Persistent Graphs and Consensus Convergence

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***Abstract*— This paper investigates the role persistent arcs play for averaging algorithms to reach a global consensus under discrete-time or continuous-time dynamics. Each (directed) arc in the underlying communication graph is assumed to be associated with a time-dependent weight function. An arc is said to be persistent if its weight function has infinite $\mathscr{L}_1$ or ℓ_1 norm for continuous-time or discrete-time models, respectively. The graph that consists of all persistent arcs is called the persistent graph of the underlying network. Three necessary and sufficient conditions on agreement or ϵ-agreement are established, by which we prove that the persistent graph fully determines the convergence to a consensus. It is also shown how the convergence rates explicitly depend on the diameter of the persistent graph.**

Keywords: Consensus, Persistent Graphs, Averaging Algorithms

I. Introduction

Recent years have witnessed wide research interest in the the study averaging algorithms throughout social science [10], [13], [11], [12], computer science [15], [36], [37] and engineering [38], [39], [32], [16], [22], [20], [21]. Agreement seeking has been extensively studied in the literature for both discrete-time and continuous-time models [16], [15], [40], [20], [18], [27], [28], [19], [17], [25], [30], [26].

The communication graph plays an important role in the study of consensus. In most existing work, the arc weights, which reflect the strength of the influence from one node to another, are assumed to either be constant whenever two nodes meet with each other [10], [18], [17], or in a compact set with positive lower and upper bounds [40], [16], [27], [28], [13]. However, in reality, the arc weights may vary in a wide range, and may even fade away since arcs may have different persistency properties. Links can be impulsive, vanishing, persistent, etc. Then an interesting question arises: are there certain arcs which are the ones that actually generate the convergence to consensus and how do their properties influence the convergence rate?

The central aim of the paper is to build a model to classify the arcs in the underlying communication graph, and then give a precise description on how the persistent arcs indeed determine the agreement seeking. We define the persistent graph as the graph having links whose weight functions have infinite $\mathscr{L}_1$ or ℓ_1 norm for continuous-time and discrete-time algorithms, respectively. Global agreement and ϵ-agreement are defined as whether the maximum state difference converges to zero and whether the convergence is exponentially fast, respectively. For the discrete-time case, a necessary and sufficient condition is obtained on ϵ-agreement under general stochasticity, self-confidence, and arc balance assumptions. Then for the continuous-time case, two necessary and sufficient conditions are established on global agreement and ϵ-agreement, respectively. In this way, we precisely state how the persistent graph plays a fundamental role in consensus seeking. Additionally, comparisons of our new conditions are given with existing results and the relations between the discrete-time and continuous-time evolutions are highlighted.

The rest of the paper is organized as follows. In Section II, we introduce the network model and define the problem of interest. Then in Sections III and IV, the main results and convergence analysis are presented for discrete-time and continuous-time dynamics, respectively. Finally concluding remarks are given in Sections V.

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II. Problem Definition

In this section, we present the network model and define the considered problem. To this end, we first introduce some basic graph theory [4].

A (simple) *digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \dots, n\}$ of nodes and an arc set $\mathcal{E}$, where each *arc* $(i, j) \in \mathcal{E}$ is an ordered pair from node $i \in \mathcal{V}$ to another node $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, k$, the sequence is called a (directed) *path* with *length* k. A path from i to j is denoted $i \to j$, and the length of $i \to j$ is denoted $|i \to j|$. A path with no repeated nodes is called a *simple* path. If there exists a path from node i to node j, then node j is said to be reachable from node i. Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. $\mathcal{G}$ is said to be *strongly connected* if it contains path $i \to j$ and $j \to i$ for every pair of nodes i and j; $\mathcal{G}$ is said to be *quasi-strongly connected* if $\mathcal{G}$ has a center [5], [25].

The *distance* from i to j, $d(i, j)$, is defined as the length of a shortest (simple) path $i \to j$ when j is reachable from i, and the *diameter* of $\mathcal{G}$ as $d_0 = \max\{d(i, j) | i, j \in \mathcal{V},\ j \text{ is reachable from } i\}$.

In this paper, we consider a network model with node set $\mathcal{V} = \{1, \dots, n\}$. Let the digraph $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ denote the *underlying* graph. The underlying graph indicates all potential interactions between nodes. Node j is said to be a *neighbor* of i at time t when there is an arc $(j, i) \in \mathcal{E}_*$; each node is supposed to be a neighbor of itself. Let $\mathcal{N}_i = \{i\} \cup \{j : (j, i) \in \mathcal{E}_*\}$ denote the neighbor set of node i.

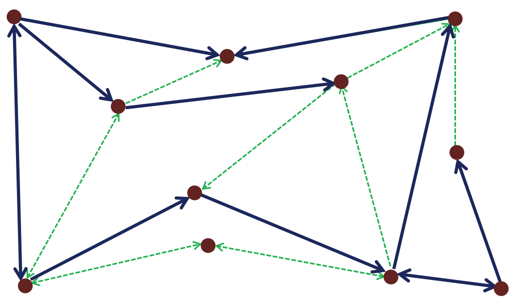

Fig. 1. The underlying graph consists of persistent arcs (solid) and vanishing arcs (dashed). The persistent graph is shown to play a fundamental role for the convergence to an agreement.

Let $x_i(t) \in \mathbb{R}$ be the *state* of node i at time t. Time is either discrete or continuous. The initial time is $t_0 \geq 0$ in both cases and each node is equipped with an initial value $x_i(t_0)$. The consensus algorithm is in discrete time:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} W_{ij}(t) x_j(t), \quad i = 1, \dots, n \tag{1}$$

and in continuous time:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} W_{ij}(t) \big[x_j(t) - x_i(t) \big], \quad i = 1, \dots, n. \tag{2}$$

Here $W_{ij}(t) : [0, \infty) \to [0, \infty)$ is a nonnegative scalar function which represents the weight of arc (j, i). Clearly $W_{ij}(t)$ describes the strength of the influence of node j on i. Since $W_{ij}(t) = 0$ may happen from time to time, the graph is indeed time-varying.

We define

$$\psi(t) \doteq \min_{i \in \mathcal{V}} \{x_i(t)\}, \quad \Psi(t) \doteq \max_{i \in \mathcal{V}} \{x_i(t)\}$$

as the minimum and maximum state value at time t, respectively. Then we introduce

$$\mathcal{H}(t) \doteq \Psi(t) - \psi(t)$$

The considered global agreement and ϵ-agreement for both the discrete-time and continuous-time updating rules are defined as follows.

Definition 2.1: (a) Global *agreement* is achieved if for any $x(t_0) \doteq (x_1(t_0) \dots x_n(t_0))^T \in \mathbb{R}^n$, we have

$$\lim_{t \to \infty} \mathcal{H}(t) = 0. \tag{3}$$

(b) Global ϵ-agreement is achieved if there exist two constants $0 < \epsilon < 1$ and $T_0 > 0$ such that for any $x(t_0) \in \mathbb{R}^n$ and $t \geq t_0$, we have

$$\mathcal{H}(t + T_0) \leq \epsilon \mathcal{H}(t). \tag{4}$$

The goal of this paper is to distinguish the arcs from the underlying graph that are *persistent* over a long time range and how they influence global agreement. To be precise, we impose the following definition for persistent arcs and persistent graphs based on the $\mathscr{L}_1$ or ℓ_1 norms of the weight functions (see Fig. 1).

Definition 2.2: (a) An arc $(j, i) \in \mathcal{G}_*$ is a persistent arc of the discrete-time updating rule (1) if

$$\sum_{t=0}^{\infty} W_{ij}(t) = \infty,$$

and a persistent arc of the continuous-time updating rule (2) if

$$\int_s^{\infty} W_{ij}(t) dt = \infty \text{ for all } s \geq 0.$$

(b) The graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$ that consists of all persistent arcs is called the persistent graph.

Next, in Sections III and IV, we will investigate the discrete-time and continuous-time updating rules, respectively. We will establish sufficient and necessary conditions on global agreement and ϵ-agreement, which illustrate that the notion of persistent graphs is critical to the convergence.

III. Discrete-time Systems

In this section, we focus on the discrete-time model (1). In order to obtain the main result, we need the following assumptions.

A1 *(Stochasticity)* $\sum_{j \in \mathcal{N}_i} W_{ij}(t) = 1$ for all $i \in \mathcal{V}$ and $t \geq 0$.

A2 *(Self-confidence)* There exists $0 < \eta < 1$ such that $W_{ii}(t) \geq \eta$ for $i \in \mathcal{V}$ and $t \geq 0$.

A3 *(Arc Balance)* There exists a constant $A > 1$ such that for any two arcs $(j, i), (m, k) \in \mathcal{E}^p$ and $t \geq 0$, we have

$$A^{-1} W_{ij}(t) \leq W_{km}(t) \leq A W_{ij}(t).$$

The main result for the discrete-time updating rule (1) on global ϵ-agreement is as follows.

Theorem 3.1: Suppose A1, A2 and A3 hold. Global ϵ-agreement is achieved for (1) if and only if

(a) $\mathcal{G}^p$ is quasi-strongly connected;

(b) there exist a constant $a_* > 0$ and an integer $T_* > 0$ such that $\sum_{s=t}^{t+T_*-1} W_{ij}(s) \geq a_*$ for all $t \geq 0$ and $(j, i) \in \mathcal{E}^p$.

In fact, if (a) and (b) hold, then we have

$$\mathcal{H}(t + d_0 T_*) \leq \Big(1 - \frac{\eta^{d_0 T_*}}{2} \cdot \big(\frac{a_*}{T_*}\big)^{d_0}\Big) \mathcal{H}(t) \tag{5}$$

for all $t \geq t_0$, where d_0 represents the diameter of $\mathcal{G}^p$.

Before we state the proof, we introduce some more notations, which will be used throughout the rest of the paper. For two sets S_1 and S_2, $S_1 \setminus S_2$ is defined as $S_1 \setminus S_2 = \{z : z \in S_1, z \notin S_2\}$. For the underlying graph $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$ and the persistent graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$, we denote

$$\theta(t) = \sum_{(j,i) \in \mathcal{E}_* \setminus \mathcal{E}^p} W_{ij}(t), \tag{6}$$

$$\xi^+(t; m) = \sum_{j \in \mathcal{N}_m \setminus \{m\}} W_{mj}(t), \tag{7}$$

and

$$\xi_0^+(t; m) = \sum_{j \in \mathcal{N}_m \setminus \{m\}, (j,m) \in \mathcal{E}^p} W_{mj}(t). \tag{8}$$

In the following two subsections, we prove the necessity and sufficiency parts of Theorem 3.1, respectively.

A. Necessity

We need to show that a global ϵ-agreement cannot be achieved without either condition (a) or (b).

The upcoming analysis relies on the following well-known lemmas.

Lemma 3.1: Suppose $0 \le p_k < 1$ for all k. Then $\sum_{k=0}^{\infty} p_k = \infty$ if and only if $\prod_{k=0}^{\infty}(1-p_k) = 0$.

Lemma 3.2: $\log(1-t) \ge -2t$ for all $0 \le t \le 1/2$.

We have the following proposition indicating that $\mathcal{G}^p$ being quasi-strongly connected is not only a necessary condition for (1) to reach global ϵ-agreement, but also necessary for (simple) global agreement, even in the absence of assumptions A2 and A3.

Proposition 3.1: Suppose A1 holds. If global agreement is achieved for (1), then $\mathcal{G}^p$ is quasi-strongly connected.

We are now in a place to present the following conclusion, which shows the necessity of condition (b) in Theorem 3.1.

Proposition 3.2: Suppose A1 and A3 hold. If global ϵ-agreement is achieved for (1), then there exist a constant $a_* > 0$ and an integer $T_* > 0$ such that $\sum_{s=t}^{t+T_*} W_{ij}(s) \ge a_*$ for all $t \ge 0$ and $(j,i) \in \mathcal{G}^p$.

The necessity claim in Theorem 3.1 follows from Propositions 3.1 and 3.2. We refer to [45] for technical details of the proofs.

B. Sufficiency

We establish a lemma on the upper and lower bounds for some particular nodes.

Lemma 3.3: Suppose A1 holds. Let $x_m(t) = \mu\psi(t) + (1-\mu)\Psi(t)$ with $0 \le \mu \le 1$. Then for any integer $T > 0$, we have:

$$x_m(t+T) \le \mu \prod_{s=t}^{t+T-1} \big(1-\xi^+(s;m)\big)\cdot\psi(t) + \Big(1-\mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\Big)\cdot\Psi(t), \quad (9)$$

and

$$x_m(t+T) \ge \mu \prod_{s=t}^{t+T-1} \big(1-\xi^+(s;m)\big)\cdot\Psi(t) + \Big(1-\mu\prod_{s=t}^{t+T-1}\big(1-\xi^+(s;m)\big)\Big)\cdot\psi(t). \quad (10)$$

Proof: When $x_m(t) = \mu\psi(t) + (1-\mu)\Psi(t)$, for time $t+1$, we have

$$x_m(t+1) \le \mu\big(1-\xi^+(t;m)\big)\cdot\psi(t) + \Big(1-\mu\big(1-\xi^+(t;m)\big)\Big)\Psi(t). \quad (11)$$

For time $t+2$, we obtain

$$x_m(t+2) \le \mu\prod_{s=t}^{t+1}\big(1-\xi^+(s;m)\big)\cdot\psi(t) + \Big(1-\mu\prod_{s=t}^{t+1}\big(1-\xi^+(s;m)\big)\Big)\cdot\Psi(t). \quad (12)$$

Continuing, we obtain (9).

In equality (10) can be easily obtained using a symmetric analysis as for (9). □

We now present the sufficiency proof of Theorem 3.1. In fact, we are going to prove a stronger statement which does not rely on the arc balance assumption A3.

Proposition 3.3: Suppose A1 and A2 hold. Global ϵ-agreement is achieved for (1) if $\mathcal{G}^p$ is quasi-strongly connected and there exist a constant $a_* > 0$ and an integer $T_* > 0$ such that $\sum_{s=t}^{t+T_*-1} W_{ij}(s) \ge a_*$ for all $t \ge 0$ and $(j,i) \in \mathcal{G}^p$.

Proof: Let $i_0 \in \mathcal{V}$ be a center of $\mathcal{G}^p$. Take $t_0 \ge 0$. Assume first that

$$x_{i_0}(t_0) \le \frac{1}{2}\psi(t_0) + \frac{1}{2}\Psi(t_0). \quad (13)$$

Then from Lemma 3.3, one has

$$\begin{aligned} x_{i_0}(t_0+T) &\le \frac{1}{2}\prod_{s=t_0}^{t_0+T-1}\big(1-\xi^+(s;i_0)\big)\cdot\psi(t_0) \\ &\quad + \Big(1-\frac{1}{2}\prod_{s=t_0}^{t_0+T-1}\big(1-\xi^+(s;i_0)\big)\Big)\cdot\Psi(t_0) \\ &\le \frac{\eta^T}{2}\psi(t_0) + \Big(1-\frac{\eta^T}{2}\Big)\Psi(t_0) \end{aligned} \quad (14)$$

for all $T = 0, 1, \dots$.

Denote $\mathcal{V}_1$ as the node set consisting of all the nodes of which i_0 is a neighbor in $\mathcal{G}^p$, i.e., $\mathcal{V}_1 = \{j : (i_0, j) \in \mathcal{E}^p\}$. Note that $\mathcal{V}_1$ is nonempty because i_0 is a center. For any $i_1 \in \mathcal{V}_1$, there exists an instance $\bar{t}_1 \in [t_0, t_0+T_*-1]$ such that $W_{i_1 i_0}(\bar{t}_1) \ge a_*/T_*$ because $\sum_{t=t_0}^{t_0+T_*-1} W_{i_1 i_0}(t) \ge a_*$. Suppose $\bar{t}_1 = t_0 + \varrho_1$ with $\varrho_1 \in [0, T_*-1]$. Then with (14), we have

$$\begin{aligned} x_{i_1}(\bar{t}_1+1) &= x_{i_1}(t_0+\varrho_1+1) \\ &\le W_{i_1 i_0}(t_0+\varrho_1)x_{i_0}(t_0+\varrho_1) \\ &\quad + \big(1-W_{i_1 i_0}(t_0+\varrho_1)\big)\Psi(t_0) \\ &\le \eta^{\varrho_1}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0) + \Big(1-\eta^{\varrho_1}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0). \end{aligned} \quad (15)$$

Based on Lemma 3.3, we can further conclude

$$x_{i_1}(t_0+\varrho_1+T) \le \eta^{\varrho_1+T-1}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0) + \Big(1-\eta^{\varrho_1+T-1}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0) \quad (16)$$

for all $T = 1, 2, \dots$, which implies

$$x_{i_1}(t_0+T_*+K) \le \eta^{T_*+K}\cdot\frac{a_*}{2T_*}\cdot\psi(t_0) + \Big(1-\eta^{T_*+K}\cdot\frac{a_*}{2T_*}\Big)\Psi(t_0) \quad (17)$$

for all $K = 0, 1, \dots$.

Next, since $\mathcal{G}^p$ is quasi-strongly connected, we can denote $\mathcal{V}_2$ as the node set consisting of all the nodes each of which has a neighbor in $\{i_0\}\cup\mathcal{V}_1$ within $\mathcal{G}^p$. For any $i_2 \in \mathcal{V}_2$, there exist a node $i_* \in \{i_0\}\cup\mathcal{V}_1$ and an instance $\bar{t}_2 = t_0 + T_* + \varrho_2$

with $\varrho_2 \in [0, T_* - 1]$ such that $W_{i_2 i_*}(\bar{t}_1) \geq a_*/T_*$. Similarly we have

$$x_{i_2}(\bar{t}_2+1)\frac{\eta^{T_*+\varrho_2}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\cdot\psi(t_0) + \Big(1-\frac{\eta^{T_*+\varrho_2}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\Big)\Psi(t_0), \quad (18)$$

and therefore

$$x_{i_2}(t_0+2T_*+K) \leq \frac{\eta^{2T_*+K}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\psi(t_0) + \Big(1-\frac{\eta^{2T_*+K}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2\Big)\Psi(t_0)$$

for all $K = 0, 1, \dots$.

Proceeding the estimate, $\mathcal{V}_3, \dots, \mathcal{V}_k$ can be similarly defined until $\big(\cup_{i=1}^k \mathcal{V}_i\big) \cup \{i_0\} = \mathcal{V}$. Moreover, it is not hard to see that i_0 can be selected so that $k = d_0$, where d_0 is the diameter of $\mathcal{G}^p$, and thus

$$\Psi(t_0+d_0T_*) \leq \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\cdot\psi(t_0) + \Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\Psi(t_0). \quad (19)$$

With (19), we eventually have

$$\mathcal{H}(t_0+d_0T_*) \leq \Big(1-\frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^{d_0}\Big)\mathcal{H}(t_0). \quad (20)$$

For the opposite case of (13) with

$$x_{i_0}(t_0) > \frac{1}{2}\psi(t_0) + \frac{1}{2}\Psi(t_0), \quad (21)$$

(20) is obtained using a symmetric argument by bounding $\psi(t_0 + d_0T_*)$ from below.

Therefore, the desired conclusion follows with $\epsilon = 1 - \frac{\eta^{d_0T_*}}{2}\cdot\big(\frac{a_*}{T_*}\big)^2$ and $T_0 = d_0T_*$ since (20) holds independent with the choice of t_0. □

IV. Continuous-time Systems

In this section, we turn to the continuous-time updating rule. We need an assumption on the continuity of each weight function $W_{ij}(t)$ for the existence of trajectories of (2).

A4 *(Continuity)* Each $W_{ij}(t),\ (j,i) \in \mathcal{E}_*$ is continuous except for a set with measure zero.

With assumption A4, each solution of (2) is considered in the sense of Caratheodory in the following [3], [9].

The upper Dini derivative of a function $h : (a,b) \to \mathbb{R}$ at t is defined as

$$D^+h(t) = \limsup_{s\to 0^+}\frac{h(t+s)-h(t)}{s}$$

The next result is useful for the calculation of Dini derivatives [6], [25].

Lemma 4.1: Let $V_i(t,x) : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R},\ i = 1,\dots,n,$ be C^1 and $V(t,x) = \max_{i=1,\dots,n} V_i(t,x)$. If $\mathcal{I}(t) = \big\{i \in \{1,\dots,n\} : V(t,x(t)) = V_i(t,x(t))\big\}$ is the set of indices where the maximum is reached at t, then $D^+V(t,x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t,x(t))$.

The following lemma establishes the monotonicity of $\Psi(t)$ and $\psi(t)$.

Lemma 4.2: For all $t \geq t_0 \geq 0$, we have $D^+\Psi(t) \leq 0$ and $D^+\psi(t) \geq 0$.

Proof: We prove $D^+\Psi(t) \leq 0$. The other part can be proved similarly.

Let $\mathcal{I}_0(t)$ represent the set containing all the agents that reach the maximum in the definition of $\Psi(t)$ at time t, i.e., $\mathcal{I}(t) = \{i \in \mathcal{V} |\ x_i(t) = \Psi(t)\}$. Then according to Lemma 4.1, we obtain

$$\begin{aligned} D^+\Psi(t) &= \max_{i\in\mathcal{I}_0(t)} \dot{x}_i(t) \\ &= \max_{i\in\mathcal{I}_0(t)} \Big[\sum_{j\in N_i} W_{ij}(t)(x_j(t)-x_i(t))\Big] \\ &\leq 0, \end{aligned} \quad (22)$$

which completes the proof. □

Lemma 4.2 implies, $\mathcal{H}(t)$ is non-increasing for all $t \geq t_0$, and therefore each (Caratheodory) trajectory of (2) is bounded within the initial states of the nodes. As a result, the trajectories exist in $[t_0, \infty)$ for any initial condition.

The main result on global consensus and ϵ-consensus is stated in the following two theorems.

Theorem 4.1: Suppose A3 and A4 hold. Global agreement is achieved for (2) if and only if $\mathcal{G}^p$ is quasi-strongly connected.

Theorem 4.2: Suppose A3 and A4 hold. Global ϵ-agreement is achieved for (2) if and only if

(a) $\mathcal{G}^p$ is quasi-strongly connected;

(b) there exists two constants $a_*,\ \tau_0 > 0$ such that $\int_t^{t+\tau_0} W_{ij}(s)ds \geq a_*$ for all $t \geq 0$ and $(j,i) \in \mathcal{G}^p$.

Moreover, if (a) and (b) hold, then we have

$$\mathcal{H}\Big(t+\tau_0\cdot\Big\lceil\frac{d_0\log 2}{a_*}\Big\rceil\Big) \leq \Big(1-\frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t), \quad (23)$$

where $m_0 = \big(\frac{\omega_0}{2}\big)^2\frac{1}{(n-1)A}$ with $\omega_0 = e^{-\int_0^\infty \theta(t)dt}$, d_0 is the diameter of $\mathcal{G}^p$, and $\lceil z\rceil$ represents the smallest integer which is no smaller than z.

Theorem 4.1 implies that the connectivity of the persistent graph $\mathcal{G}^p$ totally determines whether an agreement can be achieved globally. Furthermore, Theorem 4.2 implies that $\int_0^T W_{ij}(t)dt = O(T)$ is a critical condition to ensure a global ϵ-consensus.

Remark 4.1: If we have $\int_{t=t_0}^T W_{ij}(t)dt = \infty, (j,i) \in \mathcal{G}^p$ for some finite T, it follows from the proof of Theorem 4.1 below that (2) will reach a global agreement in finite time when t tends to T.

A. Preliminaries

We establish two lemmas which describe the boundaries of how much each individual arc affects the nodes' dynamics. We refer to [45] for the technical proofs.

Lemma 4.3: Suppose $x_m(s) \leq \mu\psi(s) + (1-\mu)\Psi(s)$ for some $s \geq t_0$ and $m \in \mathcal{V}$ with $0 \leq \mu \leq 1$ a giving constant. Then we have

$$x_m(t) \leq \mu e^{-\int_s^t \xi^+(\tau;m)d\tau}\psi(s) + \big[1-\mu e^{-\int_s^t \xi^+(\tau;m)d\tau}\big]\Psi(s) \quad (24)$$

for all $t \geq s$.

Lemma 4.4: Suppose $(l, m) \in \mathcal{E}_*$ and there exists a constant $0 < \mu < 1$ such that

$$x_l(t) \leq \mu\psi(s_0) + (1-\mu)\Psi(s_0), \quad t \in [s_0, s]$$

for $t_0 \leq s_0 < s$. Then for all $t \in [s_0, s]$, we have

$$\begin{aligned} x_m(t) \leq \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du \cdot \psi(s_0) \\ + \Big[1 - \mu \int_{s_0}^{t} e^{-\int_u^t \xi^+(\tau;m)d\tau} W_{ml}(u)du\Big]\Psi(s_0). \end{aligned}$$

B. Proof of Theorem 4.1

Let $i_0 \in \mathcal{V}$ be a center of $\mathcal{G}^p$. Assume first that

$$x_{i_0}(t_0) \leq \frac{1}{2}\psi(t_0) + \frac{1}{2}\Psi(t_0). \tag{25}$$

Denote $\omega_0 = e^{-\int_0^\infty \theta(t)dt}$. Then we have $0 < \omega_0 \leq 1$. Thus, based on Lemma 4.3 and noting the fact that $\psi(t_0) \leq \Psi(t_0)$, we have

$$\begin{aligned} x_{i_0}(t) \leq \frac{\omega_0}{2} e^{-\int_{t_0}^{t} \xi_0^+(\tau;i_0)d\tau} \psi(t_0) \\ + \Big[1 - \frac{\omega_0}{2} e^{-\int_{t_0}^{t} \xi_0^+(\tau;i_0)d\tau}\Big]\Psi(t_0). \end{aligned}$$

Define

$$\hat{t}_1 = \inf\Big\{t \geq t_0 : \ e^{-\int_{t_0}^{t} \xi_0^+(\tau;i_0)d\tau} = \frac{1}{2}\Big\}. \tag{26}$$

We see that $\hat{t}_1$ is finite from the definition of $\mathcal{E}^p$. As a result, we obtain

$$x_{i_0}(t) \leq \frac{\omega_0}{4}\psi(t_0) + \big[1 - \frac{\omega_0}{4}\big]\Psi(t_0), \quad t \in [t_0, \hat{t}_1]. \tag{27}$$

Next, we denote the node set consisting of all the nodes of which i_0 is a neighbor in $\mathcal{G}^p$ as $\mathcal{V}_1$, i.e., $\mathcal{V}_1 = \{j : (i_0, j) \in \mathcal{E}^p\}$. Note that $\mathcal{V}_1$ is nonempty because i_0 is a center. Then for any $i_1 \in \mathcal{V}_1$, we see from Lemma 4.4 that

$$\begin{aligned} x_{i_1}(\hat{t}_1) \leq \frac{\omega_0^2}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \cdot \psi(t_0) \\ + \Big[1 - \frac{\omega_0^2}{4} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du\Big]\Psi(s_0). \end{aligned} \tag{28}$$

The arc balance assumption A3 implies that

$$\int_u^{\hat{t}_1} \xi_0^+(t;i_1)dt \leq \int_u^{\hat{t}_1} (n-1)AW_{i_1 i_0}(t)dt,$$

which yields

$$\begin{aligned} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \\ \geq \frac{1}{(n-1)A} \cdot \Big[1 - e^{-(n-1)A\int_{t_0}^{\hat{t}_1} W_{i_1 i_0}(\tau)d\tau}\Big]. \end{aligned} \tag{29}$$

On the other hand, we also have

$$\int_{t_0}^{\hat{t}_1} \xi_0^+(t;i_0)dt \leq \int_{t_0}^{\hat{t}_1} (n-1)AW_{i_1 i_0}(t)dt.$$

Thus, we know from (29) and the definition of $\hat{t}_1$ that

$$\begin{aligned} \int_{t_0}^{\hat{t}_1} e^{-\int_u^{\hat{t}_1} \xi_0^+(\tau;i_1)d\tau} W_{i_1 i_0}(u)du \\ \geq \frac{1}{(n-1)A} \cdot \Big[1 - e^{-\int_{t_0}^{\hat{t}_1} \xi_0^+(\tau;i_0)d\tau}\Big] \\ = \frac{1}{2(n-1)A}. \end{aligned} \tag{30}$$

Equations (28) and (30) result in

$$x_{i_1}(\hat{t}_1) \leq \frac{m_0}{2}\psi(t_0) + (1 - \frac{m_0}{2})\Psi(t_0) \tag{31}$$

for all $i_1 \in \mathcal{V}_1$, where $m_0 = \big(\frac{\omega_0}{2}\big)^2 \frac{1}{(n-1)A}$.

Since $\mathcal{G}^p$ has a center, we can proceed the estimation to nodes in $\mathcal{V}_2, \dots, \mathcal{V}_k$ until $\big(\cup_{j=1}^k \mathcal{V}_j\big) \cup \{i_0\} = \mathcal{V}$ with $\hat{t}_2, \dots, \hat{t}_k$ such that

$$x_i(\hat{t}_k) \leq \frac{m_0^k}{2}\psi(t_0) + (1 - \frac{m_0^k}{2})\Psi(t_0) \tag{32}$$

for all $i \in \mathcal{V}$, which leads to

$$\Psi(\hat{t}_k) \leq \frac{m_0^k}{2}\psi(t_0) + (1 - \frac{m_0^k}{2})\Psi(t_0). \tag{33}$$

We see that i_0 can be chosen so that $k \leq d_0$ always holds, where d_0 is the diameter of $\mathcal{G}^p$. Denoting $t_1 = \hat{t}_k$, we eventually arrive at

$$\begin{aligned} \mathcal{H}(t_1) \leq \frac{m_0^{d_0}}{2}\psi(t_0) + \big(1 - \frac{m_0^{d_0}}{2}\big)\Psi(t_0) - \psi(t_0) \\ = \Big(1 - \frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t_0). \end{aligned} \tag{34}$$

Although the analysis up to now is based on assumption (25), we see that (34) also holds for the other case with $x_{i_0}(t_0) > \frac{1}{2}\psi(t_0) + \frac{1}{2}\Psi(t_0)$ using a symmetric argument by investigating the lower bound of $\psi(t_1)$.

Similar estimate can be carried out for $t_k, k = 2, 3, \dots$, which leads to

$$\mathcal{H}(t_{k+1}) \leq \Big(1 - \frac{m_0^{d_0}}{2}\Big)\mathcal{H}(t_k) \tag{35}$$

for all $t_k, k = 1, 2, \dots$, which yields

$$\mathcal{H}(t_k) \leq \Big(1 - \frac{m_0^{d_0}}{2}\Big)^k \mathcal{H}(t_0). \tag{36}$$

Therefore, we can now conclude that $\lim_{t\to\infty} \mathcal{H}(t) = 0$ because $\mathcal{H}(t)$ is non-increasing and $0 < m_0 < 1$. The sufficiency statement of Theorem 4.1 is thus proved.

The necessity part follows the same line as the proof of Proposition 3.1, and therefore omitted.

C. Proof of Theorem 4.2

The necessity statement follows from a similar argument as the proof of Proposition 3.2. The sufficiency part can be obtained based on the convergence analysis in Theorem 4.1. We refer to [45] for technical details.

V. CONCLUSIONS

This paper studied persistent graphs under discrete-time and continuous-time consensus algorithms. Sufficient and necessary conditions were established on the persistent graph for the network to reach global agreement or ϵ-agreement. It was shown that the persistent graph essentially determines both the convergence and convergence rate to an agreement.

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