta_j; v^m) > U_j^m(q, \theta_j; v^m)$ , implying  $y^m \in A_{\mathcal{D}}^m(q, \theta; v^m)$ . But by joint continuity, we also have  $U_i^m(y^m, \theta_i; v^m) > U_i^m(\pi_i^m(q, \theta; v^m), \theta_i; v^m)$  for high enough  $m$ , contradicting the fact that  $\pi_i^m(q, \theta; v^m)$  solves  $\mathcal{P}_i^m(\mathcal{D}, q, \theta; v^m)$ . For the second case, suppose  $\pi_i^*(q, \theta; v^*) = q$ . Then  $\pi_i^*(q, \theta; v^*) = q \in A_{\mathcal{D}}^m(q, \theta; v^m)$  for all  $m$ . By joint continuity, we have  $U_i^m(q, \theta; v^m) > U_i^m(\pi_i^m(q, \theta; v^m), \theta_i; v^m)$  for high enough  $m$ , again contradicting the fact that  $\pi_i^m(q, \theta; v^m)$  solves  $\mathcal{P}_i^m(\mathcal{D}, q, \theta; v^m)$ . This establishes the claim.

We next claim that for all  $i$  and all  $\theta$ ,  $\{U_i^m(\cdot, \theta; v^m)\}$  converges uniformly to  $U_i^*(\cdot, \theta; v^*)$  on each compact set  $Y \subseteq \mathfrak{R}^d$ . If not, then there exists  $\epsilon > 0$  and a sequence  $\{x^m\}$  in  $Y$  such that

$$|(1 - \delta_i^m)u_i^m(x^m, \theta_i) + \delta_i^m v_i^m(x^m) - (1 - \delta_i^*)u_i^*(x^m, \theta_i) - \delta_i^* v_i^*(x^m)| \geq \epsilon$$

for all  $m$ . By compactness of  $Y$ , we may go to a convergent subsequence, indexed by  $m$ , with  $x^m \rightarrow x \in Y$ . But  $v^m \rightarrow v$  uniformly, and with continuity of our parameterization, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} (1 - \delta_i^m)u_i^m(x^m, \theta_i) + \delta_i^m v_i^m(x^m) &= (1 - \delta_i^*)u_i^*(x, \theta_i) + \delta_i^* v_i^*(x) \\ &= \lim_{m \rightarrow \infty} (1 - \delta_i^*)u_i^*(x^m, \theta_i) + \delta_i^* v_i^*(x^m), \end{aligned}$$

a contradiction. This establishes the claim.

Finally, let  $\hat{v}^m = \psi(v^m, \gamma^m)$  and  $\hat{v}^* = \psi(v^*, \gamma^*)$ . Let  $\partial$  denote a derivative operator with respect to the coordinates of  $x$  of any order  $s = 0, 1, \dots, r$ . Consider any compact set  $Y \subseteq \mathfrak{R}^d$ . We must show that  $\partial \hat{v}^m$  converges uniformly to  $\partial \hat{v}^*$  on  $Y$ . If not, then there exists  $\epsilon > 0$ , a subsequence  $\{\partial \hat{v}^m\}$ , still indexed by  $m$ , and a corresponding sequence  $\{x^m\}$  in  $Y$  such that for all  $m$ ,  $\|\partial \hat{v}^m(x^m) - \partial \hat{v}^*(x^m)\| \geq \epsilon$ . By compactness of  $Y$ , we may go to a further subsequence, still indexed by  $m$ , such that  $x^m \rightarrow x$  for some  $x \in Y$ . Then [Aliprantis and Burkinshaw's \(1990\) Theorem 20.4](#) implies that for all  $i$  and all  $m$ ,

$$\partial \hat{v}_i^m(x^m) = \int_q \int_{\theta} \sum_j p_j^m U_i^m(\pi_j^m(q, \theta; v^m), \theta_i; v^m) f^m(\theta) \partial g^m(q|x^m) d\mu.$$

Consider the generic case of  $(q, \theta)$  such that for all  $j$ ,  $\pi_j^m(q, \theta; v^m) \rightarrow \pi_j^*(q, \theta; v^*)$ . By uniform convergence, from our preceding claim,  $U_i^m(\pi_j^m(q, \theta; v^m), \theta_i; v^m) \rightarrow U_i^*(\pi_j^*(q, \theta; v^*), \theta_i; v^*)$ .

This gives us pointwise convergence of the integrand of  $\partial\hat{v}_i^m(x^m)$  for almost all  $(q, \theta)$ :

$$\sum_j p_j^m U_i^m(\pi_j^m(q, \theta; v^m), \theta_i; v^m) f^m(\theta) \partial g^m(q|x^m) \rightarrow \sum_j p_j^* U_i^*(\pi_j^*(q, \theta; v^*), \theta_i; v^*) f^*(\theta) \partial g^*(q|x).$$

Since  $\partial g^m(q|x^m)$  is zero outside  $X$  and since  $v^m \in \mathcal{V}$ , the terms in the above sequence are bounded in norm by the integrable function  $b_f b_g b_h I_X$ . By Lebesgue's dominated convergence theorem, and using [Aliprantis and Burkinshaw's \(1990\) Theorem 20.4](#), we therefore have

$$\partial\hat{v}_i^m(x^m) \rightarrow \int_q \int_\theta \sum_j p_j^* U_i^*(\pi_j^*(q, \theta; v^*), \theta_i; v^*) f^*(\theta) \partial g^*(q|x) d\mu = \partial\hat{v}_i^*(x).$$

By continuity of  $\partial\hat{v}_i^*$ , we also have  $\partial\hat{v}_i^*(x^m) \rightarrow \partial\hat{v}_i^*(x)$ , but then  $|\partial\hat{v}_i(x^m) - \partial\hat{v}_i^*(x^m)| \rightarrow 0$ . Since  $i$  was arbitrary, we have  $\|\partial\hat{v}^m(x^m) - \partial\hat{v}^*(x)\| \rightarrow 0$ , a contradiction. We conclude that  $\{\partial\hat{v}^m\}$  converges to  $\partial\hat{v}^*$  uniformly on  $Y$ , and therefore  $\hat{v}^m \rightarrow \hat{v}^*$ , as required.  $\blacksquare$

We can at last turn to the proof of [Theorem 1](#).

**Proof of [Theorem 1](#)** The statement of [Theorem 1](#) implicitly fixes a model  $\gamma \in \Gamma$ . By part 1 of [Lemma 5](#),  $\mathcal{V}$  is nonempty, convex, and compact. By part 3 of [Lemma 5](#),  $\psi^\gamma$  maps  $\mathcal{V}$  to  $\mathcal{V}$  and the mapping  $\psi^\gamma: \mathcal{V} \rightarrow \mathcal{V}$  is continuous. Therefore, [Glicksberg's \(1952\) theorem](#) yields a fixed point  $v^* \in \mathcal{V}$  such that  $\psi^\gamma(v^*) = v^*$ . We then construct equilibrium strategies as follows: for all  $i$ , we specify  $\pi_i(q, \theta) = \pi_i^\mathcal{Q}(q, \theta; v^*)$ , and we specify  $\alpha_i(y, q, \theta) = 1$  if  $y \in A_i(q, \theta; v^*)$  and  $\alpha_i(y, q, \theta) = 0$  otherwise. Evidently, the strategy profile  $\sigma = (\pi_i, \alpha_i)_{i \in N}$  so defined is a pure stationary legislative equilibrium. Part 1 of [Theorem 1](#) follows from  $v^* \in \mathcal{V}$ , and parts 2, 3, and 4 follow from part 3 of [Lemma 1](#), part 3 of [Lemma 3](#), and part 1 of [Lemma 3](#), respectively.  $\blacksquare$

The proof of [Theorem 1](#) relied on the fact that every fixed point of  $\psi^\gamma$  corresponds to a stationary legislative equilibrium in model  $\gamma$ . Our final lemma establishes the converse.

**Lemma 6** *For all  $(v, \gamma) \in M(\mathfrak{R}^d, \mathfrak{R}^n) \times \Gamma$ , if  $v \in E(\gamma)$ , then  $v \in \mathcal{V}$  and  $v = \psi^\gamma(v)$ .*

**Proof** Let  $(v, \gamma) \in M(\mathfrak{R}^d, \mathfrak{R}^n) \times \Gamma$  be such that  $v \in E(\gamma)$ , and let  $\sigma$  be the stationary legislative equilibrium generating  $v$ , so that  $v = v(\cdot; \sigma)$ . As in the proof of part 3 of [Lemma 5](#), define the measurable mapping  $w: \mathfrak{R}^d \rightarrow \mathfrak{R}^n$  by [\(6\)](#) for all  $i$  and all  $q$ , so that  $v(x) = \int_q w(q) g^\gamma(q|x) d\mu_q$  for all  $x$ . As argued in the proof of part 3 of [Lemma 5](#), we then have  $v \in \mathcal{V}$ . Part 3 of [Lemma 1](#) therefore implies that for all  $i$  and almost all  $(q, \theta)$ , we have  $\pi_i(q, \theta) = \pi_i^\gamma(q, \theta; v)$ . This in turn implies that  $v = \psi(v, \gamma)$ .  $\blacksquare$

We now complete the proofs of [Theorems 2–4](#).

**Proof of [Theorem 2](#)** The statement of [Theorem 2](#) implicitly fixes a model  $\gamma \in \Gamma$ , which we suppress notationally. Consider an arbitrary mixed stationary legislative equilibrium  $\bar{\sigma}$ , and let the measurable mapping  $v: \mathfrak{R}^d \rightarrow \mathfrak{R}^n$  be defined by the equilibrium continuation values as  $v(x) = (v_1(x; \bar{\sigma}), \dots, v_n(x; \bar{\sigma}))$ . To facilitate the proof, define

$$W_i(y, q, \theta; \bar{\sigma}) = \bar{\alpha}(y, q, \theta; \bar{\sigma}) U_i(y, \theta; \bar{\sigma}) + (1 - \bar{\alpha}(y, q, \theta; \bar{\sigma})) U_i(q, \theta; \bar{\sigma})$$

as the objective function of the proposer given strategy profile  $\bar{\sigma}$ .

Now consider any  $q$ , and set  $\Theta(q) = \Theta_1(q; v) \cup \Theta_4(q; v)$ . Consider any  $\theta \notin \Theta(q)$ . Since  $\theta \notin \Theta_1(q; v)$ , part 3 of Lemma 1 implies that  $\pi_i^{\mathcal{D}}(q, \theta; v)$  is the unique solution to  $\mathcal{P}_i(\mathcal{D}, q, \theta; v)$ . We consider two cases. First, suppose that  $\pi_i^{\mathcal{D}}(q, \theta; v) = q$ . If we have  $\int_{X \setminus \{q\}} \bar{\alpha}(y, q, \theta; \bar{\sigma}) \bar{\pi}_i(q, \theta)(dy) > 0$ , then there is a set  $Y \subseteq X \setminus \{q\}$  such that  $\bar{\pi}_i(q, \theta)(Y) > 0$  and for all  $y \in Y$ ,  $\bar{\alpha}(y, q, \theta; \bar{\sigma}) > 0$ . By definition of equilibrium, the latter implies  $Y \subseteq A_{\mathcal{D}}(q, \theta; v)$ . Then  $U_i(q, \theta; \bar{\sigma}) > U_i(y, \theta; \bar{\sigma})$  for all  $y \in Y$ , which implies  $W_i(y, q, \theta; \bar{\sigma}) < U_i(q, \theta; \bar{\sigma}) = W_i(q, q, \theta; \bar{\sigma})$  for all  $y \in Y$ , contradicting the fact that  $\bar{\pi}_i$  places probability one on maximizers of  $W_i(\cdot, q, \theta; \bar{\sigma})$ . Therefore,  $\int_{X \setminus \{q\}} \bar{\alpha}(y, q, \theta; \bar{\sigma}) \bar{\pi}_i(q, \theta)(dy) = 0$ . Second, suppose  $\pi_i^{\mathcal{D}}(q, \theta; v) \neq q$ . We claim that

$$\sup_{y \in X} W_i(y, q, \theta; \bar{\sigma}) \geq U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v).$$

To see this, note that since  $\theta \notin \Theta_4(q; v)$ , Lemma 4 yields a sequence  $\{y^m\}$  in  $X$  such that  $y^m \rightarrow \pi_i^{\mathcal{D}}(q, \theta; v)$  and for all  $m$ , there is a decisive coalition  $C^m$  satisfying  $U_j(y^m, \theta_j, v) > U_j(q, \theta_j; v)$  for all  $j \in C^m$ . By definition of equilibrium, it then follows that  $\bar{\alpha}_j(y^m, q, \theta_j) = 1$  for all  $j \in C^m$ , which implies  $\bar{\alpha}(y^m, q, \theta; \bar{\sigma}) = 1$ . By continuity, we then have  $W_i(y^m, q, \theta; \bar{\sigma}) = U_i(y^m, \theta; v) \rightarrow U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v)$ , as claimed. Thus, by definition of equilibrium, the mixed proposal strategy  $\bar{\pi}_i$  must achieve an expected payoff of at least  $U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v)$ . Next, we claim that  $\bar{\pi}_i(q, \theta)(\{\pi_i^{\mathcal{D}}(q, \theta; v)\}) = 1$  and  $\bar{\alpha}(\pi_i^{\mathcal{D}}(q, \theta; v), q, \theta; \bar{\sigma}) = 1$ . Consider any  $y \neq \pi_i^{\mathcal{D}}(q, \theta; v)$ , and note that if  $y \notin A_{\mathcal{D}}(q, \theta; v)$ , then  $\bar{\alpha}(y, q, \theta; \bar{\sigma}) = 0$ , which implies  $W_i(y, q, \theta; \bar{\sigma}) = U_i(q, \theta; v) < U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v)$ . And if  $y \in A_{\mathcal{D}}(q, \theta; v) \setminus \{q\}$ , then  $U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v) > \max\{U_i(y, \theta; v), U_i(q, \theta; v)\}$ , which implies the inequality  $W_i(y, q, \theta; \bar{\sigma}) < U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v)$ . Therefore, we conclude that  $\bar{\pi}_i(q, \theta)$  indeed puts probability one on  $\pi_i^{\mathcal{D}}(q, \theta; v)$ . If we had  $\bar{\alpha}(\pi_i^{\mathcal{D}}(q, \theta; v), q, \theta; \bar{\sigma}) < 1$ , then the inequality  $U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v) > U_i(q, \theta; v)$  would imply  $W_i(\pi_i^{\mathcal{D}}(q, \theta; v), q, \theta; \bar{\sigma}) < U_i(\pi_i^{\mathcal{D}}(q, \theta; v), \theta; v)$ , contradicting our previous claim. Thus, conclude  $\bar{\alpha}(\pi_i^{\mathcal{D}}(q, \theta; v), q, \theta; \bar{\sigma}) = 1$ , as desired.

Finally, we specify pure proposal strategies by  $\pi_i(q, \theta) = \pi_i^{\mathcal{D}}(q, \theta; v)$ , and we specify pure voting strategies by  $\alpha_i(y, q, \theta) = 1$  if  $y \in A_i^{\mathcal{D}}(q, \theta; v)$  and  $\alpha_i(y, q, \theta) = 0$  otherwise. The pure stationary strategy profile  $\sigma = (\pi_i, \alpha_i)_{i \in N}$  generates the same policy outcomes as  $\bar{\sigma}$  for almost all  $(q, \theta)$  and, therefore, the same continuation values. By construction, proposal and voting strategies satisfy the equilibrium conditions of Section 3, and therefore  $\sigma$  is a pure stationary legislative equilibrium. Evidently,  $\bar{\sigma}$  is equivalent to  $\sigma$ , and by Lemma 6, the equilibrium continuation value function  $v$  lies in  $\mathcal{V}$  and is a fixed point of  $\psi^\gamma$ . Then the property of part 1 of Theorem 1 follows immediately, and the properties of parts 2, 3, and 4 follow from part 3 of Lemma 1, part 3 of Lemma 3, and part 1 of Lemma 3, respectively. ■

**Proof of Theorem 3** Consider sequences  $\{\gamma^m\}$  in  $\Gamma$  and  $\{v^m\}$  in  $C^r(\mathbb{R}^d, \mathbb{R}^n)$  such that  $\gamma^m \rightarrow \gamma \in \Gamma$ ,  $v^m \rightarrow v \in C^r(\mathbb{R}^d, \mathbb{R}^n)$ , and for all  $m$ ,  $v^m \in E(\gamma^m)$ . By Lemma 6, we have  $v^m = \psi(v^m, \gamma^m)$  for all  $m$ . Taking limits, we have  $v^m \rightarrow v$  and, by part 3 of Lemma 5,  $\psi(v^m, \gamma^m) \rightarrow \psi(v, \gamma)$ . Thus,  $v = \psi(v, \gamma)$ , which implies  $v \in E(\gamma)$ , establishing closed graph of  $E$ . By Lemma 6, the range of  $\psi$  lies in  $\mathcal{V}$ , a compact space, and therefore closed graph of  $E$  implies upper hemicontinuity. ■

**Proof of Theorem 4** Let  $\sigma^*$  be a stationary legislative equilibrium. For all  $x \in X$  and all measurable  $Z \subseteq X \times \Theta$ , let  $Q(x, Z) = \int_q \int_\theta I_Z(q, \theta) f(\theta) g(q|x) d\mu$  denote the probability

that next period's  $(q, \theta)$  lies in  $Z$ , conditional on policy choice  $x$  this period. To verify Doeblin's condition, define the finite Borel measure  $\eta$  on  $\mathfrak{R}^d$  by

$$\eta(Y) = \int_{\pi_i^{-1}(Y) \cap (X \times \Theta)} f(\theta) d\mu.$$

Set  $\epsilon = \frac{1}{1+b}$ , and consider any  $x \in \mathfrak{R}^d$  and any measurable  $Y \subseteq \mathfrak{R}^d$ . Note that  $\eta(Y) \leq \epsilon$  implies  $b\eta(Y) \leq \frac{b}{1+b}$ , and furthermore, we have

$$P(x, Y) = \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y)) = \sum_{j \in N} p_j Q(x, \pi_j^{-1}(Y) \cap (X \times \Theta)) \leq b\eta(Y) \leq 1 - \epsilon,$$

where we use the assumption that the support of  $g(\cdot|x)$  lies in  $X$ . Therefore,  $P(x, Y) \leq 1 - \epsilon$ , establishing Doeblin. By Futia's (1982) Theorem 4.9, the Markov operator  $T$  is quasi-compact, and it follows that the adjoint  $T^*$  is also quasi-compact. (See Futia (1982), proof of Theorem 3.3.) For an arbitrary initial distribution  $\mu$ , Futia's (1982) Theorems 3.2 and 3.4 then yield convergence to an invariant distribution  $\mu^*$  at the rate claimed in part 1.

Suppose that for all  $x \in X$ , we have  $g(x|x) > 0$ . Let  $C_1, \dots, C_\beta$  be pairwise disjoint, measurable sets such that for all  $j = 1, \dots, \beta$  and all  $x \in C_j$ , we have  $P(x, C_{j+1 \bmod \beta}) = 1$ , and suppose  $\beta > 2$ . Let  $\overline{C}_j$  denote the closure of  $C_j$ , which is compact. We first claim that for  $j \neq \ell$ , we have  $\overline{C}_j \cap \overline{C}_\ell = \emptyset$ . Otherwise, consider  $x \in \overline{C}_j \cap \overline{C}_\ell$ . Since  $g(x|x) > 0$  and  $g$  is continuous, we can choose  $x_j \in C_j$ ,  $x_\ell \in C_\ell$ , and an open set  $G$  containing  $x$  such that for all  $q \in X$ , we have  $g(q|x_j) > 0$  and  $g(q|x_\ell) > 0$ . Let  $i$  satisfy  $p_i > 0$ . Because  $P(x_j, C_{j+1 \bmod \beta}) = 1$ , it follows that  $G \times \Theta$  contains a measure zero set  $Z_j$  such that for all  $(q, \theta) \in (G \times \Theta) \setminus Z_j$ , we have  $\pi_i(q, \theta) \in C_{j+1 \bmod \beta}$ . Similarly,  $G \times \Theta$  contains a measure zero set  $Z_\ell$  such that for all  $(q, \theta) \in (G \times \Theta) \setminus Z_\ell$ , we have  $\pi_i(q, \theta) \in C_{\ell+1 \bmod \beta}$ . But  $G \times \Theta$  has positive measure, so there exists  $(q, \theta) \in G \times \Theta$  such that  $\pi_i(q, \theta) \in C_{j+1 \bmod \beta} \cap C_{\ell+1 \bmod \beta}$ , a contradiction. By Mas-Colell's (1985) Theorem I.3.1, there exists  $\theta_i$  with positive marginal density such that  $U_i(y, \theta_i; \sigma^*)$  has a unique maximizer on the set  $\bigcup_{j=1}^\beta \overline{C}_j$ , say  $x^* \in \overline{C}_j$ . In particular, we have shown

$$U_i(x^*, \theta_i; \sigma^*) > \max \left\{ U_i(y, \theta_i; \sigma^*) : y \in \bigcup_{\ell: \ell \neq j} \overline{C}_\ell \right\}, \quad (7)$$

and continuity of  $U_i(\cdot; \sigma^*)$  yields an open set  $G$  containing  $\theta_i$  such that the strict inequality in (7) continues to hold for all  $\theta'_i \in G$ . Furthermore, since  $g(x^*|x^*) > 0$  and  $g$  is continuous, there is an open set  $H$  containing  $x^*$  such that for all  $\theta'_i \in G$  and all  $q \in H$ , we have  $g(q|x^*) > 0$  and

$$U_i(q, \theta'_i; \sigma^*) > \max \left\{ U_i(y, \theta_i; \sigma^*) : y \in \bigcup_{\ell: \ell \neq j} \overline{C}_\ell \right\}. \quad (8)$$

We claim that for all  $(q, \theta')$  such that  $\theta'_i \in G$  and  $q \in H$ , we have  $\pi_i(q, \theta') \notin C_{j+1 \bmod \beta}$ . Indeed, optimality of  $\pi_i(q, \theta')$  implies  $U_i(\pi_i(q, \theta'), \theta'_i; \sigma^*) \geq U_i(q, \theta'_i; \sigma^*)$ , and then (8) implies  $\pi_i(q, \theta') \notin \overline{C}_{j+1 \bmod \beta}$ . Using continuity of  $g$  and  $x^* \in \overline{C}_j$ , there exists  $\tilde{x} \in C_j$  such that for all  $q \in H$ , we have  $g(q|\tilde{x}) > 0$ . But then with probability  $Q(\tilde{x}, H \times G \times \mathfrak{R}^{(n-1)d}) > 0$ , we have  $\pi_i(q, \theta') \notin C_{j+1 \bmod \beta}$ , contradicting  $P(\tilde{x}, C_{j+1 \bmod \beta}) = 1$ . Thus,  $P$  is aperiodic, and



Doob's (1953) Case (f) obtains. Since Doeblin's condition holds, we can partition  $X$  into a finite number of ergodic sets  $E_1, \dots, E_\alpha$  and a transient set  $(X \setminus \bigcup_{j=1}^\alpha E_j)$ , and for every ergodic set  $E_j$ , there is a unique invariant probability measure  $\xi_j$  such that  $\xi_j(E_j) = 1$  (Doob (1953), pp. 210–211). By Doob's (1953) equation (5.13), we have for all measurable  $Y \subseteq X$ ,

$$\lim_{t \rightarrow \infty} P^t(x, Y) = \sum_{j=1}^{\alpha} \lim_{t \rightarrow \infty} P^t(x, E_j) \xi_j(Y),$$

where  $\xi_x(\cdot) = \sum_{j=1}^{\alpha} \lim_{t \rightarrow \infty} P^t(x, E_j) \xi_j(\cdot)$  is an invariant probability measure. In fact, the limit is uniform and approached exponentially fast (see his explanation below equation (5.15)), and therefore there exist  $c'$  and  $\rho < 1$  such that for all  $x \in X$  and all  $t$ ,

$$\sup_Y |P^t(x, Y) - \xi_x(Y)| \leq c' \rho^t,$$

where the supremum is over measurable subsets of  $X$ . Given an initial probability measure  $\xi$  on  $X$ , define  $\xi^*$  by  $\xi^*(Y) = \int \xi_x(Y) \xi(dx)$ . Then Jensen's inequality yields

$$|T^{*t} \xi(Y) - \xi^*(Y)| \leq \int |P^t(x, Y) - \xi_x(Y)| \xi(dx) \leq c' \rho^t.$$

Defining  $c = 2c'$ , we therefore have  $\|T^{*t} \xi(\cdot) - \xi^*(\cdot)\| = 2 \sup_Y |T^{*t} \xi(Y) - \xi^*(Y)| \leq c \rho^t$ , as required for part 2.

Since Doeblin's condition holds, we again partition  $X$  into a finite number of ergodic sets  $E_1, \dots, E_\alpha$  and a transient set. We will first show that there is just one ergodic set  $E$ , i.e.,  $\alpha = 1$ . Suppose there are distinct ergodic sets,  $E$  and  $E'$ , and consider any  $x \in E$  and  $x' \in E'$ . By assumption, there exists  $(\tilde{q}, \tilde{\theta})$  such that  $g(\tilde{q}|x)g(\tilde{q}|x')f(\tilde{\theta}) > 0$ , and continuity then implies  $Z = \{(q, \theta) : g(q|x)g(q|x')f(\theta) > 0\}$  is a nonempty, open subset of  $X \times \Theta$ . Let  $i$  satisfy  $p_i > 0$ . Because  $P(x, E) = 1$ , there is a measure zero set  $W \subseteq Z$  such that for all  $(q, \theta) \in Z \setminus W$ , we have  $\pi_i(q, \theta) \in E$ . But similarly, there is a measure zero set  $W' \subseteq Z$  such that for all  $(q, \theta) \in Z \setminus W'$ , we have  $\pi_i(q, \theta) \in E'$ . But  $Z$  has positive measure, so there exists  $(q, \theta) \in Z \setminus (W \cup W')$ , and we have  $\pi_i(q, \theta) \in E \cap E'$ , a contradiction. Thus, there is a unique ergodic set  $E \subset X$ . An identical argument establishes that  $E$  cannot be partitioned into cyclically moving subclasses, that is, pairwise disjoint, measurable subsets  $C_1, \dots, C_\beta \subseteq E$ , with  $\beta \geq 2$ , such that for all  $j = 1, \dots, \beta$  and all  $x \in C_j$ , we have  $P(x, C_{j+1 \bmod \beta}) = 1$ . If such a partition were possible, then for each  $x \in C_1$  and  $x' \in C_2$ , there would exist  $(q, \theta)$  such that  $\pi_i(q, \theta) \in C_1 \cap C_2$ , a contradiction. Thus, a strong version of Doeblin's condition holds, delivering part 3 (see Doob (1953), page 221, on condition  $(D_0)$ ).  $\blacksquare$

We now prove Theorem 5. Recall that we have fixed a canonical model  $(k, (\hat{x}^i)_{i \in N}, \delta)$  satisfying  $\hat{x}^k \in X$ . Given model  $\gamma \in \Gamma^\infty$ , strategies  $\sigma$ , and a pair  $(\tilde{x}, \tilde{\theta}) \in X \times \Theta$  in any period, we write the expected payoff  $U_i(\tilde{x}, \tilde{\theta}_i; \sigma)$  as the integral of  $u_i(x, \theta_i)$  with respect to a Borel probability measure  $\mu_i$  on  $X \times \Theta$  as follows. We define  $\mu^0$  as the unit mass on  $(\tilde{x}, \tilde{\theta})$ ,  $\mu^1$  by

$$\mu^1(Y \times H) = \sum_j p_j \int_q \int_\theta I_Y(\pi_j(q, \theta)) I_H(\theta) f(\theta) g(q|\tilde{x}) d\mu \quad (9)$$

for all open  $Y \subseteq \mathfrak{R}^d$  and open  $H \subseteq \mathfrak{R}^{mn}$ , and for  $t \geq 2$ , we define  $\mu^t$  by

$$\mu^t(Y \times H) = \int_x \int_\theta \sum_j p_j \left[ \int_q \int_{\theta'} I_Y(\pi_j(q, \theta')) I_H(\theta') f(\theta') g(q|x) d\mu \right] d\mu^{t-1}. \quad (10)$$

Thus, given policy outcome  $\tilde{x}$  and preference shocks  $\tilde{\theta}$  in the current period,  $\mu^t$  is the joint distribution on policies and preference shocks  $t$  periods hence. We notationally suppress the dependence of  $\mu^t$  on  $(\tilde{x}, \tilde{\theta})$ , but we make this dependence clear from context. We then define  $\mu_i = (1 - \delta_i) \sum_{t=0}^{\infty} \delta_i^t \mu^t$ . As is well-known, the probability measures  $\mu^t$  and  $\mu_i$  extend uniquely to the Borel sigma-algebra on  $X \times \Theta$ . We refer to  $\mu_i$  as the *continuation distribution* of  $(\tilde{x}, \tilde{\theta})$  at  $\sigma$  in  $\gamma$  for legislator  $i$ . Note that the definition of  $\mu^t$  is independent of  $i$ , while  $\mu_i$  depends on  $i$  through the discount factor  $\delta_i$ . When discount factors are common, the continuation distribution is common to all legislators, and in this case we write  $\mu_c$  for  $(1 - \delta) \sum_{t=0}^{\infty} \delta^t \mu^t$ . Note that for all  $t \geq 1$ , the marginal probability measure  $\mu_\theta^t$  on  $\Theta$  is given by  $f$  and that the supports of the conditionals  $\mu^t(\cdot|\theta)$  lie in  $X$ .

The next result gives us a partial characterization of equilibria of an  $\epsilon$ -canonical model  $\gamma \in \Gamma^\infty$  in terms of dynamic utilities. In the sequel, let  $\beta^0(\epsilon)$  be the supremum of  $|u_i(x, \theta_i) - u_i(x, 0)|$ , and let  $\beta^1(\epsilon)$  be the supremum of  $|u_i(x, \theta_i)|$ , over  $i \in N$ ,  $x \in X$ , and  $\theta \in B_\epsilon(0)$ . Note that  $\limsup_{\epsilon \rightarrow 0} \beta^0(\epsilon) = 0$  and  $\limsup_{\epsilon \rightarrow 0} \beta^1(\epsilon) < \infty$ .

**Lemma 7** *Assume  $\mathcal{D}$  is proper and strong. For all  $\lambda > 0$ , there exists  $\bar{\epsilon}(\lambda) > 0$  such that for all  $\epsilon > 0$  with  $\epsilon < \bar{\epsilon}(\lambda)$ , all  $\epsilon$ -canonical models  $\gamma \in \Gamma^\infty$ , all stationary strategy profiles  $\sigma$ , all  $\theta \in \text{supp}f$ , and all  $y, z \in X$ , the following hold: (i) if  $U_k(y, \theta_k; \sigma) > U_k(z, \theta_k; \sigma) + \lambda$ , then  $\{i \in N : U_i(y, \theta_i; \sigma) > U_i(z, \theta_i; \sigma)\} \in \mathcal{D}$ , and (ii) if  $\{i \in N : U_i(y, \theta_i; \sigma) \geq U_i(z, \theta_i; \sigma)\} \in \mathcal{D}$ , then  $U_k(y, \theta_k; \sigma) > U_k(z, \theta_k; \sigma) - \lambda$ .*

**Proof** To prove the lemma, we show (i), as (ii) follows analogously. If (i) does not hold, then there exist  $\lambda > 0$ , a canonical sequence  $\{\gamma^m\}$ , and corresponding stationary strategy profiles  $\sigma^m$ , shocks  $\theta^m \in \text{supp}f^m$ , and policies  $y^m, z^m \in X$  such that  $U_{k^m}^m(y^m, \theta_{k^m}^m; \sigma^m) > U_{k^m}^m(z^m, \theta_{k^m}^m; \sigma^m) + \lambda$  but  $C^m = \{i \in N : U_i^m(z^m, \theta_i^m; \sigma^m) \geq U_i^m(y^m, \theta_i^m; \sigma^m)\} \in \mathcal{D}$ . For each  $m$ , let  $\mu^{m,0}$  be the unit mass on  $(y^m, \theta^m)$ , and let  $\mu^{m,t}$  denote the joint distribution on policies and preference shocks  $t \geq 1$  periods in the future defined as in (9) and (10). Let  $\mu_i^m = (1 - \delta_i^m) \sum_{t=0}^{\infty} (\delta_i^m)^t \mu^{m,t}$  be the continuation distribution for legislator  $i$ , and let  $\mu_c^m = (1 - \delta) \sum_{t=0}^{\infty} (\delta)^t \mu^{m,t}$  be the corresponding continuation distribution in the canonical model. Likewise, define  $\nu_c^m$  to be the continuation distribution in the canonical model corresponding to policy choice and shock  $(z^m, \theta^m)$ . Using compactness of  $X$ , and going to a further subsequence (still indexed by  $m$ ) if necessary, we may assume that  $\mu_c^m \rightarrow \mu_c$  and  $\nu_c^m \rightarrow \nu_c$  weak\*. Obviously,

$$\left| \int u_i^c(x) d\mu_c^m - \int u_i^c(x) d\mu_c \right| \rightarrow 0 \quad \text{and} \quad \left| \int u_i^c(x) d\nu_c^m - \int u_i^c(x) d\nu_c \right| \rightarrow 0$$

by weak\* convergence. Defining  $I_i^{m,t} = \int_{x,\theta} u_i^m(x, \theta_i) d\mu^{m,t}$ , we verify the following inequali-

ties:<sup>25</sup>

$$\begin{aligned}
& \left| U_i^m(y^m, \theta_i^m; \sigma^m) - \int_{x,\theta} u_i^c(x) d\mu_c^m \right| \\
& \leq \left| \int_{x,\theta} u_i^m(x, \theta_i) d\mu_i^m - \int_{x,\theta} u_i^m(x, \theta_i) d\mu_c^m \right| \\
& \quad + \left| \int_{x,\theta} (u_i^m(x, \theta_i) - u_i^m(x, 0)) d\mu_c^m \right| + \left| \int_{x,\theta} (u_i^m(x, 0) - u_i^c(x)) d\mu_c^m \right| \\
& \leq \left| (1 - \delta) \left[ \sum_{t=1}^{\infty} ((\delta_i^m)^t - \delta^t) I_i^{m,t} \right] - (\delta_i^m - \delta) \left[ \sum_{t=0}^{\infty} (\delta_i^m)^t I_i^{m,t} \right] \right| + \beta^0(\epsilon^m) + \epsilon^m \\
& = |\delta_i^m - \delta| \left| \left[ (1 - \delta) \sum_{t=1}^{\infty} \left( \frac{(\delta_i^m)^t - \delta^t}{\delta_i^m - \delta} \right) I_i^{m,t} \right] - \left[ \sum_{t=0}^{\infty} (\delta_i^m)^t I_i^{m,t} \right] \right| + \beta^0(\epsilon^m) + \epsilon^m \\
& \leq |\delta_i^m - \delta| \left[ \frac{\beta^1(\epsilon^m)}{1 - \delta_i^m} + \frac{\beta^1(\epsilon^m)}{1 - \delta_i^m} \right] + \beta^0(\epsilon^m) + \epsilon^m \\
& \leq 2\epsilon^m \beta^1(\epsilon^m) + \beta^0(\epsilon^m) + \epsilon^m.
\end{aligned}$$

We conclude that  $|U_i^m(y^m, \theta_i^m; \sigma^m) - \int_{x,\theta} u_i^c(x) d\mu_c| \rightarrow 0$ . By a similar argument for  $(z^m, \theta^m)$ , it follows that

$$\lim_{m \rightarrow \infty} U_i^m(y^m, \theta_i^m; \sigma^m) = \int_{x,\theta} u_i^c(x) d\mu_c \quad \text{and} \quad \lim_{m \rightarrow \infty} U_i^m(z^m, \theta_i^m; \sigma^m) = \int_{x,\theta} u_i^c(x) d\nu_c. \quad (11)$$

Thus,  $\int u_k^c(x) d\mu_c \geq \int u_k^c(x) d\nu_c + \lambda$ , and Lemma 2.1 of [Banks and Duggan \(2006b\)](#) implies

$$C' = \left\{ i \in N : \int_{x,\theta} u_i^c(x) d\mu_c > \int_{x,\theta} u_i^c(x) d\nu_c \right\} \in \mathcal{D}. \quad (12)$$

Since  $\mathcal{D}$  is proper, we may select  $i \in C \cap C'$ , but the inequality in (12), with (11), yields  $U_i^m(y^m, \theta_i^m; \sigma^m) > U_i^m(z^m, \theta_i^m; \sigma^m)$  for sufficiently high  $m$ , contradicting  $i \in C$ .  $\blacksquare$

We next use Lemma 7 to deduce a lower bound for the equilibrium dynamic utility of the core legislator  $k$  from  $k$ 's own proposal and from the proposals of all other legislators. In the sequel, given  $\lambda > 0$ ,  $\bar{\epsilon}(\lambda)$  is chosen as in Lemma 7.

**Lemma 8** *Assume  $\mathcal{D}$  is proper and strong. For all  $\lambda > 0$ , all  $\epsilon < \bar{\epsilon}(\lambda)$ , all  $\epsilon$ -canonical models  $\gamma \in \Gamma^\infty$ , all stationary legislative equilibria  $\sigma \in E(\gamma)$ , all  $q \in X$ , and all  $\theta \in \text{supp} f$ , the core legislator  $k$ 's dynamic utility satisfies (i) for all  $y \in X(q)$ ,  $U_k(\pi_k(q, \theta), \theta_k; \sigma) \geq U_k(y, \theta_k; \sigma) - \lambda$ , and (ii) for all  $j \neq k$ ,  $U_k(\pi_j(q, \theta), \theta_k; \sigma) \geq U_k(q, \theta_k; \sigma) - \lambda$ .*

**Proof** For (i), note that if  $U_k(y, \theta_k) - \lambda > U_k(q, \theta_k; \sigma)$ , then Lemma 7(i) implies  $y \in A(q, \theta; \sigma)$ , i.e.,  $y$  will pass if proposed, generating a dynamic utility of  $U_k(y, \theta_k; \sigma)$ . Otherwise, if  $U_k(y, \theta_k; \sigma) - \lambda \leq U_k(q, \theta_k; \sigma)$ , then  $k$  can propose the status quo and obtain at least  $U_k(y, \theta_k; \sigma) - \lambda$ . For (ii), Lemma 7(ii) ensures that any proposal  $y$  that receives the approval of a winning coalition yields at least  $U_k(q, \theta_k; \sigma) - \lambda$  for the core legislator, as claimed.  $\blacksquare$

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<sup>25</sup>The third inequality in what follows uses the identity  $(1 - \delta) \sum_{t=1}^{\infty} \left( \frac{\delta_i^t - \delta^t}{\delta_i - \delta} \right) = \frac{1}{1 - \delta_i}$ .

Next, we extend Lemma 8 to deduce a bound in terms of stage utilities. Given an  $\epsilon$ -canonical model, let  $\beta^2(\epsilon)$  be the supremum of  $|u_i(x, 0) - u_i(y, 0)|$  over  $i \in N$ ,  $x \in X$ , and  $y \in B_\epsilon(x)$ . Note that  $\limsup_{\epsilon \rightarrow 0} \beta^0(\epsilon) = \limsup_{\epsilon \rightarrow 0} \beta^2(\epsilon) = 0$ .

**Lemma 9** *Assume  $\mathcal{D}$  is proper and strong. For all  $\zeta > 0$ , there exist  $\bar{\epsilon} > 0$  such that for all  $\epsilon$  with  $\epsilon < \bar{\epsilon}$ , all  $\epsilon$ -canonical models  $\gamma \in \Gamma^\infty$ , all stationary legislative equilibria  $\sigma \in E(\gamma)$ , all  $\theta \in \text{suppf}$ , and all  $y \in X$ , we have  $U_k(y, \theta_k; \sigma) \geq u_k(y, 0) - \frac{\zeta}{1-\delta_k}$ , and  $v_k(y; \sigma) \geq u_k(y, 0) - \frac{\zeta}{1-\delta_k}$ .*

**Proof** Fix  $\zeta > 0$ . Let  $\lambda > 0$  satisfy  $\lambda < \zeta/2$ , and choose  $\bar{\epsilon}$  so that  $0 < \bar{\epsilon} < \bar{\epsilon}(\lambda)$  and  $\lambda + \beta^0(\bar{\epsilon}) + \beta^2(\bar{\epsilon}) \leq \zeta/2$ . Consider any  $\epsilon > 0$  with  $\epsilon < \bar{\epsilon}$ , any  $\epsilon$ -canonical model  $\gamma$ , and any equilibrium  $\sigma \in E(\gamma)$ . Define  $\psi = \max\{u_k(x, 0) - U_k(x, \theta_k; \sigma) : x \in X, \theta_k \in \bar{B}_\epsilon(0)\}$ , where  $\bar{B}_\epsilon(0)$  is the closure of  $B_\epsilon(0)$ , and let  $(x^*, \theta_k^*)$  solve this program. Note that for all  $y \in X$  and all  $\theta_k \in \bar{B}_\epsilon(0)$ , we must have  $U_k(y, \theta_k; \sigma) \geq u_k(y, 0) - \psi$ . Since  $\epsilon < \bar{\epsilon}(\lambda)$ , Lemma 8(ii) implies

$$\begin{aligned} U_k(x^*, \theta_k^*; \sigma) &\geq (1 - \delta_k)[u_k(x^*, 0) - \beta^0(\epsilon)] + \delta_k \int_{\theta} \int_q [U_k(q, \theta_k; \sigma) - \lambda] f(\theta) g(q|x^*) d\mu \\ &\geq (1 - \delta_k)[u_k(x^*, 0) - \beta^0(\epsilon)] + \delta_k \int_q [u_k(q, 0) - \psi - \lambda] g(q|x^*) d\mu_q \\ &\geq (1 - \delta_k)[u_k(x^*, 0) - \beta^0(\epsilon)] + \delta_k [u_k(x^*, 0) - \beta^2(\epsilon) - \psi - \lambda], \end{aligned}$$

where the third inequality uses  $\text{suppg}(\cdot|x^*) \subseteq B_\epsilon(x^*)$ . Since  $U_k(x^*, \theta_k^*; \sigma) = u_k(x^*, 0) - \psi$ , the foregoing implies

$$\psi \leq \frac{\delta_k(\lambda + \beta^2(\epsilon))}{1 - \delta_k} + \beta^0(\epsilon),$$

and we conclude that for all  $y \in X$ ,

$$U_k(y, \theta_k; \sigma) \geq u_k(y, 0) - \psi \geq u_k(y, 0) - \frac{\lambda + \beta^2(\epsilon) + \beta^1(\epsilon)}{1 - \delta_k} \geq u_k(y, 0) - \frac{\zeta}{2(1 - \delta_k)},$$

which delivers the first part of the theorem. By Lemma 8, we then have

$$\begin{aligned} v_k(y) &\geq \int_q \int_{\theta} [U_k(q, \theta_k; \sigma) - \lambda] f(\theta) g(q|y) d\mu \\ &\geq u_k(y, 0) - \lambda - \frac{\zeta}{2(1 - \delta_k)} - \beta^2(\epsilon) \\ &\geq u_k(y, 0) - \frac{\zeta}{1 - \delta_k}, \end{aligned}$$

completing the proof. ■

Finally, we prove Theorem 5 on core convergence of stationary legislative equilibria.

**Proof of Theorem 5** Part 1 follows directly from Lemma 9. To prove part 2, let  $\{\gamma^m\}$  be a canonical sequence, let  $\{\sigma^m\}$  be a sequence of stationary legislative equilibria, and let  $\{\xi^m\}$  be any selection of invariant distributions generated by  $\{\sigma^m\}$ . For each  $m$ ,

$\gamma^m$  is  $\epsilon^m$ -canonical, where  $\epsilon^m \rightarrow 0$ , and we write  $P_m$  and  $P_m^t$  for the transition probability and  $t$ -periods transition, respectively, corresponding to  $\sigma^m$  in model  $\gamma^m$ . We must show that  $\{\xi^m\}$  converges weakly to the unit mass on  $\hat{x}_k$ , i.e., for all  $\eta > 0$ , we have  $\xi^m(B_\eta(\hat{x}_k)) \rightarrow 1$ . To prove this, we fix  $\eta > 0$  and proceed in four steps.

First, we claim that the core legislator's proposals become uniformly close to the limiting core policy  $\hat{x}_k$ , i.e., there exists a sequence  $\{\zeta^m\}$  with  $\zeta^m \downarrow 0$  such that for all  $m$ , all  $q \in X$ , and all  $\theta \in \text{supp} f^m$ , we have  $\pi_k^m(q, \theta) \in B_{\zeta^m}(\hat{x}_k)$ . Since  $\epsilon^m \rightarrow 0$ , we can choose a sequence  $\{\lambda^m\}$  with  $\lambda^m \downarrow 0$  such that for all but finitely many  $m$ , we have  $\epsilon^m < \bar{\epsilon}(\lambda^m)$ . Let

$$\psi^m = \max\{u_k^m(y, 0) - U_k^m(y, \theta_k; \sigma^m) : y \in X, \theta_k \in \bar{B}_{\epsilon^m}(\hat{x}_k)\},$$

where  $\bar{B}_{\epsilon^m}(\hat{x}_k)$  is the closed ball. Since  $\{\delta_k^m\}$  has limit less than one, Lemma 9 implies that  $\psi^m \rightarrow 0$ . For each  $m$ , define

$$\zeta^m = \sqrt{\frac{\epsilon^m + (\epsilon^m)^2 + \beta^0(\epsilon^m) + \psi^m + \lambda^m}{1 - \delta_k^m}},$$

and note that  $\zeta^m \rightarrow 0$ . Now fix any  $m$ , any  $q \in X^m$ , any  $\theta \in \text{supp} f^m$ , and any  $x^m \in X^m(q) \cap B_{\epsilon^m}(\hat{x}_k)$ , and consider any  $y \notin B_{\zeta^m}(\hat{x}_k)$ . We show that  $y$  is not the optimal proposal for legislator  $k$ , and since it is an arbitrary policy outside  $B_{\zeta^m}(\hat{x}_k)$ , the claim is established. Note that  $u_k^m$  is uniformly within  $\epsilon^m$  of  $u_k^c$ , which has a maximum value of zero, implying  $\epsilon^m \geq u_k^m(x^m, 0)$ . Since  $x^m \in B_{\epsilon^m}(\hat{x}_k)$ , i.e.,  $u_k^c(x^m) \geq -(\epsilon^m)^2$ , we then have

$$\begin{aligned} U_k^m(x^m, \theta_k; \sigma^m) &\geq u_k^m(x^m, 0) - \psi^m \\ &\geq (u_k^m(x^m, 0) - u_k^c(x^m)) + u_k^c(x^m) - \psi^m \\ &\geq -\epsilon^m - (\epsilon^m)^2 - \psi^m. \end{aligned}$$

Using  $v_k^m(y; \sigma^m) \leq \max\{u_k^m(x, 0) + \beta^0(\epsilon^m) : x \in X\} \leq \epsilon^m + \beta^0(\epsilon^m)$ , we also have

$$\begin{aligned} U_k^m(y, \theta_k; \sigma^m) &\leq (1 - \delta_k^m)[u_k^c(y) + \epsilon^m + \beta^0(\epsilon^m)] + \delta_k^m v_k^m(y; \sigma^m) \\ &\leq (1 - \delta_k^m)u_k^c(y) + \epsilon^m + \beta^0(\epsilon^m). \end{aligned}$$

Since  $y \notin B_{\zeta^m}(\hat{x}_k)$ , i.e.,  $-(\zeta^m)^2 \geq u_k^c(y)$ , we deduce

$$\begin{aligned} U_k^m(x^m, \theta_k; \sigma^m) - U_k^m(y, \theta_k; \sigma^m) &\geq -\epsilon^m - (\epsilon^m)^2 - \psi^m - [(1 - \delta_k^m)u_k^c(y) + \epsilon^m + \beta^0(\epsilon^m)] \\ &\geq (1 - \delta_k^m)(\zeta^m)^2 - [2\epsilon^m + (\epsilon^m)^2 + \beta^0(\epsilon^m) + \psi^m] \\ &> \lambda^m. \end{aligned}$$

In particular,  $U_k^m(x^m, \theta; \sigma^m) - \lambda^m > U_k^m(y, \theta; \sigma^m)$ . Since  $\epsilon^m < \bar{\epsilon}(\lambda^m)$ , Lemma 8(i) ensures

$$U_k^m(\pi_k^m(q, \theta), \theta_k; \sigma^m) \geq U_k^m(x^m, \theta_k; \sigma^m) - \lambda^m > U_k^m(y, \theta_k; \sigma^m),$$

implying  $\pi_k^m(q, \theta) \neq y$ , as desired. As we can take  $\{\zeta^m\}$  to be decreasing, the claim is fulfilled.

Our second claim is that given a current policy outcome close to the core, the probability that the outcome next period is close to the core goes to one. To be precise, define

$B(m) = B_{\zeta^m}(\hat{x}^k)$ ; then we claim that for all  $\rho > 0$ , there exists  $\bar{m}$  such that for all  $m \geq \bar{m}$  and all  $x \in B(\bar{m})$ , we have  $P_m(x, B_\rho(\hat{x}_k)) = 1$ . By Lemmas 8 and 9, we can choose a sequence  $\{\lambda^m\}$  with  $\lambda^m \rightarrow 0$  and such that for all but finitely many  $m$ ,  $\epsilon^m < \bar{\epsilon}(\lambda^m)$ , and such that for all  $x \in X$ , we have  $U_k^m(x, \theta_k; \sigma^m) \geq u_k^m(x, 0) - \lambda^m$  and  $v_k(x; \sigma^m) \geq u_k^m(x, 0) - \lambda^m$ . In particular, since  $\epsilon^m < \bar{\epsilon}(\lambda^m)$ , it follows that for all  $i \in N$ , all  $q \in X$ , and all  $\theta \in \text{supp} f^m$ , we have  $U_k^m(\pi_i^m(q, \theta), \theta_k; \sigma^m) \geq U_k^m(q, \theta_k; \sigma^m) - \lambda^m$ . Furthermore, since  $\epsilon^m \rightarrow 0$ ,  $\beta^0(\epsilon^m) \rightarrow 0$ ,  $\zeta^m \rightarrow 0$ , and  $\delta_k^m \rightarrow \delta < 1$ , we can pick large enough  $\bar{m}$  such that for all  $m \geq \bar{m}$ , the following holds:

$$(1 - \delta_k^m)(-\rho^2 + \epsilon^m + \beta^0(\epsilon^m)) + \delta_k^m(\epsilon^m + \beta^0(\epsilon^m)) < -(\epsilon^m + \zeta^m)^2 - \epsilon^m - 2\lambda^m.$$

Now consider any  $m \geq \bar{m}$ , any  $x \in B(\bar{m})$ , any  $y \notin B_\rho(\hat{x}_k)$ , any  $q \in \text{supp} g^m(\cdot|x)$ , and any  $\theta \in \text{supp} f^m$ . We then have

$$\begin{aligned} U_k(q, \theta_k; \sigma^m) - \lambda^m &\geq u_k^m(q, 0) - 2\lambda^m \\ &\geq u_k^c(q) - \epsilon^m - 2\lambda^m \\ &\geq -(\epsilon^m + \zeta^m)^2 - \epsilon^m - 2\lambda^m \\ &> (1 - \delta_k^m)(-\rho^2 + \epsilon^m + \beta^0(\epsilon^m)) + \delta_k^m(\epsilon^m + \beta^0(\epsilon^m)) \\ &\geq (1 - \delta_k^m)(u_k^c(y) + \epsilon^m + \beta^0(\epsilon^m)) + \delta_k^m v_k(y; \sigma^m) \\ &\geq (1 - \delta_k^m)(u_k^m(y, \theta_k) + \delta_k^m v_k(y; \sigma^m)) \\ &\geq U_k(y, \theta_k; \sigma^m), \end{aligned}$$

and since  $U_k(\pi_i^m(q, \theta), \theta_k; \sigma^m) \geq U_k(q, \theta_k; \sigma^m) - \lambda^m$  for all  $i$ , this implies  $y \neq \pi_i^m(q, \theta)$ . As  $y \notin B_\rho(\hat{x}_k)$  was arbitrary, we conclude that  $\pi_i^m(q, \theta) \in B_\rho(\hat{x}_k)$ , and the claim follows.

Third, we claim that given current policy outcome close to the core, the probability that future policies will be close to the core goes to one; formally, for all  $t \geq 1$ , there exists  $\bar{m}(t)$  such that for all  $m \geq \bar{m}(t)$  and for all  $x \in B(\bar{m}(t))$ , we have  $P_m^t(x, B_\eta(\hat{x}_k)) = 1$ . We prove the claim by induction on  $t$ . The case  $t = 1$  follows immediately by setting  $\rho = \eta$  in the second claim. Suppose the claim holds for some  $t = \tau \geq 1$ , i.e., the induction hypothesis is that there exists  $\bar{m}(\tau)$  such that for all  $m \geq \bar{m}(\tau)$  and all  $x \in B(\bar{m}(\tau))$ ,  $P_m^\tau(x, B_\eta(\hat{x}_k)) = 1$ . We will show that the claim also holds for  $t = \tau + 1$ . By the second claim, we can choose  $\bar{m}(\tau + 1) \geq \bar{m}(\tau)$  such that for all  $m \geq \bar{m}(\tau + 1)$  and all  $x \in B(\bar{m}(\tau + 1))$ ,  $P_m(x, B(\bar{m}(\tau))) = 1$ . Consequently, for all  $m \geq \bar{m}(\tau + 1)$  and all  $x \in B(\bar{m}(\tau + 1))$ , we have

$$\begin{aligned} P_m^{\tau+1}(x, B_\eta(\hat{x}_k)) &= \int P_m^\tau(y, B_\eta(\hat{x}_k)) P_m(x, dy) \\ &\geq \int_{B(\bar{m}(\tau))} P_m^\tau(y, B_\eta(\hat{x}_k)) P_m(x, dy) \\ &= \int_{B(\bar{m}(\tau))} P_m(x, dy) \\ &= 1, \end{aligned}$$

where the second to last equality uses the induction hypothesis with  $B(\bar{m}(\tau + 1)) \subseteq B(\bar{m}(\tau))$ , and the last follows from the second claim. This completes the proof of the claim.

Finally, we complete the proof. Part 1 of Theorem 4 shows that for all  $m$ , the transition  $P_m(x, Y)$  satisfies Doeblin's condition, so there is a finite number of ergodic sets,  $E_1^m, \dots, E_{\ell_m}^m$ , and each  $E_j^m$  admits a unique invariant distribution  $\xi_j^m$ . For all  $m$ , let  $E_{j_m}^m$  satisfy

$$\xi_{j_m}^m(B_\eta(\hat{x}_k)) = \min\{\xi_j^m(B_\eta(\hat{x}_k)) \mid j = 1, \dots, \ell_m\}.$$

By Theorem 5.7 of Doob (1953), every invariant distribution  $\xi^m$  corresponding to equilibrium  $\sigma^m$  is a convex combination of the invariant distributions  $\{\xi_j^m\}_{j=1, \dots, \ell_m}$ , and it follows that  $\xi^m(B_\eta(\hat{x}_k)) \geq \xi_{j_m}^m(B_\eta(\hat{x}_k))$ . Thus, in order to prove the theorem, it suffices to show that  $\lim_{m \rightarrow \infty} \xi_{j_m}^m(B_\eta(\hat{x}_k)) = 1$ . Suppose otherwise in order to derive a contradiction. Going to a subsequence (still indexed by  $m$ ), we may assume  $\lim_{m \rightarrow \infty} \xi_{j_m}^m(B_\eta(\hat{x}_k)) = 1 - \phi < 1$  and  $\lim_{m \rightarrow \infty} p_k^m = p > 0$ . Choose a natural number  $T$  such that  $\phi > (1 - p)^T$ . Since  $\xi_{j_m}^m$  is an invariant distribution, we must have

$$\begin{aligned} \xi_{j_m}^m(B_\eta(\hat{x}_k)) &= \int_x P_m^T(x, B_\eta(\hat{x}_k)) \xi_{j_m}^m(dx) \\ &= \int_x \sum_{t=0}^{T-1} \int_y p_k^m \left[ \int_q \int_\theta P_m^{T-t}(\pi_k^m(q, \theta), B_\eta(\hat{x}_k)) f^m(\theta) g^m(q|y) d\mu \right] \tilde{P}^t(x, dy) \xi_{j_m}^m(dx) \\ &\quad + \int_x \tilde{P}_m^T(x, B_\eta(\hat{x}_k)) \xi_{j_m}^m(dx), \end{aligned} \tag{13}$$

where  $\tilde{P}_m^t$  is defined inductively by setting  $\tilde{P}_m^0(x, \cdot)$  to be the unit mass on  $x$ , letting

$$\tilde{P}_m^1(x, Y) = \int_q \int_\theta \sum_{i \neq k} p_i^m I_Y(\pi_i^m(q, \theta)) f^m(\theta) g^m(q|x) d\mu,$$

and then defining  $\tilde{P}_m^t(x, Y) = \int \tilde{P}_m^{t-1}(y, Y) \tilde{P}_m^1(x, dy)$  for  $t > 1$ . The last line of (13) reflects paths of play such that the core legislator  $k$  is not recognized in any period  $t = 1, \dots, T$ , while the previous line collects all paths of play such that the core legislator is recognized for the first time in the  $t$ -th period,  $t = 1, \dots, T$ . By our first claim,  $\pi_k^m(q, \theta) \in B(m)$  for all  $m$ , all  $q \in X$ , and all  $\theta \in \text{supp} f^m$ . We can then define  $\bar{m} = \max\{\bar{m}(1), \dots, \bar{m}(T)\}$  and apply our third claim to deduce that for all  $m \geq \bar{m}$ , all  $t = 1, \dots, T$ , all  $q \in X$ , and all  $\theta \in \text{supp} f^m$ , we have  $P_m^t(\pi_k^m(q, \theta), B_\eta(\hat{x}_k)) = 1$ . Furthermore, note that  $\int \tilde{P}_m^t(x, dy) = (1 - p_k^m)^t$ . As a consequence, (13) implies that for all  $m \geq \bar{m}$ , we have

$$\xi_{j_m}^m(B_\eta(\hat{x}_k)) \geq \int_x \sum_{t=0}^{T-1} \int_y p_k^m \tilde{P}^t(x, dy) \xi_{j_m}^m(dx) = \sum_{t=0}^{T-1} p_k^m (1 - p_k^m)^t.$$

Taking the limit as  $m \rightarrow \infty$ , we deduce that

$$1 - \phi \geq \sum_{t=0}^{T-1} p(1 - p)^t = 1 - (1 - p)^T,$$

contradicting  $\phi > (1 - p)^T$ . We conclude that  $\lim_{m \rightarrow \infty} \xi_{j_m}^m(B_\eta(\hat{x}_k)) = 1$ , as desired.  $\blacksquare$

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