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CNOIDAL WAVE SOLUTIONS IN SHALLOW WATER AND SOLITARY WAVE LIMIT

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Abstract

Equations of motion in (X, Y) frame moving with the waves are considered for steady, incompressible flow. The boundary conditions at the bottom $Y = 0$ and at the free surface $Y = \eta(X)$ are used for solving shallow water wave problems. The remaining boundary conditions are also taken from Navier-Stokes equation of motion. Using these boundary conditions, three nonlinear ordinary differential equations are formulated, which can be solved by using series expansion method. We consider that all variations in X is relatively slow and can be expressed in terms of dimensionless variable $\frac{\alpha X}{h}$, where α is a small quantity and h is the trough depth of fluid. Then approaching on series expansion method, two types of nonlinear ordinary differential equations are formulated. Using Jacobi elliptic function, first and second order cnoidal wave solutions have been derived. Then mean value of Jacobi elliptic function and the solitary wave limit of cnoidal wave solutions are also formulated.

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1. Introduction

The general case of water wave motion is the disturbance that propagates in varying directions over water of possible non-uniform density, which flows on a shear current. A convenient set of approximations is assumed that the bed is impermeable and flat, the propagation of disturbances is collinear and they are of infinite length transverse to the direction of propagation such that the flow is two dimensional, homogeneous, and incompressible. Under these approximations, it is possible to obtain analytical solutions which correspond to a single periodic wave train, which propagates steadily without change of form. This is the steady wave problem from which convenient model is obtained.

Throughout coastal and ocean engineering, the convenient model of a steadily progressing wave train is used to derive fluid velocities and surface elevations caused by waves. The steady wave problems be solved in terms of three physical length scales only: water depth (h), wave length (λ), and wave height (H). The main theories and methods for the steady wave problem which have been used are: Stokes theory, an explicit theory based on an assumption that the waves are not very steep and which is best suited in deeper water; cnoidal theory, an explicit theory for waves in shallow water; and Fourier approximation methods which are capable of high accuracy but which solve the problem numerically and require computationally expensive matrix techniques. A review and comparison of the methods is given in Sobey et al. [1] and Fenton [3]. For relatively simple solution methods that are explicit in nature, Stokes and cnoidal theories play an important role in this field.

Fenton [2] presented a fifth order cnoidal theory where boundary condition comes from Bernoulli's equation, which was both apparently complicated, requiring the presentation of many coefficients as unattractive floating point numbers, and also gave poor results for fluid velocities under high waves. In a later work, Fenton [3], however, the author showed that instead of fluid velocities being expressed as expansions in wave height, if the original spirit of cnoidal theory were retained and they be written as series in shallowness, then the results are considerably more accurate.

Cnoidal theory obtained its name in 1895 when Korteweg and de Vries [4] obtained their eponymous equation for the propagation of waves over a flat bed. They obtained periodic solutions which they termed “cnoidal”, because the surface elevation is proportional to the square of the Jacobian elliptic function $cn(\theta/m)$. The cnoidal solution shows the familiar long flat troughs and narrow crests of real wave in shallow water. A second order cnoidal theory was presented in a formal manner by Laitone [5, 6], who provided a number of results, re-casting the series in terms of the wave height/depth. The next approximation was obtained by Chappelear [7], as one of a remarkable sequence of papers on nonlinear waves. He obtained the third order solution and expressed the results as series in a parameter directly proportional to shallowness: $(\text{depth/wavelength})^2$.

Tsuchiya and Yasuda [8] obtained a third order solution with the introduction of another definition of wave celerity based on assumptions concerning the Bernoulli constant. Nishimura et al. [9] devised procedures for generating higher order theories for both Stokes and cnoidal theories, making extensive use of recurrence relations. The authors concentrated on questions of the convergence of the series. They computed a 24th order solution; however, few detailed formulae for application were given. Nishimura et al. [9] continued the work of Nishimura et al. [10] and presented a unified view of Stokes and cnoidal theories. Karabut [13] solved an ordinary quadratic nonlinear differential difference equation of the first order containing an unknown function under certain conditions. Halasz [14] discussed on higher order corrections for shallow water solitary waves. Hongquie et al. [15] studied the cnoidal wave solutions of the Boussinesq systems in two different techniques by using the Jacobi elliptic function series. Carter et al. [16] discussed the kinematics and stability of solitary and cnoidal wave solutions of the Serre equations, which are a pair of strongly nonlinear, weakly, dispersive, Boussinesq type partial differential equations. They also described the model of the surface elevation and the depth averaged horizontal velocity of an inviscid, irrotational, incompressible shallow water. In this study, shallow water wave problems have been solved by using boundary conditions at the bottom $Y = 0$ and at the free surface $Y = \eta(X)$. Also, the boundary conditions are taken from Navier-Stokes equation of motion, which generates cnoidal wave solutions.

2. Cnoidal Wave Theory

In fluid dynamics, a cnoidal wave is a nonlinear and exact periodic wave solution of the Korteweg and de Vries equation. These solutions are in terms of the Jacobi elliptic function $cn(\theta/m)$. They are used to describe surface gravity waves of fairly long wavelength, as compared to the water depth. The cnoidal wave solutions were derived by Korteweg and de Vries, [4] in which they also propose their dispersive long wave equation, known as the Korteweg-de Vries equation.

Consider the wave as shown in Figure 1, with a stationary frame of reference (x, y), x in the direction of propagation of the waves and y vertically upwards with the origin on the flat bed. The waves travel in the x direction at speed c relative to this frame. Consider also a frame of reference (X, Y) moving with velocity c in positive X direction, such that $x = X + ct$, where t is time and $y = Y$. The fluid velocity in the (x, y) frame is (u, ν) and that in the (X, Y) frame is (U, V). The velocities are related by $u = U + c$ and $\nu = V$.

In the (X, Y) frame, all fluid motion is steady and consists of a flow in the negative X direction, roughly of the magnitude of the wave speed, underneath the stationary wave profile. The mean horizontal fluid velocity in this frame, for a constant value of Y over one wavelength λ is denoted by $-\overline{U}$. It is negative because the apparent flow is in the $-X$ direction. For the convenience of our calculation, the velocities in this frame are used to obtain the solutions.

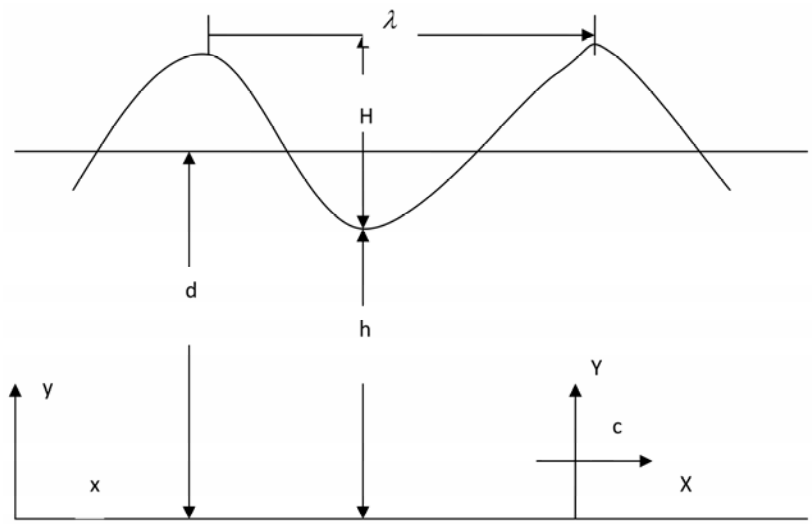

Figure 1. Wave train, showing important dimensions and coordinates.

3. Equations of Motion in a Frame Moving with the Wave

Consider the equation of motion in (X, Y) frame moving with the wave for steady, incompressible flow. There exists a stream function $\psi(X, Y)$ such that the velocity components (U, V) are given by

$$U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X}. \tag{1}$$

For irrotational flow, ψ satisfies Laplace equation

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = 0. \tag{2}$$

The boundary conditions at the bottom $Y = 0$ is a stream line on which $\psi(X, Y)$ is constant and at the free surface $Y = \eta(X)$ is also a stream line.

$$\therefore \psi(X, 0) = 0 \text{ (taking zero constant) and } \psi(X, \eta(X)) = -Q, \tag{3}$$

where Q is the volume flux underneath the wave train per unit span. The negative sign is for the flow which is in the negative X-direction, such that the wave will also propagate in the positive X-direction.

The remaining boundary condition from Navier-Stokes equation for steady incompressible flow

$$(\underline{\nu}.\nabla)\underline{\nu} = \underline{g} - \frac{1}{\rho}\nabla p + \upsilon\nabla^2\underline{\nu}, \tag{4}$$

where $\underline{\nu}$ is fluid velocity, $\underline{g} = (0, g)$ is acceleration of gravity, ρ is density, p is pressure, and υ is viscosity of the fluid.

For two components, Equation (4) can be rewritten as

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \upsilon\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right), \text{ on free surface } Y = \eta(X), \text{ pressure } p = 0, \tag{5}$$

and

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = g + \upsilon\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right). \tag{6}$$

Using Taylor expansion for ψ about the bed of the following form:

$$\psi(X, Y) = -\left(\sin Y\frac{d}{dX}\right)f(X) = \left\{-Y\frac{d}{dX} + \frac{Y^3}{3!}\frac{d^3}{dX^3} - \frac{Y^5}{5!}\frac{d^5}{dX^5} + \dots\right\}f(X), \tag{7}$$

as in Fenton [2], where $\frac{df}{dX}$ is the horizontal velocity on the bed. Using infinite Taylor series, the differential operator $\left(\sin Y \frac{d}{dX}\right)$ can be written

$$\left(\sin Y \frac{d}{dX}\right) = \left\{Y \frac{d}{dX} - \frac{Y^3}{3!}\frac{d^3}{dX^3} + \frac{Y^5}{5!}\frac{d^5}{dX^5} - \ldots\right\}. \tag{8}$$

Now, the velocity components anywhere in the fluid are

$$U = \frac{\partial\psi}{\partial Y} = -\left(\cos Y \frac{d}{dX}\right)f'(X),$$

$$V = -\frac{\partial\psi}{\partial X} = \left(\sin Y \frac{d}{dX}\right)f'(X).$$

The field Equation (2) is satisfied identically by the stream function, which is expressed in Equation (7). From Equations (5) and (6), we get

$$\left(\left(\cos Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos Y \frac{d}{dX}\right)f''(X)\right) + \left(\left(\sin Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin Y \frac{d}{dX}\right)f''(X)\right) = 0, \tag{9}$$

and

$$-\left(\left(\cos Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin Y \frac{d}{dX}\right)f''(X)\right) + \left(\left(\sin Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos Y \frac{d}{dX}\right)f''(X)\right) = g. \tag{10}$$

At the free surface $Y = \eta(X)$, Equations (3), (9), and (10) become

$$\left(\sin \eta \frac{d}{dX}\right)f(X) = Q, \tag{11}$$

$$\left(\left(\cos Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos Y \frac{d}{dX}\right)f''(X)\right) + \left(\left(\sin Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin Y \frac{d}{dX}\right)f''(X)\right) = 0, \tag{12}$$

and

$$-\left(\left(\cos Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin Y \frac{d}{dX}\right)f''(X)\right)$$

$$+\left(\left(\sin Y \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos Y \frac{d}{dX}\right)f''(X)\right) = g. \qquad (13)$$

Differentiating Equation (11), we get

$$\left(\sin \eta \frac{d}{dX}\right)f'(X) = -\frac{d\eta}{dX}\left(\cos \eta \frac{d}{dX}\right)f'(X). \qquad (14)$$

Substituting this value in Equations (12) and (13), we have

$$\left(\left(\cos \eta \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos \eta \frac{d}{dX}\right)f''(X)\right)$$

$$-\frac{d\eta}{dX}\left(\left(\cos \eta \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin \eta \frac{d}{dX}\right)f''(X)\right) = 0, \qquad (15)$$

and

$$\left(\left(\cos \eta \frac{d}{dX}\right)f'(X)\right)\left(\left(\sin \eta \frac{d}{dX}\right)f''(X)\right)$$

$$+\frac{d\eta}{dX}\left(\left(\cos \eta \frac{d}{dX}\right)f'(X)\right)\left(\left(\cos \eta \frac{d}{dX}\right)f''(X)\right) = -g. \qquad (16)$$

Equations (11), (15), and (16) are three nonlinear ordinary differential equations in the unknowns $\eta(X)$, $f'(X)$, and $f''(X)$. These ordinary differential equations can be solved by using power series method.

4. Power Series Solution

Assuming constant depth $\eta(X) = h$ and $f'(X) = U$, the derived equations can be solved by using series expansion method about the state of a uniform critical flow. Let the scaled horizontal variable be $\theta = \frac{\alpha X}{h}$, $\eta = \eta_* h$, and $f = f_* Q$.

Equations (11), (15), and (16) can be rewritten in terms of these dimensionless quantities as

$$\frac{1}{\alpha}\left(\sin\eta_*\alpha\frac{d}{d\theta}\right)f_*(\theta)-1=0 \text{ as } \frac{1}{h}\approx 1, \tag{17}$$

$$\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*'(\theta)\right)\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*''(\theta)\right)$$
$$-\alpha\frac{d\eta_*}{d\theta}\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*'(\theta)\right)\left(\left(\sin\eta_*\alpha\frac{d}{d\theta}\right)f_*''(\theta)\right)=0, \tag{18}$$

and

$$\frac{d\eta_*}{d\theta}\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*'(\theta)\right)\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*''(\theta)\right)$$
$$+\frac{1}{\alpha}\left(\left(\cos\eta_*\alpha\frac{d}{d\theta}\right)f_*'(\theta)\right)\left(\left(\sin\eta_*\alpha\frac{d}{d\theta}\right)f_*''(\theta)\right)=-g_*,$$
$$\text{as } \frac{1}{\alpha^2}\approx 1 \text{ and } \frac{gh}{Q^2}=g_*. \tag{19}$$

Equations (17), (18), and (21) suggest that α^2 is taken as the expansion parameter, we write the series expansions

$$\eta_* = 1+\sum_{j=1}^{N}\alpha^{2j}Y_j(\theta), \tag{20}$$

$$f_*' = 1+\sum_{j=1}^{N}\alpha^{2j}F_j(\theta), \tag{21}$$

$$g_* = 1+\sum_{j=1}^{N}\alpha^{2j}g_j, \tag{22}$$

where N is the order of solution required.

5. First Order Cnoidal Wave Solution

Using the above series expansion taking $o(\alpha^6)$, Equations (17), (18), and (19) can be written as, respectively,

$$\alpha^2(Y_1 + F_1) + \alpha^4\left(Y_2 + F_1Y_1 + F_2 - \frac{1}{6}F_1''\right)$$
$$+ \alpha^6\left(Y_3 + Y_2F_1 + Y_1F_2 + F_3 - \frac{1}{6}F_2'' - \frac{1}{2}Y_1F_1'' + \frac{1}{120}F_1^{iv}\right) = 0, \quad (23)$$

$$\alpha^2F_1' + \alpha^4\left(F_2' - \frac{F_1'''}{2} + F_1F_1'\right)$$
$$+ \alpha^6\begin{pmatrix} F_3' - Y_1F_1''' - \frac{1}{2}F_2''' + \frac{1}{24}F_1^{v} + F_1F_2' - \frac{1}{2}F_1F_1''' + F_1'F_2 \\ -\frac{1}{2}F_1'F_1'' - Y_1'F_1'' \end{pmatrix} = 0; \quad (24)$$

and

$$\alpha^2F_1'' + \alpha^4\left(Y_1'F_1' + F_2'' + Y_1F_1'' - \frac{1}{6}F_1^{iv} + F_1F_1''\right)$$
$$+ \alpha^6\begin{pmatrix} F_1'Y_2' + F_2'Y_1' - \frac{1}{2}F_1''Y_1' + F_1F_1'Y' + F_3'' + Y_1F_2'' + Y_2F_1'' - \frac{1}{6}F_2^{iv} \\ -\frac{1}{2}Y_1F_1^{iv} + \frac{1}{120}F_1^{vi} + F_1F_2'' - \frac{1}{6}F_1F_1^{iv} + F_2F_1'' + Y_1F_1F_1'' - \frac{1}{2}F_1''^2 \end{pmatrix}$$
$$= -\left(1 + \alpha^2g_1 + \alpha^4g_2 + \alpha^6g_3\right). \quad (25)$$

Equating the coefficients of α^2 and α^4 from Equations (23), (24), and (25), we have

$$\left.\begin{aligned} F_1 + Y_1 &= 0, \\ F_1' &= 0, \\ F_1'' + g_1 &= 0. \end{aligned}\right\}, \quad (26)$$

$$\left.\begin{aligned} &F_2 + Y_2 + Y_1F_1 - \frac{1}{6}F_1'' = 0, \\ &F_1F_1' + F_2' - \frac{F_1'''}{2} = 0, \\ &F_2'' + Y_1'F_1' + Y_1F_1'' + F_1F_1'' - \frac{1}{6}F_1^{iv} + g_2 = 0. \end{aligned}\right\}. \tag{27}$$

Using Equation (26), Equation (27) can be written as

$$\left.\begin{aligned} &F_2 + Y_2 - F_1^2 - \frac{1}{6}F_1'' = 0, && \text{(a)} \\ &F_1F_1' + F_2' - \frac{F_1'''}{2} = 0, && \text{(b)} \\ &F_2'' - F_1'^2 - \frac{1}{6}F_1^{iv} + g_2 = 0. && \text{(c)} \end{aligned}\right\}. \tag{28}$$

Differentiating (28a), we have

$$F_2' + Y_2' - 2F_1F_1' - \frac{1}{6}F_1''' = 0. \tag{29}$$

Again, differentiating the above equation

$$F_2'' + Y_2'' - 2F_1F_1'' - 2F_1'^2 - \frac{1}{6}F_1^{iv} = 0. \tag{30}$$

Applying Equations (28b)-(29), we get

$$3F_1F_1' - Y_2' - \frac{1}{3}F_1''' = 0.$$

Again, differentiating the above equation

$$\therefore\ 3F_1F_1'' + 3F_1'^2 - Y_2'' - \frac{1}{3}F_1^{iv} = 0. \tag{31}$$

Again, (28c)-(30), we get

$$F_2'' - F_1'^2 - \frac{1}{6}F_1^{iv} + g_2 - F_2'' - Y_2'' + 2F_1F_1'' + 2F_1'^2 + \frac{1}{6}F_1^{iv} = 0$$

$$\Rightarrow Y_2'' - 2F_1F_1'' - F_1'^2 - g_2 = 0. \tag{32}$$

Adding (31) and (32), we obtain

$$\frac{1}{3}F_1^{iv} - 2F_1'^2 + g_1F_1 + g_2 = 0. \tag{33}$$

This is a nonlinear ordinary differential equation of fourth order. According to Abramowitz and Stegun [11], the solution for F_1 in terms of $cn^2(\theta/m)$, the Jacobian elliptic function can be obtained as

$$F_1 = A_1 cn^2(\theta / m), \tag{34}$$

where A_1 is independent of θ and m is the parameter of elliptic function. According to Abramowitz and Stegun [11], we get

$$sn^2(\theta / m) = 1 - cn^2(\theta / m), \tag{35}$$

and

$$dn^2(\theta / m) = 1 - m + mcn^2(\theta / m), \tag{36}$$

$$F_1' = \frac{d}{d\theta}\left(A_1 cn^2(\theta / m)\right) = -2A_1 cn(\theta / m) sn(\theta / m)\, dn(\theta / m),$$

$$F_1'^2 = 4A_1^2\left[cn^2(\theta / m)\left\{1 - m + 2mcn^2(\theta / m) - cn^2(\theta / m) - mcn^4(\theta / m)\right\}\right], \tag{37}$$

$$\therefore\ F_1'' = A_1\left[(2 - 2m) + (8m - 4)cn^2(\theta / m) - 6mcn^4(\theta / m)\right], \tag{38}$$

$$\therefore\ F_1''' = A_1\left[\left\{-2(8m - 4)cn(\theta / m) + 24mcn^3(\theta / m)\right\} \times \left\{\left(1 - m - cn^2(\theta / m) + 2mcn^2(\theta / m) - mcn^4(\theta / m)\right)\right\}^{\frac{1}{2}}\right], \tag{39}$$

and

$$F_1^{iv} = 8A_1\left[-\left(1 - 3m + 2m^2\right) + \left(2 - 17m + 17m^2\right)cn^2(\theta / m) + m(15 - 30m)cn^4(\theta / m) + 15m^2cn^6(\theta / m)\right]. \tag{40}$$

Substituting these values in Equation (33), we get

$$\frac{8}{3}A_1\left[-\left(1-3m+2m^2\right)+\left(2-17m+17m^2\right)cn^2(\theta/m)\right.$$

$$\left.+\ m(15-30m)cn^4(\theta/m)+15m^2cn^6(\theta/m)\right]$$

$$-8A_1{}^2\left[cn^2(\theta/m)-mcn^2(\theta/m)+2mcn^4(\theta/m)-cn^4(\theta/m)-mcn^6(\theta/m)\right]$$

$$+g_1A_1cn^2(\theta/m)+g_2=0.$$

Collecting coefficients of like powers of $cn^2(\theta/m)$, we have

$$A_1=-5m,\quad g_1=-\frac{16}{3}\left(m^2-m+1\right),\text{ and } g_2=-\frac{40}{3}m(1-2m)(1-m).$$

Hence, the first order cnoidal solution from Equations (20), (21), and (22) becomes

$$\eta_*=1+5m\alpha^2cn^2(\theta/m), \tag{41}$$

$$f_*'=1-5m\alpha^2cn^2(\theta/m), \tag{42}$$

and

$$g_*=1-\frac{16}{15}.5\alpha^2\left(m^2-m+1\right). \tag{43}$$

Here, the above series have been in terms of α^2 where neglected terms are at least of the order of α^4. It is simpler to express the series in terms of δ, where

$$\delta=5\alpha^2, \tag{44}$$

as suggested by the results of Equations (41), (42), and (43). The function $cn(\theta/m)$ has a real period of $4K(m)$, where K is complete elliptic integral of first kind. According to Abramowitz and Stegun [11], it is easily shown that $cn^2(\theta / m)$ has a period of $2K(m)$, i.e., $\alpha x = K(m)$ and as the wave has a wavelength λ, $x = \frac{\lambda}{2h}$, therefore, the elementary geometric relation holds $\alpha \frac{\lambda}{h} = 2K(m)$.

Using the boundary condition at the wave crest $\eta_*(0) = 1 + \frac{H}{h}$, then Equation (41) can be written as

$$1 + \frac{H}{h} = 1 + 5m\alpha^2 cn^2(0) = 1 + 5m\alpha^2$$

$$\Rightarrow \alpha = \left(\frac{1}{5m}\frac{H}{h}\right)^{\frac{1}{2}}. \tag{45}$$

Therefore, Equation (41) becomes

$$\eta_* = 1 + \left(\frac{H}{h}\right) cn^2(\theta / m). \tag{46}$$

6. Second Order Cnoidal Wave Solution

From Equations (23), (24), and (25), equating the like powers of α^6, we have

$$\left.\begin{aligned}
&Y_3 + Y_2 F_1 + Y_1 F_2 + F_3 - \frac{1}{6} F_2'' - \frac{1}{2} Y_1 F_1'' + \frac{1}{120} F_1^{iv} = 0,\\
&F_3' - Y_1 F_1''' - \frac{1}{2} F_2''' + \frac{1}{24} F_1^{v} + F_1 F_2' - \frac{1}{2} F_1 F_1''' + F_1' F_2 - \frac{1}{2} F_1' F_1'' - Y_1' F_1'' = 0,\\
&F_1' Y_2' + F_2' Y_1' - \frac{1}{2} F_1'' Y_1' + F_1 F_1' Y_1' + F_3'' + Y_1 F_2'' + Y_2 F_1'' - \frac{1}{6} F_2^{iv} - \frac{1}{2} Y_1 F_1^{iv} + \frac{1}{120} F_1^{vi} + F_1 F_2''\\
&- \frac{1}{6} F_1 F_1^{iv} + F_2 F_1'' + Y_1 F_1 F_1'' - \frac{1}{2} F_1''^2 + g_3 = 0.
\end{aligned}\right\}.$$

Using Equation (26), we get

$$\left.\begin{aligned}
&Y_3 + Y_2F_1 - F_1F_2 + F_3 - \frac{1}{6}F_2'' + \frac{1}{2}F_1F_1'' + \frac{1}{120}F_1^{iv} = 0, && \text{(a}'\text{)}\\
&F_3' - \frac{1}{2}F_2''' + \frac{1}{24}F_1^{v} + F_1F_2' + \frac{1}{2}F_1F_1''' + F_1'F_2 + \frac{1}{2}F_1'F_1'' = 0, && \text{(b}'\text{)}\\
&F_1'Y_2' - F_2'F_1' + \frac{1}{2}F_1'''F_1' + F_1F_1'^2 + F_3'' + Y_2F_1'' - \frac{1}{6}F_2^{iv} + \frac{1}{3}F_1F_1^{iv} + \frac{1}{120}F_1^{vi}\\
&+ F_2F_1'' - F_1^2F_1'' - \frac{1}{2}F_1''^2 + g_3 = 0. && \text{(c}'\text{)}
\end{aligned}\right\}. \tag{47}$$

Now, differentiating Equation $(47\text{a}')$, we get

$$Y_3' + Y_2'F_1 + Y_2F_1' - F_1'F_2 - F_1F_2' + F_3' - \frac{1}{6}F_2''' + \frac{1}{2}F_1'F_1'' + \frac{1}{2}F_1F_1''' + \frac{1}{120}F_1^{v} = 0. \tag{48}$$

Again, differentiating Equation (48), we get

$$Y_3'' + Y_2''F_1 + 2Y_2'F_1' + Y_2F_1'' - 2F_1'F_2' - F_1''F_2 - F_1F_2'' + F_3'' - \frac{1}{6}F_2^{iv} + F_1'F_1''' + \frac{1}{2}F_1''^2 + \frac{1}{2}F_1F_1^{iv} + \frac{1}{120}F_1^{vi} = 0. \tag{49}$$

Subtracting Equation (48) from Equation $(47\text{b}')$, we get

$$Y_3' + Y_2'F_1 + Y_2F_1' - 2F_1'F_2 - 2F_1F_2' + \frac{1}{3}F_2''' - \frac{1}{30}F_1^{v} = 0. \tag{50}$$

Again, differentiating (50), we get

$$Y_3'' + Y_2''F_1 + 2Y_2'F_1' + Y_2F_1'' - 2F_1''F_2 - 4F_1'F_2' - 2F_1F_2'' + \frac{1}{3}F_2^{iv} - \frac{1}{30}F_1^{vi} = 0. \tag{51}$$

Subtracting Equation (51) from Equation (49), we get

$$F_3'' + F_1''F_2 + F_1F_2'' + 2F_1'F_2' + F_1'F_1''' + \frac{1}{2}F_1''^2 + \frac{1}{2}F_1F_1^{iv} - \frac{1}{2}F_2^{iv} + \frac{1}{24}F_1^{vi} = 0. \tag{52}$$

Finally, subtracting Equation (52) from Equation $(47\text{c}')$, we get

$$\frac{1}{3}F_2^{iv} - F_1F_2'' - 3F_1'F_2' + F_1'Y_2' + \frac{1}{2}F_1'''F_1' - F_1F_1'^2 + Y_2F_1'' - F_1^2F_1'' - F_1'F_1'''$$

$$- F_1''^2 - \frac{1}{6}F_1F_1^{iv} - \frac{1}{30}F_1^{vi} + g_3 = 0. \quad (53)$$

Substituting the values of Y_2 and Y_2' from Equation (28a) and using Equation (26) in Equation (53), we have

$$\frac{1}{3}F_2^{iv} - F_1F_2'' - F_1''F_2 - \frac{5}{6}F_1''^2 - \frac{1}{6}F_1F_1^{iv} - \frac{1}{30}F_1^{vi} + g_3 = 0, \quad (54)$$

which is a nonlinear ordinary differential equation. According to Fenton [2], we also assume that

$$F_2 = a_1 + a_2cn^2(\theta/m) + a_3cn^4(\theta/m), \quad (55)$$

where a_1, a_2, and a_3 are independent of θ and m.

Differentiating Equation (55) four times and using Equations (35) and (36), we have

$$\therefore F_2' = \left\{-2a_2cn(\theta/m) - 4a_3cn^3(\theta/m)\right\}sn(\theta/m)dn(\theta/m);$$

$$F_2'' = 2a_2(1-m) + \{12a_3(1-m) - 4a_2(1-2m)\}cn^2(\theta/m) + (-16a_3(1-2m)$$

$$-6a_2m)cn^4(\theta/m) - 20a_3mcn^6(\theta/m); \quad (56)$$

$$F_2''' = \begin{bmatrix} -2\{12a_3(1-m) - 4a_2(1-2m)\}cn(\theta/m) + 4\{16a_3(1-2m) \\ +6a_2m\}cn^3(\theta/m) + 120a_3mcn^5(\theta/m) \end{bmatrix}$$

$$sn(\theta/m)dn(\theta/m);$$

$$F_2^{iv} = 24a_3\left(1-2m+m^2\right) - 8a_2\left(1-3m+2m^2\right)$$

$$+\left\{-240a_3\left(1-3m+2m^2\right) + 8a_2\left(2-17m+17m^2\right)\right\}cn^2(\theta/m)$$

$$+\left\{32a_3\left(8-53m+53m^2\right) + 120a_2m(1-2m)\right\}cn^4(\theta/m)$$

$$+\left\{1040a_3m(1-2m) + 120a_2m^2\right\}cn^6(\theta/m) + 840a_3m^2cn^8(\theta/m). \quad (57)$$

Differentiating Equation (40), we have

$$\therefore\ F_1^{\nu} = 8A_1\Big[-2\left(2-17m+17m^2\right)cn(\theta/m) - 4m(15-30m)\,cn^3(\theta/m) - 90m^2cn^5(\theta/m)\Big]sn(\theta/m)dn(\theta/m).$$

Again, differentiating above equation and using Equations (35) and (36), we have

$$F_1^{\nu i} = -40m$$

$$\begin{bmatrix} 2\left(2-19m+34m^2-17m^3\right) - 8\left(1-33m+93m^2-62m^3\right)cn^2(\theta/m) \\ -4m\left(63-378m+378m^2\right)cn^4(\theta/m) - 840m^2(1-2m)cn^6(\theta/m) - 630m^3cn^8(\theta/m) \end{bmatrix}. \tag{58}$$

Also squaring Equation (38), we have

$$F_1''^2 = 100m^2$$

$$\begin{bmatrix} \left(1-2m+m^2\right) - 4\left(2m^2-3m+1\right)cn^2(\theta/m) + 2\left(11m^2-11m+2\right) \\ \times\ cn^4(\theta/m) + 12m(1-2m)cn^6(\theta/m) + 9m^2cn^8(\theta/m) \end{bmatrix}. \tag{59}$$

Substituting these values in Equation (54), we get

$$\begin{Bmatrix} 8a_3\left(1-2m+m^2\right) - \frac{8}{3}a_2\left(1-3m+2m^2\right) + 5a_1m(2-2m) - \frac{250}{3}m^2\left(1-2m+m^2\right) \\ +\frac{8}{3}m\left(2-19m+34m^2-17m^3\right) + g_3 \end{Bmatrix}$$

$$+\begin{bmatrix} -80a_3\left(1-3m+2m^2\right) + \frac{8}{3}a_2\left(2-17m+17m^2\right) + 10ma_2(1-m) + 5a_1m(8m-4) \\ +\ 5a_2m(2-2m) + \frac{1000}{3}m^2\left(2m^2-3m+1\right) + \frac{100}{3}m^2\left(1-3m+2m^2\right) \\ -\frac{32}{3}m\left(1-33m+93m^2-62m^3\right) \end{bmatrix}$$

$$cn^2(\theta/m)$$

$$+\left[\begin{array}{l}\left\{\frac{32}{3}a_3\left(8-53m+53m^2\right)+40a_2m(1-2m)\right\}+5m\{12a_3(1-m)-4a_2(1-2m)\} \\ -30a_1m^2+5a_2m(8m-4)+5a_3m(2-2m)-\frac{500}{3}m^2\left(11m^2-11m+2\right) \\ -\frac{100}{3}m^2\left(2-17m+17m^2\right)-\frac{16}{3}m^2\left(63-378m+378m^2\right)\end{array}\right]$$

$$cn^4(\theta/m)$$

$$+\left[\begin{array}{l}\frac{1040}{3}a_3m(1-2m)+40a_2m^2-5m\{16a_3(1-2m)+6a_2m\}-30a_2m^2 \\ +5a_3m(8m-4)-1000m^3(1-2m)-500m^3(1-2m)-1120m^3(1-2m)\end{array}\right]$$

$$cn^6(\theta/m)$$

$$+\left(280a_3m^2-100a_3m^2-30a_3m^2-750m^4-500m^4-840m^4\right)cn^8(\theta/m).$$

Now equating the coefficients of like power of $cn^2(\theta/m)$ from the above equation, we obtain the values of a_1, a_2, and a_3, i.e.,

$$a_1=\frac{1}{675}\left(10192-55927m+55927m^2\right), \tag{60}$$

$$a_2=\left(\frac{1838}{45}\right)m(1-2m), \tag{61}$$

and

$$a_3=\frac{209}{15}m^2. \tag{62}$$

Substituting the values a_1, a_2, and a_3 in Equation (55), we get

$$F_2=\frac{1}{675}\left(10192-55927m+55927m^2\right)+\frac{1838}{45}m(1-2m)cn^2(\theta/m)$$
$$+\frac{209}{15}m^2cn^4(\theta/m). \tag{63}$$

Therefore, Equation (28a) becomes

$$Y_2 = -\frac{10192}{675} + \frac{54802}{675}m - \frac{54802}{675}m^2 - \frac{1688}{45}m(1-2m)\,cn^2(\theta/m)$$

$$+ \frac{241}{15}m^2 cn^4(\theta/m). \tag{64}$$

Hence, the second order cnoidal solutions are obtained from Equations (20), (21), and (22), respectively,

$$\eta_* = 1 + \alpha^2 Y_1 + \alpha^4 Y_2 = 1 + 5m\alpha^2 cn^2(\theta/m)$$
$$+ \alpha^4 \left\{ \begin{aligned} & -\frac{10192}{675} + \frac{54802}{675}m - \frac{54802}{675}m^2 - \frac{1688}{45}m(1-2m)cn^2(\theta/m) \\ & + \frac{241}{15}m^2 cn^4(\theta/m) \end{aligned} \right\}, \tag{65}$$

$$f'_* = 1 + \alpha^2 F_1 + \alpha^4 F_2 = 1 - 5m\alpha^2 cn^2(\theta/m)$$
$$+ \alpha^4 \left\{ \begin{aligned} & \frac{1}{675}\left(10192 - 55927m + 55927m^2\right) + \frac{1838}{45}m(1-2m)cn^2(\theta/m) \\ & + \frac{209}{15}m^2 cn^4(\theta/m) \end{aligned} \right\}, \tag{66}$$

and

$$g_* = 1 + \alpha^2 g_1 + \alpha^4 g_2$$
$$= 1 - \frac{80}{15}\alpha^2\left(m^2 - m + 1\right) - \frac{40}{3}\alpha^4 m(1-2m)(1-m). \tag{67}$$

Again, using boundary condition $\eta_*(0) = 1 + \frac{H}{h}$, then Equation (65) can be written as

$$1 + \frac{H}{h} = 1 + 5m\alpha^2 + \alpha^4\left\{ -\frac{10192}{675} + \frac{54802}{675}m - \frac{54802}{675}m^2 - \frac{1688}{45}m(1-2m) + \frac{241}{15}m^2 \right\}$$

$$\Rightarrow \alpha^2 = \frac{1}{5m}\frac{H}{h}\left[1 + \frac{1}{25m^2}\frac{H}{h}\left(\frac{10192}{675} - \frac{29482}{675}m - \frac{6683}{675}m^2\right)\right],$$

$$\because \alpha^2 = \frac{1}{5m}\left(\frac{H}{h}\right),$$

and neglecting higher order of $\left(\frac{H}{h}\right)$.

$$\therefore \alpha = \left(\frac{1}{5m}\frac{H}{h}\right)^{\frac{1}{2}}\left[1 + \frac{H}{h}\left(\frac{10192 - 29482m - 6683m^2}{33750m^2}\right)\right]. \tag{68}$$

Substituting this value in Equation (65) and neglecting higher order of $\left(\frac{H}{h}\right)$, we have

$$\Rightarrow \eta^* = 1 + \left(\frac{H}{h}\right)cn^2(\theta/m)$$

$$+ \frac{1}{25m^2}\left(\frac{H}{h}\right)^2\left[\begin{array}{l}\left(-\frac{10192}{675} + \frac{54802}{675}m - \frac{54802}{675}m^2\right) \\ -\left\{-\frac{10192}{675} + \frac{54802}{675}m - \frac{43957}{675}m^2\right\}cn^2(\theta/m) \\ +\frac{241}{15}m^2cn^4(\theta/m)\end{array}\right]. \tag{69}$$

7. Mean Value of Jacobi Elliptic Function

If I_j is the mean value of $cn^{2j}(\theta/m)$, then

$$I_j = \overline{cn^{2j}(\theta/m)} = \frac{1}{K}\int_0^K \left[cn^2(\theta/m)\right]^j d\theta. \tag{70}$$

Putting $j = 0, 1, 2, \ldots,$ respectively, we have

$$I_0 = \frac{1}{K}\int_0^K \left[cn^2(\theta/m)\right]^0 d\theta = 1,$$

and

$$I_1 = \frac{1}{K}\int_0^K \left[cn^2(\theta/m)\right]d\theta. \tag{71}$$

Now, elliptic integral of the first kind is

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}.$$

Putting $\sin\theta = x,$ then

$$K(m) = \int_0^1 \frac{dx}{\sqrt{1 - mx^2}\sqrt{1 - x^2}}$$

$$= \int_0^1 \left(1 - x^2\right)^{-\frac{1}{2}}\left(1 - mx^2\right)^{-\frac{1}{2}} dx.$$

Again, putting $x^2 = t,$ we have

$$K(m) = \frac{1}{2}\int_0^1 (1 - t)^{-\frac{1}{2}}(1 - mt)^{-\frac{1}{2}}(t)^{-\frac{1}{2}} dt. \tag{72}$$

Also, using hypergeometric function, we have

$${}_2F_1(\alpha,\ \beta,\ \gamma;\ m) = \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma\gamma - \beta}\int_0^1 (1 - t)^{\beta-1}(1 - mt)^{\gamma-\beta-1}(t)^{-\alpha} dt$$

$$= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} m^n = 1 + \frac{\alpha\beta}{\gamma} m + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)2!} m^2 + \dots,$$

$$\therefore \int_0^1 (1 - t)^{\frac{1}{2}-1}(1 - mt)^{1-\frac{1}{2}-1}(t)^{-\frac{1}{2}} dt = \frac{\Gamma\frac{1}{2}\,\Gamma 1 - \frac{1}{2}}{\Gamma 1}\,{}_2F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1;\ m\right)$$

$$= 1 + \frac{\frac{1}{2}.\frac{1}{2}}{1} m + \frac{\frac{1}{2}\left(\frac{1}{2} + 1\right)\frac{1}{2}\left(\frac{1}{2} + 1\right)}{1(1 + 1)2!} m^2 + \dots.$$

$$\therefore {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; m\right) = \pi\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1.3}{2.4}\right)^2 m^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 m^3 + \ldots\right]. \quad (73)$$

Using Equation (50), Equation (72) becomes for $m << 1$

$$K(m) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1.3}{2.4}\right)^2 m^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 m^3 + \ldots\right].$$

Similarly, elliptic integral of second kind is

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \theta} d\theta = \frac{1}{2}\int_0^1 (1-t)^{-\frac{1}{2}}(1-mt)^{-\frac{1}{2}}(t)^{\frac{1}{2}} dt$$

$$= \frac{\sqrt{\frac{1}{2}\Big)1 - \frac{1}{2}}}{\Big)1}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; m\right)$$

$$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{m^2}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{m^3}{5} - \ldots\right].$$

Therefore,

$$e(m) = \frac{E(m)}{K(m)} = \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{m^2}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{m^3}{5} - \ldots\right]$$

$$\times \left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1.3}{2.4}\right)^2 m^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 m^3 + \ldots\right]^{-1}. \quad (74)$$

Now,

$$\frac{1}{K(m)} = \frac{2}{\pi}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1.3}{2.4}\right)^2 m^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 m^3 + \ldots\right]^{-1}$$

$$= \frac{2}{\pi}\left[1 - \frac{1}{4} m - \frac{5}{64} m^2 - \frac{11}{256} m^3 - \ldots\right].$$

Hence Equation (74) becomes

$$e(m) = \frac{E(m)}{K(m)} = \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{m^2}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{m^3}{5} - \ldots\right]$$

$$\times \left[1 - \frac{1}{4} m - \frac{5}{64} m^2 - \frac{11}{256} m^3 - \ldots\right]$$

$$= 1 - \frac{1}{2} m - \frac{1}{16} m^2 - \frac{1}{32} m^3 - \ldots.$$

Using the Fourier series expansion, the Jacobi elliptic function can be written as in the following form:

$$cn(\theta / m) = \frac{\pi}{\sqrt{m}K(m)} \sum_{n=0}^{\infty} \sec h\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right) \cos\left((2n+1)\frac{\pi z}{2K(m)}\right), \quad (75)$$

where $K'(m) = K(1-m)$ is the imaginary quarter period.

Now

$$\sec h\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right) = 1 - \frac{\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right)^2}{2!} + 5\frac{\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right)^4}{4!}$$

$$- 61\frac{\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right)^6}{6!} + \ldots. \quad (76)$$

Also, the nome q is given by

$$q = \exp\left(-\pi\frac{K'(m)}{K(m)}\right),$$

$$\therefore q^{\left(\frac{2n+1}{2}\right)} = \exp\left(-\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)$$

$$= 1 - \frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)} + \frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!}$$

$$- \frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} + \frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} - \ldots,$$

and

$$q^{(2n+1)} = \exp\left(-\pi(2n+1)\frac{K'(m)}{K(m)}\right)$$

$$= 1 - \pi(2n+1)\frac{K'(m)}{K(m)} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!}$$

$$+ \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} - \ldots,$$

$$\therefore 1 + q^{(2n+1)} = 2 - \pi(2n+1)\frac{K'(m)}{K(m)} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!}$$

$$+ \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} - \ldots$$

$$= 2\left[1 - \frac{1}{2}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!}\right.\right.$$

$$\left.\left. - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\}\right],$$

$$\therefore \left[1 + q^{(2n+1)}\right]^{-1} = \frac{1}{2}\left[1 - \frac{1}{2}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!}\right.\right.$$

$$\left.\left. + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\}\right]^{-1}$$

$$= \frac{1}{2}\left\{\begin{array}{l} 1 + \frac{1}{2}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} \right. \\ \qquad\qquad \left. - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\} \\ + \frac{1}{4}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} \right. \\ \qquad\qquad \left. - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\}^2 \\ + \frac{1}{8}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} \right. \\ \qquad\qquad \left. - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\}^3 \\ + \frac{1}{16}\left\{\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!} + \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!} \right. \\ \qquad\qquad \left. - \frac{\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!} + \ldots\right\}^4 + \ldots \end{array}\right.$$

$$= \frac{1}{2}\left\{1 + \frac{1}{2}\,\pi(2n+1)\frac{K'(m)}{K(m)} - \frac{1}{24}\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3 + \ldots\right\},$$

$$\therefore\ q^{(2n+1)/2}\left[1+q^{(2n+1)}\right]^{-1}$$

$$=\left[\begin{array}{l}1-\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}+\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!}-\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!}\\ +\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^3}{4!}-\ldots\end{array}\right]$$

$$=\frac{1}{2}\left\{1+\frac{1}{2}\pi(2n+1)\frac{K'(m)}{K(m)}-\frac{1}{24}\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3+\ldots\right\},$$

$$\therefore\ 2q^{(2n+1)/2}\left[1+q^{(2n+1)}\right]^{-1}$$

$$=\left[\begin{array}{l}1-\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}+\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!}-\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^3}{3!}\\ +\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!}+\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}-\frac{1}{4}\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^2\\ +\frac{1}{16}\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3-\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^4}{3!}-\frac{1}{24}\left(\pi(2n+1)\frac{K'(m)}{K(m)}\right)^3\\ +\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^4}{3}-\ldots\end{array}\right]$$

$$=\left[1-\frac{\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^2}{2!}+\frac{5\left(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)}\right)^4}{4!}-\ldots\right]. \tag{77}$$

Hence, from Equations (76) and (77), we get

$$\sec h\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right)=2\,\frac{q^{(2n+1)/2}}{1+q^{(2n+1)}}. \tag{78}$$

The nome q has the following form for small m according to Abramowitz and Stegun [11]:

$$q = \frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + 992\left(\frac{m}{16}\right)^4 + \ldots.$$

Hence, Equation (78) can be written as

$$\sec h\left((2n+1)\frac{\pi K'(m)}{2K(m)}\right) = 2\frac{\left[\frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + 992\left(\frac{m}{16}\right)^4 + \ldots\right]^{(2n+1)/2}}{1 + \left[\frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + 992\left(\frac{m}{16}\right)^4 + \ldots\right]^{(2n+1)}}. \tag{79}$$

Putting $n = 0$

$$\sec h\left(\frac{\pi K'(m)}{2K(m)}\right) = 2\frac{\left[\frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + 992\left(\frac{m}{16}\right)^4 + \ldots\right]^{\frac{1}{2}}}{1 + \left[\frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + 992\left(\frac{m}{16}\right)^4 + \ldots\right]}$$

$$= 2\left(\frac{m}{16}\right)^{\frac{1}{2}}\left[1 + 3\left(\frac{m}{16}\right) + 23\left(\frac{m}{16}\right)^2 + 129\left(\frac{m}{16}\right)^3 + \ldots\right],$$

$$\therefore \frac{\pi}{\sqrt{m}K(m)} \sec h\left(\frac{\pi K'(m)}{2K(m)}\right) = \frac{\pi}{\sqrt{m}}\frac{2}{\pi}\left[1 - \frac{1}{4}m - \frac{5}{64}m^2 - \frac{11}{256} - m^3 - \ldots\right]$$

$$\times 2\left(\frac{m}{16}\right)^{\frac{1}{2}}\left[1 + 3\left(\frac{m}{16}\right) + 23\left(\frac{m}{16}\right)^2 + 129\left(\frac{m}{16}\right)^3\right] = 1 - \frac{1}{16}m - \frac{9}{256}m^2 - \ldots.$$

Similarly, for $n = 1$,

$$\frac{\pi}{\sqrt{m}K(m)} \sec h\left(\frac{3\pi K'(m)}{2K(m)}\right) = \frac{1}{16}m + \frac{1}{32}m^2 + \ldots,$$

and for $n = 2$,

$$\frac{\pi}{\sqrt{m}K(m)} \sec h\left(\frac{5\pi K'(m)}{2K(m)}\right) = \frac{1}{256} m^2 + \ldots.$$

Therefore, for $m << 1$, the Jacobi elliptic function of Equation (75) can be written as

$$cn(\theta / m) = \left(1 - \frac{1}{16} m - \frac{9}{256} m^2 + \ldots\right) \cos \alpha\theta + \left(\frac{1}{16} m + \frac{1}{32} m^2 + \ldots\right) \cos 3\alpha\theta$$
$$+ \left(\frac{1}{256} m^2 +\right) \cos 5\alpha\theta + \ldots,$$

with $\alpha = \dfrac{\pi}{2K(m)}$.

Hence, its square term becomes

$$cn^2(\theta / m) = \left(\frac{1}{2} - \frac{1}{16} m - \frac{1}{32} m^2 + \ldots\right) + \left(\frac{1}{2} - \frac{3}{512} m^2 + \ldots\right) \cos 2\alpha\theta$$
$$+ \left(\frac{1}{16} m + \frac{1}{32} m^2 + \ldots\right) \cos 4\alpha\theta + \left(\frac{3}{512} m^2 + \ldots\right) \cos 6\alpha\theta + \ldots. \quad (80)$$

Substituting this value in Equation (71), we have

$$I_1 = \frac{1}{K} \int_0^K$$
$$\begin{bmatrix} \left(\frac{1}{2} - \frac{1}{16} m - \frac{1}{32} m^2 + \ldots\right) + \left(\frac{1}{2} - \frac{3}{512} m^2 + \ldots\right) \cos 2\alpha\theta \\ + \left(\frac{1}{16} m + \frac{1}{32} m^2 + \ldots\right) \cos 4\alpha\theta + \left(\frac{3}{256} m^2 + \ldots\right) \cos 6\alpha\theta + \ldots \end{bmatrix} d\theta$$

$$= \frac{1}{K} \begin{bmatrix} \left(\frac{1}{2} - \frac{1}{16} m - \frac{1}{32} m^2 + \ldots\right)\theta + \frac{1}{2}\left(\frac{1}{2} - \frac{3}{512} m^2 + \ldots\right) \sin 2\alpha\theta \\ + \frac{1}{4}\left(\frac{1}{16} m + \frac{1}{32} m^2 + \ldots\right) \sin 4\alpha\theta + \frac{1}{6}\left(\frac{3}{256} m^2 + \ldots\right) \sin 6\alpha\theta + \ldots \end{bmatrix}_0^K,$$

$$\therefore I_1 = \frac{1}{m} \{-1 + m + e(m)\}. \quad (81)$$

The other values may be computed in the same way.

8. Solitary Wave Limit

For every long nonlinear waves with the parameter m close to one, i.e., $m \to 1,$ the Jacobi elliptic function $cn(\theta / m)$ can be written as [11]

$$cn(\theta / m) \approx \sec h(\theta) - \frac{1}{4}(1-m)[\sinh(\theta)\cosh(\theta) - \theta]\tanh(\theta)\sec\ h(\theta), \quad (82)$$

with $\sec\ h(\theta) = \dfrac{1}{\cosh(\theta)}$, where sinh, cosh, tanh, and sech are hyperbolic functions. In the limit $m \to 1,$ $cn(\theta / m) \to \sec h(\theta).$

In the limit $m \to 1,$ from Equations (63) and (64), the values of F_2 and Y_2 become

$$F_2 = \frac{10192}{675} - \frac{1838}{45}\sec h^2(\theta) + \frac{209}{15}\sec h^4(\theta), \quad (83)$$

$$Y_2 = -\frac{10192}{675} + \frac{1688}{45}\sec h^2(\theta) + \frac{241}{15}\sec h^4(\theta). \quad (84)$$

For solitary wave limit $(m \to 1),$ Equations (41), (42), and (43), i.e., first order cnoidal wave solutions can be written as

$$\left.\begin{aligned} \eta^* &= 1 + 5\alpha^2 \sec h^2(\theta), \\ f'^* &= 1 - 5\alpha^2 \sec h^2(\theta), \text{ and} \\ g^* &= 1 - \frac{80}{15}\alpha^2. \end{aligned}\right\}. \quad (85)$$

Again, for solitary wave limit $(m \to 1),$ Equations (65), (66), and (67), i.e., second order cnoidal wave solutions can be written as

$$\left.\begin{aligned} \eta^* &= 1 + 5\alpha^2 \sec h^2(\theta) + \alpha^4\left\{-\frac{10192}{675} + \frac{1688}{45}\sec h^2(\theta) + \frac{241}{15}\sec h^4(\theta)\right\}, \\ f'^* &= 1 - 5\alpha^2 \sec h^2(\theta) + \alpha^4\left\{\frac{10192}{675} - \frac{1838}{45}\sec h^2(\theta) + \frac{209}{15}\sec h^4(\theta)\right\}, \text{ and} \\ g^* &= 1 - \frac{80}{15}\alpha^2. \end{aligned}\right\}. \quad (86)$$

9. Conclusion

Cnoidal wave is a non linear and exact periodic wave solution of the Korteweg and de Vries (KdV) equation. These solutions are in terms of the Jacobi elliptic function $cn(\theta / m)$. Here, first and second order cnoidal wave solutions have been developed by using boundary conditions at the bottom $Y = 0$ and at the free surface $Y = \eta(X)$. Also, the boundary conditions are taken from Navier-Stokes equation of motion. To get accurate results, mean value of Jacobi elliptic functions are also derivedin terms of elliptic parameter. Using hypergeometric function and nome, elliptic integral of first and second kinds are converted to elliptic parameter. In solitary wave limit, i.e., elliptic parameter approaches to one, first and second order cnoidal wave solutions have also been generalized.

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