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# Torsional Shock Waves in a Viscoelastic Rod[1]

*The propagation of torsional shock waves in a thin circular viscoelastic rod is investigated theoretically. An analysis is carried out based on the approximate equations previously derived. Two typical viscoelastic models are considered, which possess, respectively, the discrete and continuous relaxation spectrum. One is the usual Maxwell-Voigt model and the other is a new model whose relaxation function is given by a power law with weak singularity. The structures of steady shock profiles are presented and compared for both types. Finally a brief discussion is included on the simplified evolution equations for a far field transient behavior.*

## 1 Introduction

A dynamic test of a viscoelastic solid at high strain rates often utilizes the wave propagation techniques consisting of the introduction of torsional or longitudinal waves in a circular rod. The torsional waves are of primary importance in evaluating a shear response. In such viscoelastic materials as polymers, the effect of finite torsion, namely, nonlinearity, plays an essential role so that the linear theory is of no use. Nevertheless, the nonlinear dynamic theory for torsional waves has been poorly cultivated so far, even for an elastic rod. Usually the effect of nonlinearity manifests itself in formation of shock waves.[2] In particular, the propagation of steady shock waves is possible as a result of balance between both the effects of nonlinearity and viscoelasticity. One of the purposes of this paper is to clarify whether the steady torsional shock waves can propagate in a circular rod and, if possible, to obtain explicit shock profiles. Our starting point is set on the approximate equations already derived in [1] for a thin, "nearly elastic" rod. The other purpose is to introduce a new type of viscoelastic character and to compare its shock profiles with the well-known model. Since the shock profiles reflect a dynamic behavior of materials, they can provide useful information in evaluating the relevant constitutive equations at a finite strain and high strain rate.

The viscoelastic behavior is often modeled by combinations of several springs and dashpots, which are usually known as Voigt model, Maxwell-Voigt model (the standard solid), etc. Such models have succeeded in explaining intuitively the inner relaxation mechanisms involved. But they are too simple to describe the actual viscoelastic behavior of materials like polymers. For a spring-dashpot model, the constitutive equations are given in the differential form. In this case, the relaxation functions are represented by a sum of several exponential functions including the delta or step function as a limiting case. Each exponential function introduces a characteristic relaxation time associated with each element involved. In this sense, the spring-dashpot model has usually several discrete relaxation times. It is known, however, that the actual viscoelastic materials have infinitely and continuously many relaxation times resulting from the inner molecular relaxation mechanisms and consequently they exhibit a slow relaxation over a long time range [2]. To describe them properly, the idea of the relaxation spectrum is introduced, so that the constitutive equations are essentially expressed in the integral (functional) form. In this paper, we consider, as a simple but promising model of relaxation function, the power function with weak singularity. This model is proposed by assuming that the continuous relaxation spectrum is approximated by an inverse power law with respect to the relaxation time.

In what follows, the approximate equations are first presented. For the relaxation function involved there, various viscoelastic models, especially of the power function type, are discussed. The conditions for existence of steady shock waves are derived, which pose restrictions on the elastic response of materials and the relaxation character. In obtaining the shock profiles, we treat two types of relaxation functions that typify the discrete and continuous relaxation spectrum, respectively. One is the usual Maxwell-Voigt model in which the relaxation function is given by the single exponential function (Type I called hereafter). The other is the power function type proposed here (Type II). Particularly for Type II, the nonlinear integral equation with the weak singular kernel is reduced to obtain the shock profiles. The numerical method solving it is demonstrated. The explicit structures of shock profiles are compared for both types. Finally a brief discussion is included on the simplified evolution equations describing a far field behavior.

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[1] A part of this paper was presented at the IUTAM symposium on "Nonlinear Deformation Waves" held at Tallinn, Estonia U.S.S.R. in 1982.

[2] The shock waves in the present context are meant not only by the first-order discontinuity of displacement but also by the smooth transition layer caused intrinsically by the nonlinearity.

Contributed by the Applied Mechanics Division for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division August; 1983, final revision, November, 1983.

## 2 Approximate Equations

The following analysis begins with the approximate equations previously derived in [1]. A finite torsional deformation couples with a longitudinal one through the normal stress effect. If the angle of torsion of a cross section and the longitudinal deformation are denoted by $\varphi$ and $w$, respectively, the approximate equations in dimensionless form are given in terms of the axial coordinate $z$ and the time $t$ as

$$\frac{\partial^2 \varphi}{\partial t^2} - G\frac{\partial^2 \varphi}{\partial z^2} - \gamma\frac{\partial^2}{\partial z^2}\int_{-\infty}^{t} K(t-t_1)\frac{\partial}{\partial t_1}\varphi(z,t_1)dt_1$$

$$= -2\epsilon^2\left[\frac{\partial}{\partial t}\left(\Phi\frac{\partial \varphi}{\partial t}\right) - G\frac{\partial}{\partial z}\left(\Phi\frac{\partial \varphi}{\partial z}\right)\right]$$

$$+\epsilon^2\frac{\partial}{\partial z}\left\{\left[a\frac{\partial w}{\partial z} + b_1\left(\frac{\partial \varphi}{\partial t}\right)^2 + b_2\left(\frac{\partial \varphi}{\partial z}\right)^2\right]\frac{\partial \varphi}{\partial z}\right\}, \quad (1)$$

$$\frac{\partial^2 w}{\partial t^2} - E\frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z}\left[c_1\left(\frac{\partial \varphi}{\partial t}\right)^2 + c_2\left(\frac{\partial \varphi}{\partial z}\right)^2\right], \quad (2)$$

with

$$\Phi = -\frac{E-2G}{2G}\frac{\partial w}{\partial z} + d_1\left(\frac{\partial \varphi}{\partial t}\right)^2 + d_2\left(\frac{\partial \varphi}{\partial z}\right)^2, \quad (3)$$

where the length and time scales are already normalized, respectively, by a characteristic wavelength $L$ (which is assumed to be sufficiently long compared with the radius of rod $R$) and time $T$ $(=L/(S/\rho_0)^{1/2})$, $S$ being a characteristic stress, and $\rho_0$ the density in the reference state; $G$ and $E$, both normalized by $S$, denote, respectively, the modulus of rigidity and Young's modulus in the equilibrium state, while the constants $a$, $b_i$, $c_i$, and $d_i$ $(i = 1, 2)$ are determined by $G$, $E$, and the higher order equilibrium elastic moduli (see Appendix 2 in [1]). Here the two small parameters $\gamma$ and $\epsilon$ $(0 < \gamma, \epsilon << 1$, and $\epsilon^4 < \gamma << 1)$ imply, respectively, the order of weakness of viscoelasticity and that of thin rod $(\epsilon \equiv R/L)$, the latter also representing the order of torsional shear strain, i.e., *nonlinearity* [1]. The viscoelastic behavior is characterized by the relaxation function $K(t)$. Strictly speaking, however, we note that $G + \gamma K(t)$ represents the shear stress relaxation function in linear viscoelasticity and hence $\gamma K(t)$ is a deviation from the equilibrium state $G$. Accordingly $K(t)$ is assumed to vanish as $t$ increases.

For the well-known spring-dashpot models, $K(t)$ is represented by a sum of several exponential functions as

$$K(t) = \sum_{\alpha} K_\alpha \exp\left(-\frac{t}{T_\alpha}\right), \quad \text{for } t \geq 0, \quad (4)$$

where $T_\alpha$ $(>0)$ is called a characteristic relaxation time, while $K_\alpha$ is a modulus contribution. In this sense, it is said that (4) has several relaxation times. Since $K(t)$ vanishes as $t$ increases, the special case with $T_\alpha = \infty$ is excluded here. But the other special case with $T_\alpha$, say $T_1 \to 0$ with $K_1 T_1$ held constant is included to represent the delta function. For actual viscoelastic materials, however, there are continuously and infinitely many relaxation times. In this case, equation (4) is extended as

$$K(t) = \int_0^\infty H(T)\exp\left(-\frac{t}{T}\right)\frac{dT}{T}, \quad \text{for } t \geq 0, \quad (5)$$

where after the convention, $H(T)$ is called the relaxation spectrum [2, 3]. In view of this, the spring-dashpot model is said to have the discrete relaxation spectrum. In identifying the viscoelastic behavior, on the other hand, there is another approach in terms of a creep function $J(t)$ or a retardation spectrum $L(T)$ [2]. Experimentally this approach is preferable to that by a relaxation function or spectrum employed in the present theoretical treatment. But since there are interrelations among them [2–4], a transformation from one to the other is a matter of integration.

Among the models having a continuous relaxation spectrum, there are known the power function models. Caputo and Mainardi [5, 6] investigated various linear models by generalizing the Maxwell-Voigt model. Among them, we propose here, as a simple but promising model for the viscoelastic "solids," $H(T)$ approximated by the inverse power law of $T$:

$$H(T) = T^{-\nu}/\Gamma(\nu), \ (0<\nu<1), \quad (6)$$

where $\Gamma(\nu)$ is the gamma function. The resulting relaxation function from (5) with (6) is also given in the form of the inverse power function of time:

$$K(t) = t^{-\nu}, \quad \text{for } t \geq 0, \ (0<\nu<1). \quad (7)$$

Here we note that $K(t)$ exhibits an infinite stiffness at $t = 0$ just as the delta function in the Voigt model. But the relaxation does not cease so quickly as the delta function but rather it relaxes more slowly than the exponential function in the Maxwell-Voigt model. In the creep formulation, the creep function $J(t)$ corresponding to (7) is given by

$$J(t) = \frac{1}{G}\left\{1 - E_\nu\left[-\frac{G}{\gamma\Gamma(1-\nu)}t^\nu\right]\right\}, \quad \text{for } t \geq 0. \quad (8)$$

where $E_\nu[\ldots]$ denotes the Mittag-Leffler function of order $\nu$[7]. From (8), it is found that $J(t)$ increases from zero and approaches the equilibrium value $G^{-1}$ as time elapses.

As another simple and plausible model, it is known that the creep function may be given by a power function of time [4–6]:

$$J(t) = J_0 + \gamma t^{\nu'}, \quad \text{for } t \geq 0, \ (0<\nu'<1), \quad (9)$$

where $J_0$ is a positive constant and $\gamma$ is also a small positive constant for a measure of weakness of viscoelasticity. Owing to the relations between the creep function $J(t)$ and the relaxation function $K'(t)$ $(=G+\gamma K(t))$ [4–6], the corresponding $K'(t)$ is given by

$$K'(t) = \frac{1}{J_0}E_{\nu'}\left[-\frac{\gamma\Gamma(1+\nu')}{J_0}t^{\nu'}\right], \quad \text{for } t \geq 0. \quad (10)$$

Since we are concerned with the case $\gamma << 1$, equation (10) is expanded as

$$K'(t) = G - \gamma G^2 t^{\nu'} + O(\gamma^2), \quad \text{for } t \geq 0, \quad (11)$$

where $G = J_0^{-1}$ and $\gamma G^2 t^{\nu'} << G$. The first term $G$ implies the constant elastic response and the second term gives $K(t)$ in our formulation, namely, $K(t) = -G^2 t^{\nu'}$. This model is also very simple. But is should be noted that as time elapses, (9) increases without limit and no equilibrium state can be expected. Such a model provides a good agreement with the experiments at a relatively slow strain rate and over a long time range [4]. It is not suitable, however, for a case with a high strain rate such as the propagation of shock waves. Indeed, as will be shown later, there exist no steady shock waves for this model. Therefore we are concerned here with the power function model given by (7). To compare with the usual spring-dashpot model, we first consider the exponential function type (Type I) corresponding to the well-known Maxwell-Voigt model and later the power function type (Type II):

Type I: $\quad K(t) = \exp(-\kappa t), \quad (12)$

Type II: $\quad K(t) = t^{-\nu}, \ (0<\nu<1), \quad (13)$

where $\kappa^{-1}$ is the characteristic relaxation time. But we note again that there exists no such time in Type II because of the continuous relaxation spectrum.

Before proceeding to the shock wave problem, we consider the linear dispersion ralation of (1). On neglecting the nonlinear terms, it can easily be obtained by setting $\varphi \propto \exp[i(kz-\omega t)]$, $k$ and $\omega$ ($>0$) being a wave number and a frequency, respectively. For Type I, the phase velocity $c(=\omega/k)$ is given by $c = [G+\gamma/(1+i\kappa/\omega)]^{1/2}$. In this case, there exist two characteristic sound speeds, the instantaneous sound speed $c_i = (G+\gamma)^{1/2}$ as $\omega \to \infty$, and the equilibrium sound speed $c_e = G^{1/2}$ as $\omega \to 0$. For Type II, on the other hand, since the relaxation function exhibits an infinite stiffness at $t=0$, the instantaneous sound speed becomes infinite. This can be seen from the dispersion relation $c = [G + \gamma\Gamma(1-\nu)(-i\omega)^{\nu}]^{1/2}$ that $c_i \to \infty$ as $\omega \to \infty$, although $c_e = G^{1/2}$.

## 3 Structure of Steady Shock Waves

Based on (1)–(3), let us investigate the steady shock waves propagating into the unstrained state far ahead ($z \to \infty$) with an equilibrium state far behind ($z \to -\infty$). Assuming $\varphi$ and $w$ depend on $\eta = t - z/\lambda$ only, $\lambda$ being a positive constant representing a shock velocity, equations (1)–(3) are written as

$$(\lambda^2 - G)\varphi' - \gamma\int_{-\infty}^{\eta} K(\eta-\eta_1 \frac{d\varphi'}{d\eta_1} d\eta_1$$

$$= -2\epsilon^2(\lambda^2-G)\Phi\varphi' + \epsilon^2\Big[-\frac{a}{\lambda}\frac{dw}{d\eta}$$

$$+\Big(b_1 + \frac{b_2}{\lambda^2}\Big)\varphi'^2\Big]\varphi' + \text{const.}, \qquad (14)$$

$$\frac{dw}{d\eta} = -\frac{c_1\lambda^2 + c_2}{\lambda(\lambda^2-E)}\varphi'^2 + \text{const.}, \qquad (15)$$

with

$$\Phi = \frac{E-2G}{2G\lambda}\frac{dw}{d\eta} + \Big(d_1 + \frac{d_2}{\lambda^2}\Big)\varphi'^2, \qquad (16)$$

where $\varphi' \equiv d\varphi/d\eta$ and const. denotes an integration constant. For the boundary condition at both infinite ends of the rod, we require that $\varphi'$ and $dw/d\eta \to 0$ as $\eta \to -\infty$, while as $\eta \to +\infty$, $\varphi' \to \varphi'_\infty$ (= constant), where note that $dw/d\eta$ as $\eta \to +\infty$ is then determined by (15) as $-\{(c_1\lambda^2+c_2)/[\lambda(\lambda^2 - E)]\}\varphi_\infty'^2$ + const. Hence the integration constants must be chosen to be zero. We eliminate $dw/d\eta$ in (14) by (15). In (14), all quantities except $\epsilon^2$ and $\gamma$ are assumed to be of order of unity. To have a significant $\varphi'$, $(\lambda^2-G)$ should be of order of $\epsilon^2$ or $\gamma$, so that $\lambda^2$ is nearly equal to $G$: $\lambda^2 = G + O(\epsilon^2, \gamma)$. Using this result, it follows upon retaining the lowest order of $\epsilon^2$ and $\gamma$ that

$$(\lambda^2 - G)\varphi' - \gamma\int_{-\infty}^{\eta} K(\eta-\eta_1)\frac{d\varphi'}{d\eta_1} d\eta_1$$

$$\cong \frac{\epsilon^2}{\lambda^2}\Big[\frac{a(c_1\lambda^2+c_2)}{\lambda^2-E} + b_1\lambda^2 + b_2\Big]\varphi'^3$$

$$\cong \frac{\epsilon^2}{G}\Big[\frac{a(c_1 G+c_2)}{G-E} + b_1 G + b_2\Big]\varphi'^3$$

$$\cong \epsilon^2 C\varphi'^3, \qquad (17)$$

where $C$ is defined by $[a(c_1G+c_2)/(G-E) + b_1G + b_2]/G$. We introduce the parameter $U = (\lambda^2-G)/\gamma$ and set $\phi = [\epsilon^2 |C|/\gamma]^{1/2}\varphi'$. Then (17) and the boundary condition are rewritten in a compact form as

$$U\phi - \text{sgn}C\phi^3 = \int_{-\infty}^{\eta} K(\eta-\eta_1)\frac{d\phi}{d\eta_1} d\eta_1, \qquad (18)$$

with $\phi \to 0$ as $\eta \to -\infty$, and $\phi \to \phi_\infty$ = constant as $\eta \to +\infty$, where the sgn $C$ takes the value 1 for $C>0$ and $-1$ for $C<0$, and $\phi_\infty \equiv [\epsilon^2 |C|/\gamma]^{1/2}\varphi'_\infty$. Here the case with $C=0$ is excluded since no steady shock waves exist in such a "linear" case. By the cubic nonlinearity in $\phi$, the solutions of (18) always have two branches $\pm\phi$ which represent the right and left twisted wave. In what follows, we are only concerned with a case $\phi_\infty > 0$. The longitudinal deformation induced, $w$, is obtained by integrating (15) if $\phi$ i.e., $\varphi'$ is solved. Finally we keep it in mind that $\phi$ represents physically the variation of the torsional shear strain or the angular velocity of a cross section, if a proper numerical factor is introduced.

**3.1 Conditions for Existence of Steady Shock Waves.** Before solving (18), we check the necessary conditions for the steady shock wave solutions to exist. On assuming the existence of $\phi_\infty$, we multiply (18) by $d\phi/d\eta$ and integrate it over the whole range of $\eta$. Then it follows that

$$U\phi_\infty^2/2 - \text{sgn}C\phi_\infty^4/4 = (2\pi)^{1/2}\int_{-\infty}^{\infty}\hat{K}(y)\,|\hat{\phi}'(y)|^2 dy \equiv A, \qquad (19)$$

where $\hat{\phi}'(y)$ and $\hat{K}(y)$ are, respectively, Fourier transform of $d\phi/d\eta$ and $K(\eta)$ defined by, for example,

$$\hat{\phi}'(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\phi}{d\eta}\exp(iy\eta)d\eta, \qquad (20)$$

where $K(\eta)$ vanishes for $\eta < 0$ and therefore it is extended as $K(|\eta|)h(\eta)$, $h(\eta)$ being the unit step function. Here each explicit expression of A for Type I and Type II is given, respectively, by

$$A = \int_{-\infty}^{\infty}\frac{\kappa}{y^2+\kappa^2}\,|\hat{\phi}'(y)|^2 dy > 0, \quad \text{for Type I}, \qquad (21)$$

and

$$A = \Gamma(1-\nu)\sin\Big(\frac{\pi\nu}{2}\Big)\int_{-\infty}^{\infty}|y|^{\nu-1}\,|\hat{\phi}'(y)|^2 dy > 0,$$

$$\text{for Type II}, \qquad (22)$$

Next, on differentiating (18) with respect to $\eta$ and multiplying it by $\phi$, it also follows after integration that [3]

$$U\phi_\infty^2/2 - 3\,\text{sgn}\,C\phi_\infty^4/4 = -A. \qquad (23)$$

Thus from (19) and (23), we should have

$$\phi_\infty^2 = U\text{sgn}\,C, \quad \text{and} \quad A = U^2\,\text{sgn}\,C/4. \qquad (24)$$

Hence for $\phi_\infty$ to exist, $A$ must be finite. To examine it, we must know the asymptotic behavior of $\hat{\phi}'(y)$ as $y \to 0$ and $|y| \to \infty$. For the former limit, it is easily seen from the definition (20) that $\hat{\phi}'(y) \to (2\pi)^{-1/2}\phi_\infty$ as $y \to 0$, provided $\phi_\infty$ be finite. For the latter limit, on the other hand, $\hat{\phi}'$ dies away at least as rapidly as $|y|^{-2}$, if $\phi$ and $d\phi/d\eta$ are continuous [8] (see next subsection) and hence $A$ is found to exist. Therefore if we are looking for the continuous solutions with $d\phi/d\eta$ inclusive, we conclude from (24) that $C$ and $U$ must be positive. Since $C$ is the material constant, its positiveness poses a restriction on possible types of elastic response. From $U>0$, the equilibrium value $\phi_\infty$ is given by $\pm U^{1/2}$. Conversely if a strength of shock wave $\phi_\infty$ is given, then the velocity of shock wave $\lambda$ is determined by

[3] Note that the Fourier transform of $\phi$ is given by $i\hat{\phi}'/y$ + const. $\delta(y)$, $\delta(y)$ being the delta function [9].

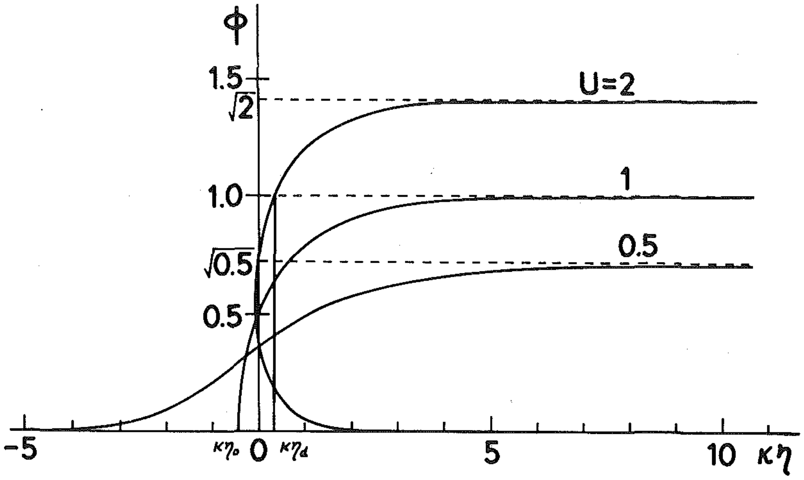


Fig. 1 Shock profiles for Type I with various values of $U$

$$\lambda^2 = G+\gamma U = G+\gamma\phi_\infty^2 = G+\epsilon^2 \, |C| \, |\varphi_\infty'|^2 > G = c_e^2. \quad (25)$$

It is thus found that $\lambda$ is always greater than the equilibrium sound speed $c_e$ and it becomes faster as $\phi_\infty$ increases.

Finally we observe that there exist no steady solutions for the relaxation function (11). In this case, $A$ is given by

$$A = G^2\Gamma(1+\nu')\sin\left(\frac{\pi\nu'}{2}\right)\int_{-\infty}^{\infty} |y|^{-\nu'-1} \, |\hat{\phi}'(y)|^2 dy. \quad (26)$$

It is readily seen from this that $A$ diverges near $y = 0$. Hence we ascertain no existence of steady solutions.

**3.2 Discontinuous Solutions.** In the preceding subsection, it was assumed that $\phi$ and $d\phi/d\eta$ are continuous. In addition to such smooth solutions, we check whether or not a discontinuous solution is possible. Let a discontinuity be located at $\eta = 0$ and let the unstrained state prevail in $\eta < 0$. Near $\eta = 0$, the discontinuous solution is assumed to be expressed as $\phi = \phi_0 h(\eta)$, $\phi_0$ being a constant jump and $h(\eta)$ the unit step function. Evaluation of (18) at $\eta = \Delta\eta > 0$ ($\Delta\eta \to 0$) justs behind the discontinuity leads to

$$U\phi_0 - \text{sgn}\, C\phi_0^3 = \int_{-\infty}^{\Delta\eta} K(\Delta\eta-\eta_1)\phi_0\delta(\eta_1)d\eta_1$$

$$= \phi_0 K(\Delta\eta) \to \phi_0 K(0), \quad (27)$$

as $\Delta\eta$ tends to zero, where $\delta(\eta_1)$ denotes the delta function. If $K(0)$ remains finite (which is the case with Type I), there exists a discontinuous solution with a jump given by

$$\phi_0^2 = [U-K(0)]/\text{sgn}\, C = [(\lambda^2-G)/\gamma - K(0)]/\text{sgn}\, C. \quad (28)$$

This gives the relation between the strength of discontinuity and its velocity[4]. On the other hand, if $K(0)$ is infinite (which is the case with Type II), there exists no discontinuous solution. Further, the higher order discontinuity in $d\phi/d\eta$ may similarly be examined. Here we only note the results that for $|K(0)| < \infty$, the discontinuity occurs when $U = K(0)$, whereas for $|K(0)| \to \infty$, no discontinuity occurs.

Finally we return to the conditions for existence of steady shock solutions when such a discontinuous solution is included. This is, of course, the case with Type I. If a discontinuity in $\phi$ is concerned, the asymptotic behavior of $\hat{\phi}'$ as $|y| \to \infty$ does not decay so quickly as $|y|^{-2}$ but it remains finite there. Even so, $A$ is guaranteed finite and the existence of a discontinuous solution is possible.

[4] It should be remarked that near the discontinuity, the assumption of thin rod becomes invalid and a new formulation is required. Therefore the discontinuous solution should be regarded as a formal one.

**3.3 Steady Shock Profiles.**

*3.3.1. Type I: Exponential Function Type.* As is suggested from the Section 3.1., it is assumed in the following analysis that $C$ and $U$ are positive in (18). For Type I, equation (18) can be reduced, by differentiating with respect to $\eta$, to the first order differential equation:

$$\frac{d\phi}{d\eta} = \kappa\frac{(U-\phi^2)\phi}{1-U+3\phi^2}. \quad (29)$$

The equilibrium points of (29) are given by $\phi^2 = \phi_\infty^2 = U > 0$ and $\phi = 0$. But since $d\phi/d\eta$ becomes infinite at $\phi_B^2 = (U-1)/3$, we must pay attention to the location of $\phi_B$. The solution to (29) can easily be obtained as

$$\frac{(1-U)}{2U}\log|\phi^2| - \frac{(1+2U)}{2U}\log|U-\phi^2| = \kappa(\eta-\eta_0), \quad (30)$$

where $\eta_0$ is an arbitrary integration constant. In Fig. 1, the explicit form of (30) is shown for the typcial values of $U$ with $\phi_\infty > 0$ where $\eta_0$ is chosen so that $\phi$ takes $\phi_\infty/2$ at $\eta = 0$. If $0 < U < 1$, (30) exhibits a smooth and monotonic transition from $\phi = 0$ at $\eta = -\infty$ to $\phi = \phi_\infty$ at $\eta = \infty$. The velocity of this shock wave is determined by the strength of shock wave $\phi_\infty$ as $\lambda^2 = G + \gamma\phi_\infty^2$. Since $0 < U < 1$, $\lambda$ is greater than the equilibrium sound speed, but less than the instantaneous one, i.e., $c_e < \lambda < c_i$. If $1 < U$, on the other hand, $\phi_B$ is always located between $\phi = 0$ and $\phi = \phi_\infty$ and both asymptotic branches of (30) as $\phi \to 0$ and $\phi \to \phi_\infty$ approach $\eta \to \infty$. Thus (30) does not satisfy the boundary condition at $\eta = -\infty$. For Type I, however, we already know that there may appear a discontinuity in the solution. Hence to fit (30) with the boundary condition, the discontinuity $\phi_0 = (U-1)^{1/2}$ given by (28) with $C > 0$ is placed at a point $\eta = \eta_d$ where $\phi_0$ equals $\phi$ by (30). In Fig. 1, the vertical line connected with the solution for $U = 2$ at $\phi = 1$ represents this discontinuity. For $\eta < \eta_d$, the solution takes the value of zero, while for $\eta > \eta_d$, the solution takes the upper branch of (30) and the branch below $U = 1$ loses its meaning. Thus we have the discontinuous shock profile followed by the monotonic relaxation region. The velocity is then given by $\lambda^2 = G + \gamma\phi_\infty^2 > G + \gamma = c_i^2$ and is faster than the instantaneous sound speed. As a special case, if $U = 1$, (30) degenerates to

$$\phi^2 = 1 - \exp[-2\kappa(\eta-\eta_0)/3]. \quad (31)$$

This solution ceases to be valid for $\eta < \eta_0$ and the boundary condition at $\eta = -\infty$ cannot be satisfied. In this case as well, for $\eta \leqq \eta_0$, the solution takes the value of zero, and for $\eta > \eta_0$, it takes (31). Consequently $d\phi/d\eta$ becomes discontinuous

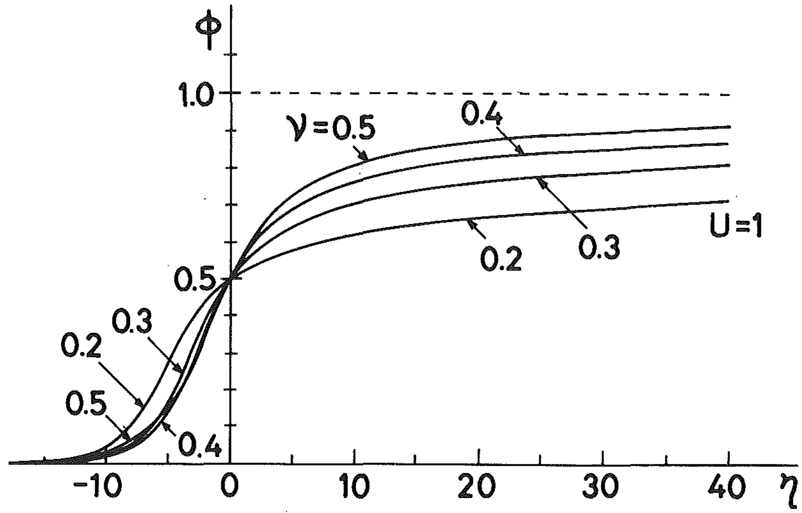


**Fig. 2 Shock profiles for Type II with $U$ = 1 and various values of $\nu$**

at $\eta = \eta_0$. This is known as the so-called acceleration wave, which propagates with the velocity $\lambda = c_i$.

*3.3.2 Type II: Power Function Type.* For this type, equation (18) remains essentially the nonlinear integral equation:

$$U\phi - \phi^3 = \int_{-\infty}^{\eta} \frac{1}{(\eta-\eta_1)^{\nu}} \frac{d\phi}{d\eta_1} d\eta_1, (0<\nu<1). \qquad (32)$$

Since an analytical solution of this equation is unknown at present, it is solved numerically by the method of quadrature. To apply this method, it is advantageous to invert (32) to remove the derivative in the integrand. Equation (32) is Abel's type of integral equation, which can be rewritten as

$$\phi = \frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \int_{-\infty}^{\eta} \frac{U\phi(\eta_1) - \phi^3(\eta_1)}{(\eta-\eta_1)^{1-\nu}} d\eta_1. \qquad (33)$$

Here we remark on two analytical results which assist the numerical computation. First, the solution is independent of the choice of the origin of coordinate axis, as this property comes from the general characteristics of the hereditary integral in (18). Secondly, though the full explicit solution of (33) is difficult to obtain analytically, the asymptotic solution as $\eta \to -\infty$ can be easily sought as

$$\phi \sim \phi^{(1)}\exp(\alpha\eta) + \frac{1}{(1-3^{\nu})U}\phi^{(1)3}\exp(3\alpha\eta) + \frac{3}{(1-3^{\nu})(1-5^{\nu})U^2}$$

$$\times \phi^{(1)5}\exp(5\alpha\eta) + \ldots, \qquad (34)$$

where $\phi^{(1)}$ is a constant and $\alpha = [U/\Gamma(1-\nu)]^{1/\nu}$. We use these results to demonstrate the numerical method of treating (33).

Since the lower bound of the integration extends to infinity and moreover the integrand diverges at the upper bound (though the integral itself exists), special care should be taken. For the treatment of the infinite lower bound, we first divide the region into two regions $(-\infty, M]$ and $[M, \eta]$, where $M$ is taken arbitrarily but fixed, say $M = 0$. Then (33) can be split into

$$\Gamma(\nu)\Gamma(1-\nu)\phi = \left(\int_{-\infty}^{M} + \int_{M}^{\eta}\right) \frac{F[\phi(\eta_1)]}{(\eta-\eta_1)^{1-\nu}} d\eta_1, \qquad (35)$$

where $F[\phi(\eta_1)] \equiv F(\phi_1) \equiv U\,\phi\,(\eta_1) - \phi^3(\eta_1)$. In the first region $(-\infty, M]$, we assume that the asymptotic solution (34) is valid by using the freedom of the choice of the origin of coordinate axis, i.e.,

$$\phi \sim \phi^{(1)}\exp[\alpha(\eta - M)] + O(\phi^{(1)3}), \qquad (36)$$

where $\phi^{(1)}$ is taken small $(0 < \phi^{(1)} << 1)$. On substituting this into the first integral, $I_1$, of (35), we obtain

$$I_1 = \int_{-\infty}^{M} \frac{U\phi^{(1)}\exp[\alpha(\eta_1 - M)]}{(\eta-\eta_1)^{1-\nu}} d\eta_1 + O(\phi^{(1)3})$$

$$= \Gamma(1-\nu)\phi^{(1)}\exp[\alpha(\eta - M)]\Gamma[\nu, \alpha(\eta-M)] + O(\phi^{(1)3}), \qquad (37)$$

where $\Gamma\,[\nu, \alpha(\eta - M)]$ denotes the incomplete gamma function. From the asymptotic behavior of $\Gamma\,[\nu, \alpha(\eta-M)] \sim [\alpha(\eta-M)]^{\nu-1} \exp\,[-\alpha(\eta-M)]$ as $\alpha(\eta-M) \to \infty$, the contribution of $I_1$ to the total integral (35) becomes small and negligible as $\alpha(\eta - M) \to \infty$.

In the second region $[M, \eta]$, on the other hand, we take the equidistant points separated by $h$ ($>0$), i.e., $\eta = \eta^{(i)} = M + h(i-1)$, $(i \geq 1)$ and evaluate the second integral in (35) by Simpson's rule. But since the integrand diverges at the upper bound, we further divide the region into two parts, $[M, \eta^{(i-2)}]$and $[\eta^{(i-2)}, \eta^{(i)}]$. In the former region, the integrand remains finite and the usual Simpson's rule can be applied. For the integral in $[\eta^{(i-2)}, \eta^{(i)}]$, however, a "modified Simpson's rule" is devised to take account of the singularity. Instead of approximating the whole integrand by the quadratic function, we approximate the numerator $F$ by the quadratic function and calculate the integral analytically. After the strightforward calculation, we have the approximation formula,

$$\int_{\eta^{(i-2)}}^{\eta^{(i)}} \frac{F(\phi_1)}{(\eta-\eta_1)^{1-\nu}} d\eta_1 = p\Big[\nu F^{(i-2)} + 4F^{(i-1)}$$

$$+ \frac{2-\nu}{\nu}F^{(i)}\Big] + O(h^{3+\nu}), \qquad (38)$$

where $p = (2h)^{\nu}/[(1+\nu)(2+\nu)]$ and $F^{(i-k)}$ $(k=0, 1, 2)$ denotes the values of $F$ at $\eta^{(i-k)}$. Thus the scheme to calculate the unknown $\phi^{(i)}$ at $\eta = \eta^{(i)}$ from the known $\phi^{(k)}$ $(k=1, 2, 3, \ldots, i-1)$ is to solve the following cubic equation for $\phi^{(i)}$ successively:

$$\phi^{(i)3} + \left[\frac{\Gamma(\nu)\Gamma(1-\nu)\nu}{(2-\nu)p} - U\right]\phi^{(i)} - \frac{\nu}{2-\nu}\Big[\nu F^{(i-2)}$$

$$+ 4F^{(i-1)} + \frac{1}{p}(I_1 + I_2)\Big] = 0, \qquad (39)$$

where $I_2$ denotes the integral in $[M, \eta^{(i-2)}]$. Here we remark that this scheme is applied to the odd $i$ $(\geq 3)$ only after giving the starting data $\phi^{(1)}$ and $\phi^{(2)}$. For the even $i$ $(\geq 2)$, the starting point $i = 1$ is shifted to $i = 0$ with $\phi^{(0)} = \phi^{(1)}\exp(-\alpha h)$ and the same scheme is applied.

Here we estimate the approximation error involved. In $I_1$, the smaller is made the error, the smaller $\phi^{(1)}$ is taken. Simpson's rule involves the error of $O(h^5)$ in the $2h$ interval, while the "modified Simpson's rule" involves that of $O(h^{3+\nu})$.[5] Therefore as $\nu$ approaches zero, the accuracy of the "modified Simpson's rule" deteriorates to that for the trapezoidal rule. To recover it, we may take the region $[\eta^{(i-4)}, \eta^{(i)}]$ and approximate $F$ by the quartic function. By this "modified Bodes's rule," the error is improved to give the order of $h^{5+\nu}$. Choosing $h = 0.02$ and $\phi^{(1)} = 0.001$ in the present analysis, the results of two modified rules are compared, but no substantial differences are recognized.

In Fig. 2, the numerical solutions are displayed for $\phi_\infty = U^{1/2} = 1$ and $\nu = 0.2, 0.3, 0.4$, and $0.5$ in which the coordinate is readjusted so that $\phi$ may $\phi_\infty/2$ at $\eta = 0$. It is seen that the shock profiles are smooth and monotonic. As suggested by the asymptotic solution (34), the step-up behavior is exponentially steep. But the very slow relaxation region appears in the trail. Therefore it takes a long time to attain the equilibrium value. This tendency becomes prominent as $\nu$ decreases. The asymmetric character of shock profiles with respect to $\eta = 0$ is remarkable, which should be compared with the results of Type I. For other positive values of $U$, the exponential step-up becomes steep as $U$ increases and the qualitative behavior is similar.

## 4 Results

We summarize the results of the steady shock wave propagation. For any materials, the steady shock waves exist only if their elastic response satisfies $C > 0$. As for their viscoelastic character, the Fourier transform of the relaxation function must satisfy the criteria shown in section 3.1. Such requirements provide useful informations in evaluating the nonlinear viscoelastic behavior. The shock waves result from a balance between the nonlinearity and the viscoelasticity represented by the two parameters $\epsilon$ and $\gamma$, respectively. This is reflected in the transformation of variable $\phi = [\epsilon^2 |C|/\gamma]^{1/2}\varphi'$. The competition between them determines the magnitude of shock waves. The velocity of shock wave $\lambda$ is given by the strength of shock wave $\phi_\infty$ $[=(\epsilon^2 |C|/\gamma)^{1/2}\varphi'_\infty]$ applied at the infinity $z = -\infty$. The velocity is always greater than the equilibrium sound speed $c_e$, and it becomes faster as $\phi_\infty$ increases, i.e., $\lambda^2 = G + \gamma\phi_\infty^2 = G + \epsilon^2 |C| {\varphi'_\infty}^2 > G = c_e^2$.

Next we consider the specific shock profiles of $\phi$ for Types I and II. We again note that $\phi$ represents the torsional shear strain if an appropriate scale factor is introduced. As is understood from the linear dispersion relations, the viscoelastic effect brings about not only the dissipation but also the dispersion. In spite of the dispersion, however, the shock profiles are always monotonic but not oscillatory. For Type I, there is the critical speed $c_i$ $[=(G+\gamma)^{1/2}]$ beyond which the shock profiles change remarkably. For $c_e < \lambda < c_i$, the shock profiles are smooth and monotonic. But for $c_i \leqq \lambda$, they contain the discontinuity followed by the monotonic relaxation region. In Fig. 1, we can define a sharp shock layer over which $\phi$ changes appreciably. The thickness of this layer, which is estimated by the characteristic relaxation time $\kappa^{-1}$, is seen to be of order of $5\lambda/\kappa$ to $10\lambda/\kappa$ for the moderate values of $U$. For the strong shock wave $U >> 1$, the discontinuity $(U-1)^{1/2}$ and the equilibrium value $U^{1/2}$ are nearly equal. Therefore the thickness is so thin that it is almost represented by the discontinuity. For the weak shock wave $U << 1$, on the other hand, the thickness becomes wide and is given from (30) by the order of $\lambda/(U\kappa)$.

For Type II, it is important that the relaxation function $K(t)$ exhibits the singularity at $t = 0$. So the instantaneous sound speed is infinitely large and also no discontinuity is allowed in the solution. From (34), there always exists for $U > 0$ the asymptotic branch emanating from $\eta = -\infty$. Consequently there is no critical speed corresponding to $c_i$ in the case of Type I. Thus the shock profiles are always smooth, which should be compared with the results of Type I. Another important characteristic of Type II is the slow relaxation. The shock profile has a steep front but a slow relaxation region behind. Therefore the thickness of shock layer is very wide compared with Type I. This is the essential difference from Type I even if it would be extended to include many (but finite) numbers of the exponential functions.

In this paper, we have sought the explicit shock profiles by assuming relevant forms of the relaxation functions. This provides useful information as a guideline. In reality, however, the wave profiles are known and the relaxation function is required. In such a case, the inverse problem of (18) must be solved to determine $K(t)$ by using the known $\phi$.

## 5 Simplified Evolution Equations

In concluding this paper, we briefly discuss the simplified evolution equations for a far field transient behavior. Equations (1)–(3) can describe the bidirectional wave propagation, namely, propagation along both directions of right and left. But if only a far field behavior is concerned along either one of the directions, the unidirectional wave propagation is sufficient. In this case, equations (1)–(3) are simplified considerably. Following the idea of the reductive perturbation method [10], we introduce the new variable $\xi = t - z/G^{1/2}$ moving with the equilibrium sound speed and the stretched space variable $\tau = \epsilon^2 z/G^{1/2}$. Rewriting (1)–(3) in terms of $\xi$ and $\tau$, and retaining the lowest order terms in $\gamma$ and $\epsilon^2$, we have the simple evolution equation for $f = -G^{-1/2}\,\partial\varphi/\partial\xi$:

$$\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} = \frac{\gamma}{2\epsilon^2 G}\frac{\partial}{\partial \xi}\int_{-\infty}^{\xi} K(\xi-\xi_1)\frac{\partial f}{\partial \xi_1}d\xi_1,$$

with

$$-G^{-1/2}\frac{\partial w}{\partial \xi} = \frac{(c_1 G + c_2)}{G(G-E)}\left(\frac{\partial \varphi}{\partial \xi}\right)^2, \tag{40}$$

where the unstrained state is assumed ahead of wave propagation. For Type I, (40) is further reduced to the differential equation:

$$\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} = -\kappa^{-1}\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} - \frac{\gamma}{2\epsilon^2 G}\frac{\partial f}{\partial \xi}\right). \tag{41}$$

In particular, if the rapid relaxation is assumed, i.e., $\kappa^{-1} \sim \epsilon^2/\gamma << 1$, (41) is further simplified to give "cubic Burgers equation":

$$\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} = \frac{\gamma}{2\epsilon^2 G\kappa}\frac{\partial^2 f}{\partial \xi^2} + O(\kappa^{-2}). \tag{42}$$

On the other hand, if the slow relaxation is assumed, i.e., $\kappa^{-1} >> 1$, (41) is reduced to

$$\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} - \frac{\gamma}{2\epsilon^2 G}\frac{\partial f}{\partial \xi} = -\frac{\gamma\kappa}{2\epsilon^2 G}f + O(\kappa^2). \tag{43}$$

After some thought, it is found that (43) describes the far field behavior moving with the instantaneous sound speed.

For Type II, equation (40) cannot be further reduced. It is called "generalized Burgers' equation" on introduction of the derivative of real order:

[5] When $\nu$ approaches unity, the order of error $h^{3+\nu}$ should agree with that of Simpson's rule. Although it might appear that the order approaches $h^4$, the coefficient of $h^4$ then vanishes and the order becomes $h^5$. Also for the modified Bode's rule" described below, the same situation occurs and the order of error approaches $h^7$ as $\nu \to 1$.

$$\frac{\partial f}{\partial \tau} - \frac{C}{2}\frac{\partial f^3}{\partial \xi} = \frac{\gamma\Gamma(1-\nu)}{2\epsilon^2 G}\frac{\partial^{1+\nu} f}{\partial \xi^{1+\nu}}, \tag{44}$$

with the definition given by [11],

$$\int_{-\infty}^{\xi} \frac{1}{(\xi-\xi_1)^\nu}\frac{\partial f}{\partial \xi_1} d\xi_1 \equiv \Gamma(1-\nu)\frac{\partial^\nu f}{\partial \xi^\nu}. \tag{45}$$


## Acknowledgments

We are grateful to Professor Y. Inoue for the stimulating discussions and also to Professor F. Mainardi at University of Bologna, Italy for informing us of his papers in compiling this manuscript. We are much obliged to the associate editor, Professor Malvern and the reviewers for their suggestions, which have led to the improvement of the original manuscript.

This work has been supported partially by the Grant-in-Aid for the Scientific Research from the Ministry of Education, Science and Culture of Japan.


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