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Stiffness of an Elastic Circular Cylinder of Finite Length

An analytical solution is proposed to determine the stiffness of an elastic circular cylinder clamped at one end, while the other end undergoes any rigid-body displacement; the lateral surface is stress-free. The validity of Saint-Venant's principle has been assessed by computing the solution of this problem for cylinders having different aspect ratios. On the other hand, the solution agrees with the thin layer theory when the aspect ratio becomes infinitely small. Therefore, the solution proposed here fills up the gap between beam theory and thin layer theory.

1 Formulation of the Problem

Consider a circular cylinder of radius R and length H, the material properties of which are assumed to be linear elastic and isotropic. The following loading conditions are applied: The curved lateral surface is stress-free, one of the ends ($\bar{z} = 0$) remains fixed (zero displacement), while the other end ($\bar{z} = H$) undergoes any rigid-body displacement; see Fig. 1. In the aforementioned configuration, flat ends of the cylinder remain circular throughout the deformation.

In order to be able to deal with more general results, the following dimensionless variables will be used

$$L = H/R \tag{1-1}$$

$$r = \bar{r}/R; \quad \theta = \theta; \quad z = \bar{z}/R \tag{1-2}$$

$$u = \bar{u}/R; \quad v = \bar{v}/R; \quad w = \bar{w}/R \tag{1-3}$$

$$\sigma_{rz} = \bar{\sigma}_{rz}/2\mu; \quad \sigma_{\theta z} = \bar{\sigma}_{\theta z}/2\mu; \quad \sigma_{zz} = \bar{\sigma}_{zz}/2\mu \tag{1-4}$$

where ($\bar{r}$, θ, $\bar{z}$) are the usual cylindrical coordinates; $\bar{u}$, $\bar{v}$, and $\bar{w}$ are the radial, tangential, and axial displacement, respectively; and μ is the shear modulus of the material.

The most general rigid-body motion of the right end ($z = L$) can be decomposed into the following basic components:

(*a*) Translation parallel to the z-axis,
(*b*) Rotation about the z-axis,
(*c*) Translation perpendicular to the z-axis,
(*d*) Rotation perpendicular to the z-axis.

The deformations corresponding to the aforementioned displacements can be interpreted, respectively, as (see Fig. 2):

(*a*) Uniaxial traction,
(*b*) Uniform torsion,
(*c*) and (*d*) Shearing and bending.

Let us choose the x-axis to be in the direction of the shear and assume that the bending moment is parallel to the y-axis. Using this frame orientation, displacements and stresses can be written as follows

$$\begin{cases} u = u_1 \cos\theta \\ v = v_0 + v_1 \sin\theta \\ w = w_0 + w_1 \cos\theta \end{cases} \tag{1-5a}$$

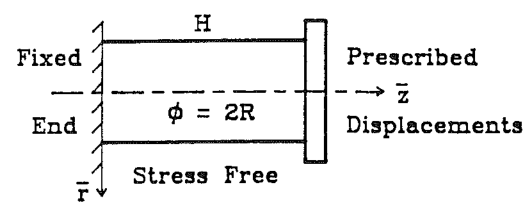

Fig. 1 Schematic configuration of the cylinder loading under study: zero displacements at $\bar{z} = 0$, stress-free lateral surface at $\bar{r} = R$, any prescribed rigid-body displacements at $\bar{z} = H$

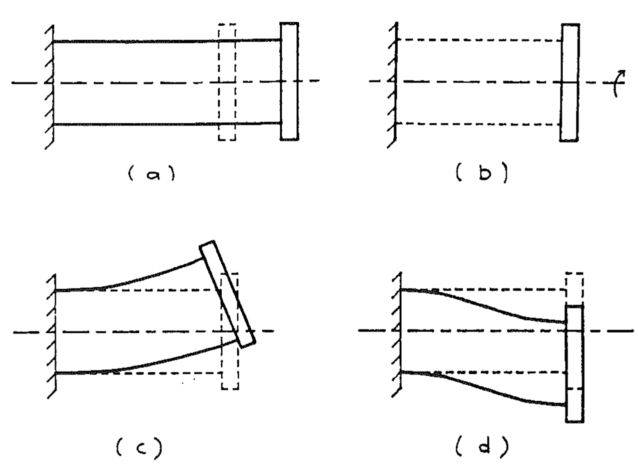

Fig. 2 Displacements prescribed at $\bar{z} = H$ schematically decomposed into: (*a*) traction; (*b*) torsion; (*c*) bending and (*d*) shearing.

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$$\begin{cases} \sigma_{rz} = (\sigma_{rz})_1 \ \cos\theta \\ \sigma_{\theta z} = (\sigma_{\theta z})_0 + (\sigma_{\theta z})_1 \ \sin\theta \\ \sigma_{zz} = (\sigma_{zz})_0 + (\sigma_{zz})_1 \cos\theta \end{cases} \tag{1-5b}$$

where subscripts 0 is associated with traction and torsion, while subscript 1 is associated with shear bending. The net forces and moments to be applied to the right end of the cylinder are therefore given by

$$½ \frac{T}{\mu} = \int_0^{2\pi} \int_0^R (\sigma_{zz})(\bar{r})\bar{r}d\bar{r}d\theta = 2\pi R^2 \int_0^1 (\sigma_{zz})_0(r)rdr,$$

$$½ \frac{C}{\mu} = \int_0^{2\pi} \int_0^R (\sigma_{\theta z})(\bar{r})\bar{r}^2 d\bar{r}d\theta = 2\pi R^3 \int_0^1 (\sigma_{\theta z})_0(r)r^2dr,$$

$$½ \frac{P}{\mu} = \int_0^{2\pi} \int_0^R (\sigma_{rz} \ \cos\theta - \sigma_{\theta z} \ \sin\theta)\bar{r}d\bar{r}d\theta$$

$$= \pi R^2 \int_0^1 [(\sigma_{rz})_1 - (\sigma_{\theta z})_1] rdrd\theta,$$

$$½ \frac{M}{\mu} = \int_0^{2\pi} \int_0^R (\sigma_{zz})(\bar{r})\bar{r}^2 \ \cos\theta \ d\bar{r}d\theta = \pi R^3 \int_0^1 (\sigma_{zz})_1(r)r^2dr.$$

where T, C, P, M represents the traction force, torque, shear force and bending moment, respectively. At the right end of the cylinder, the prescribed displacements can be written as

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \delta_T \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} + \delta_C \begin{bmatrix} 0 \\ 2Lr \\ 0 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \delta_B \begin{bmatrix} -½ \ L^2 \\ ½ \ L^2 \\ L \ r \end{bmatrix} + \delta_S \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}$$

where subscripts T, C, B and S correspond to traction, torque, bending and shearing, respectively; see Fig. 2. Note that for large strains, the displacements associated with the bending deformation δ_B do not conserve the circular shape of the flat end. We therefore assume here that the bending rotation of the flat end is small enough to neglect the second-order displacements in the x-direction, which is consistent with the small strain theory of elasticity used herein.

The net forces and moments to apply at the right end can, therefore, be expressed in the following form

$$½ \frac{1}{\mu} \begin{bmatrix} T \\ C \\ M \\ P \end{bmatrix} = \mathbf{E} \times \begin{bmatrix} \delta_T \\ \delta_C \\ \delta_B \\ \delta_S \end{bmatrix} \tag{1-6}$$

where $\mathbf{E}$ is the stiffness matrix of the cylinder. Due to the uncoupled effects of the displacements, the stiffness relations can be written as

$$½ \frac{1}{\mu} \begin{bmatrix} T \\ C \\ M \\ P \end{bmatrix} = \begin{bmatrix} E_{TT} & 0 & 0 & 0 \\ 0 & E_{CC} & 0 & 0 \\ 0 & 0 & E_{MB} & E_{MS} \\ 0 & 0 & 0 & E_{PS} \end{bmatrix} \times \begin{bmatrix} \delta_T \\ \delta_C \\ \delta_B \\ \delta_S \end{bmatrix} \tag{1-7}$$

Thus the problem is now reduced to finding the foregoing five unknowns and to study their evolution with respect to the aspect ratio $L = H/R$ of the cylinder. It is already known that for a large aspect ratio, Saint-Venant's principle can be used so that the cylinder problem can be treated within the framework of the beam theory. On the other hand, for a small aspect ratio, the cylinder can be considered as a thin layer for which analytical solutions can also be easily derived. the analytical results given here are applicable for any aspect ratio and, thereby, make the bridge between beam and thin layer theories.

2 Analytical Resolution

The resolution method used for this problem was first described in Power and Childs (1971) and Duncan-Fama (1972), then improved in Gerhardt and Cheng (1981) and extended in Robert and Keer (1987) for the asymmetric case. This method is based on an eigenfunction expansion technique for which the stress-free condition on the lateral surface is automatically satisfied. The determination of the coefficients in this expansion, also called participating factors, is made through the displacements prescribed on the right and left interfaces.

With the formulation used in the previously mentioned references, a new set of unknowns $\eta_m(r, z)$ and $\xi_m(r, z)$ are introduced in such a way that the equilibrium equations in absence of body forces can be written as

$$\mathbf{L}_m(\eta_m) + \frac{\partial^2}{\partial z^2}(\eta_m) = 0 \tag{2-1a}$$

$$\mathbf{L}_m^*(\xi_m) + \frac{\partial^2}{\partial z^2}(\xi_m) = 0 \tag{2-1b}$$

$\mathbf{L}_m$ and $\mathbf{L}_m^*$ are two formally adjoints linear operators in the radial variable r

$$\mathbf{L}_m(\eta) = \begin{bmatrix} \frac{\partial^2 \eta^1}{\partial r^2} + \left(\frac{1-4m^2}{4r^2}\right)\eta^1 \\ \frac{\partial^2 \eta^2}{\partial r^2} - \left(\frac{3+4m^2}{4r^2}\right)\eta^2 + \frac{\partial \eta^1}{\partial r} - \frac{\eta^1}{2r} - \frac{2m}{r^2}\eta^3 \\ \frac{\partial^2 \eta^3}{\partial r^2} - \left(\frac{3+4m^2}{4r^2}\right)\eta^3 - \frac{2m}{r}\eta^1 - \frac{2m}{r^2}\eta^2 \end{bmatrix} \tag{2-2a}$$

$$\mathbf{L}_m(\xi) = \begin{bmatrix} \frac{\partial^2 \xi^1}{\partial r^2} + \left(\frac{1-4m^2}{4r^2}\right)\xi^1 - \frac{\partial \xi^2}{2r} - \frac{\xi^2}{2r} - \frac{m}{r}\xi^3 \\ \frac{\partial^2 \xi^2}{\partial r^2} - \left(\frac{3+4m^2}{4r^2}\right)\xi^2 - \frac{2m}{r^2}\xi^3 \\ \frac{\partial^2 \xi^3}{\partial r^2} - \left(\frac{3+4m^2}{4r^2}\right)\xi^3 - \frac{2m}{r^2}\xi^2 \end{bmatrix} \tag{2-2b}$$

and η_m and ξ_m are given by

$$\eta_m(r,z) = r^{½} \begin{bmatrix} \frac{1}{1-\nu}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_m) + (\sigma_{zz})_m + \frac{m}{r}v_m\right\} \\ u_m \\ v_m \end{bmatrix} \tag{2-3a}$$

$$
\xi_m(r,z) = r^{1/2} \begin{bmatrix} (1-\nu)\, w_m \\ \dfrac{\partial w_m}{\partial r} - (\sigma_{rz})_m \\ -(\sigma_{\theta z})_m - \dfrac{m}{r}\, w_m \end{bmatrix} \qquad (2\text{-}3b)
$$

Equations (2.1) are now solved by separation of variables and η_m and ξ_m are sought in the form

$$
\eta_m(r,z) = \eta^*_m(r,z) + \sum_{\lambda_{mk} \neq 0} a_{mk} e^{-\lambda_{mk} z} \eta_{mk}(r) \qquad (2\text{-}4a)
$$

$$
\xi_m(r,z) = \xi^*_m(r,z) + \sum_{\lambda_{mk} \neq 0} a_{mk} e^{-\lambda_{mk} z} \xi_{mk}(r) \qquad (2\text{-}4b)
$$

where $\eta_{mk}(r)$ and $\xi_{mk}(r)$ are the nonzero radial eigenvalue solution vectors and $\eta^*_m(r, z)$ and $\xi^*_m(r, z)$ represent the solution vectors corresponding to the case $\lambda_m = 0$.

For each case ($m = 0$ and $m = 1$), the corresponding sets of eigenvalues $\{\lambda_{mk}\}$ are determined such that the stress-free conditions

$$
(\sigma_{rr})_m = (\sigma_{rz})_m = (\sigma_{r\theta})_m = 0 \qquad (2\text{-}5)
$$

expressed in terms of $\eta_{mk}(r)$ or $\xi_{mk}(r)$ are satisfied on the lateral surface ($r = 1$). The eigenvalue equations, as well as the radial eigenfunctions, have been derived in Robert and Keer (1987) and are given next.

(*a*) Axially Symmetric Part of the Deformation (m = 0). The eigenvalues λ_k must satisfy the following transcendental equation

$$
\lambda^2 J_2(\lambda)\left(\lambda^2\left[J_0^2(\lambda) + J_1^2(\lambda)\right] - 2(1-\nu)J_1^2(\lambda)\right) = 0 \qquad (2\text{-}6)
$$

The first set of radial eigenfunctions is associated with the zeros of J_2 and can be expressed as

$$
\Phi'_{0_k}(r) = \begin{bmatrix} u'_{0_k} \\ w'_{0_k} \\ (\sigma_{zz})'_{0_k} \\ (\sigma_{rz})'_{0_k} \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \Psi'_{0_k} = \begin{bmatrix} v'_{0_k} \\ (\sigma_{\theta z})'_{0_k} \end{bmatrix} = \begin{bmatrix} J_1(\delta_k r) \\ -\delta_k J_1(\delta_k r) \end{bmatrix}
$$

where $J_2(\delta_k) = 0$ for $k = 1, 2, 3, \ldots$.

The second set of solutions are associated with the roots of

$$
\lambda^2\left[J_0^2(\lambda) + J_1^2(\lambda)\right] - 2(1-\nu)J_1^2(\lambda) = 0 \qquad (2\text{-}7)
$$

and are given

$$
\begin{cases} u_{0_k}(r) = [\lambda_k^2 J_{0_k} + 2(1-\nu)\lambda_k J_{1_k}]J_1(\lambda_k r) - \lambda_k^2 J_{1_k} r J_0(\lambda_k r) \\ w_{0_k}(r) = [\lambda_k^2 J_{0_k} - 2(1-\nu)\lambda_k J_{1_k}]J_0(\lambda_k r) + \lambda_k^2 J_{1_k} r J_1(\lambda_k r) \end{cases} \qquad (2\text{-}8a)
$$

$$
\begin{cases} (\sigma_{zz})_{0_k}(r) = [-\lambda_k^3 J_{0_k} + 2\lambda_k^2 J_{1_k}]J_0(\lambda_k r) - \lambda_k^3 J_{1_k} r J_1(\lambda_k r) \\ (\sigma_{rz})_{0_k}(r) = -\lambda_k^3 J_{0_k} J_1(\lambda_k r) + \lambda_k^3 J_{1_k} r J_0(\lambda_k r) \end{cases} \qquad (2\text{-}8b)
$$

with

$$
v_{0_k}(r) = (\sigma_{\theta z})_{0_k}(r) = \mathbf{0}
$$

where

$$
J_{0_k} = J_0(\lambda_k) \quad \text{and} \quad J_{1_k} = J_1(\lambda_k)
$$

Finally, the homogeneous solution corresponding to the zero eigenvalue $\lambda_0 = 0$ is given by

$$
\Phi^*_0(r,z) = \begin{bmatrix} u^*_0 \\ w^*_0 \\ (\sigma_{zz})^*_0 \\ (\sigma_{rz})^*_0 \end{bmatrix} = \alpha^*_{01} \begin{bmatrix} -\nu r \\ z \\ 1+\nu \\ 0 \end{bmatrix} + \alpha^*_{03} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (2\text{-}9a)
$$

and

$$
\Psi^*_0(r,z) = \begin{bmatrix} v^*_0 \\ (\sigma_{\theta z})^*_0 \end{bmatrix} = \alpha^*_{02} \begin{bmatrix} 2rz \\ r \end{bmatrix} + \alpha^*_{04} \begin{bmatrix} r \\ 0 \end{bmatrix} \qquad (2\text{-}9b)
$$

where the constants α^*_{01}, α^*_{02}, α^*_{03} and α^*_{04} correspond, respectively, to a uniform traction in the z-direction, a uniform torsion, a rigid-body translation parallel to the z-axis and a rigid-body rotation about the z-axis.

(*b*) Asymmetric Part of the Deformation (m = 1). Let γ be an eigenvalue for the asymmetric case ($m = 1$); γ has to verify the following equation

$$
\gamma^2\Big(\gamma^2 J_1^3(\gamma) - 4\gamma J_1^2(\gamma) J_2(\gamma) + (\gamma^2 + 2\nu + 2)J_1(\gamma)J_2^2(\gamma) - 2\gamma J_2^3(\gamma)\Big) = 0 \qquad (2\text{-}10)
$$

It is of interest to note that in the asymmetric case ($m = 1$), it is not possible to uncouple the real roots from the complex roots, which was easily done in the axisymmetric case by taking into account first the torsion problem (corresponding to the real roots), and then the problem without torsion (corresponding to the complex roots).

The following radial eigenfunctions are obtained for $\gamma_k \neq 0$

$$
\begin{cases} (u+v)_{1_k}(r) = -\gamma_k J_{1_k} J_{2_k} r J_1(\gamma_k r) + [3\gamma_k J_{1_k}^2 - 2(1+\nu)J_{1_k} J_{2_k} + 2\gamma_k J_{2_k}^2] J_2(\gamma_k r) \\ (u-v)_{1_k}(r) = -\gamma_k J_{1_k} J_{2_k} r J_1(\gamma_k r) + [\gamma_k J_{1_k}^2 - 2(2-\nu)J_{1_k} J_{2_k} + 2\gamma_k J_{2_k}^2] J_0(\gamma_k r) \\ w_{1_k}(r) = \gamma_k J_{1_k} J_{2_k} r J_2(\gamma_k r) + J_{1_k}[\gamma_k J_{1_k} - 2(2-\nu)J_{2_k}] J_1(\gamma_k r) \end{cases} \qquad (2\text{-}11a)
$$

$$
\begin{cases} (\sigma_{rz}+\sigma_{\theta z})_{1_k}(r) = \gamma_k^2 J_{1_k} J_{2_k} r J_1(\gamma_k r) - \gamma_k[2\gamma_k J_{1_k}^2 - 2J_{1_k} J_{2_k} + \gamma_k J_{2_k}^2] J_2(\gamma_k r) \\ (\sigma_{rz}-\sigma_{\theta z})_{1_k}(r) = \gamma_k^2 J_{1_k} J_{2_k} r J_1(\gamma_k r) - \gamma_k^2 J_{2_k}^2 J_0(\gamma_k r) \\ (\sigma_{zz})_{1_k}(r) = -\gamma_k^2 J_{1_k} J_{2_k} r J_2(\gamma_k r) + \gamma_k J_{1_k}[-\gamma_k J_{1_k} + 4J_{2_k}] J_1(\gamma_k r) \end{cases} \qquad (2\text{-}11b)
$$

where

$$
J_{1_k} = J_1(\gamma_k) \quad \text{and} \quad J_{2_k} = J_2(\gamma_k).
$$

The zero-eigenvalue solutions obtained when $\gamma_0 = 0$ are given by

$$
\begin{cases} (u+v)^*_1(r) = \alpha^*_{11}(-\nu z r^2) + \alpha^*_{12}(-\nu r^2) \\ (u-v)^*_1(r) = \alpha^*_{11}\left(-\dfrac{1}{3}z^3 + \dfrac{3+2\nu}{2}z\right) + \alpha^*_{12}(-2z^2) + \alpha^*_{13}(-2z) + \alpha^*_{14} \\ w^*_1(r) = \alpha^*_{11}\left(\dfrac{1}{2}z^2 r - \dfrac{1}{4}r^3\right) + \alpha^*_{12} z r + \alpha^*_{13} r \end{cases} \qquad (2\text{-}12a)
$$

$$\begin{cases} (\sigma_{zz})_1^*(r) &= \alpha_{11}^*(1+\nu)zr+\alpha_{12}^*(1+\nu)r \\ (\sigma_{rz}+\sigma_{\theta z})_1^*(r) &= \alpha_{11}^*\left(-\dfrac{1+2\nu}{4}\,r^2\right) \\ (\sigma_{rz}-\sigma_{\theta z})_1^*(r) &= \alpha_{11}^*\left(\dfrac{3+2\nu}{4}-\dfrac{1}{2}\,r^2\right) \end{cases} \tag{2-12b}$$

where the constants α_{11}^*, α_{12}^*, α_{13}^* and α_{14}^* correspond, respectively, to a "mixed shearing" (see Love (1944), page 329), a pure bending, a rigid-body rotation about the y-axis, and a rigid-body translation parallel to the x-axis.

(*c*) **Net Forces and Moments.** As shown in Robert and Keer (1987) using a biorthogonal product between radial eigenfunctions the following relations hold

$$\int_0^1 (\sigma_{zz})_{0_k} r dr=0 \qquad \text{for all } \lambda_k \neq 0 \tag{2-13a}$$

$$\int_0^1 (\sigma_{\theta z})'_{0_k} r^2 dr=0 \qquad \text{for all } \delta_k \neq 0 \tag{2-13b}$$

$$\int_0^1 (\sigma_{zz})_{1_k} r^2 dr=0 \qquad \text{for all } \gamma_k \neq 0 \tag{2-13c}$$

$$\int_0^1 (\sigma_{rz}-\sigma_{\theta z})_{1_k} r dr=0 \qquad \text{for all } \gamma_k \neq 0 \tag{2-13d}$$

The foregoing equations, respectively, mean that the nonzero eigenvalue solution does not produce any net force in the z-direction, any torque, any bending moment nor any net force in the x-direction. Therefore, the stiffness matrix **E** will depend only upon the homogeneous solution associated with the given prescribed displacements

$$½\,\frac{T}{\mu}=2\pi R^2\int_0^1 (\sigma_{zz})_0^*(r) r dr=\pi R^2(1+\nu)\alpha_{01}^*; \tag{2-14a}$$

$$½\,\frac{C}{\mu}=2\pi R^3\int_0^1 (\sigma_{\theta z})_0^*(r) r^2 dr=½\,\pi R^3\alpha_{02}^* \tag{2-14b}$$

$$½\,\frac{P}{\mu}=\pi R^2\int_0^1 [(\sigma_{rz})_1^*-(\sigma_{\theta z})_1^*] r dr d\theta=¼\,\pi R^2(1+\nu)\alpha_{11}^* \tag{2-14c}$$

$$½\,\frac{M}{\mu}=\pi R^3\int_0^1 (\sigma_{zz})_1^*(r) r^2 dr=¼\,\pi R^3(1+\nu)(L\,\alpha_{11}^*+\alpha_{12}^*) \tag{2-14d}$$

where the constants α_{11}^*, α_{12}^*, α_{13}^* and α_{14}^* have to be solved for different aspect ratios in terms of the rigid-body displacements coefficients δ_T, δ_C, δ_B and δ_S.

3 Closed-Form Solution for Two Limit Cases

The two limit cases are obtained when the aspect ratio $L = H/R$ goes either to zero or to infinity. In the first case, thin layer theory can be applied while, in the second case, the stiffness of the cylinder can be determined within the framework of the beam theory.

(*a*) **Infinitely Small Aspect Ratio.** In this case, the cylinder can be thought as an infinite circular layer ($R = \infty$) of thickness unity ($H = 1$) for which the contribution of the stress near the edge can be neglected. Therefore, the following solutions are valid only for r less than one, since they do not satisfy the stress-free condition on the curved surface of this infinite layer.

Traction and Torsion Problems

$$\Phi_0(r,z)=\begin{bmatrix} u_0 \\ w_0 \\ (\sigma_{zz})_0 \\ (\sigma_{rz})_0 \end{bmatrix}=\delta_T\begin{bmatrix} 0 \\ z \\ \left(\dfrac{1-\nu}{1-2\nu}\right) \\ 0 \end{bmatrix};$$

$$\Psi_0(r,z)=\begin{bmatrix} v_0 \\ (\sigma_{\theta z})_0 \end{bmatrix}=\delta_C\begin{bmatrix} 2rz \\ r \end{bmatrix} \tag{3-1}$$

Bending and Shearing Problems

$$\Phi_1(r,z)=\delta_B\begin{bmatrix} -½\,z^2 \\ zr \\ \left(\dfrac{1-\nu}{1-2\nu}\right)r \\ 0 \end{bmatrix}+\delta_S\begin{bmatrix} z \\ 0 \\ 0 \\ ½ \end{bmatrix};$$

$$\Psi_1(r,z)=\delta_B\begin{bmatrix} ½z^2 \\ 0 \end{bmatrix}+\delta_s\begin{bmatrix} -z \\ -½ \end{bmatrix} \tag{3-2}$$

Therefore, the net forces and moments are given by

$$½\,\frac{T}{\mu}=\pi R^2\left(\frac{1-\nu}{1-2\nu}\right)\delta_T \qquad \text{or} \qquad \frac{T}{S}=(\lambda+2\mu)\delta_T \tag{3-3a}$$

$$½\,\frac{C}{\mu}=½\,\pi R^3\delta_C \qquad \text{or} \qquad \frac{R\,C}{I}=4\mu\delta_C \tag{3-3b}$$

$$½\,\frac{P}{\mu}=½\,\pi R^2\delta_S \qquad \text{or} \qquad \frac{P}{S}=\mu\delta_S \tag{3-3c}$$

$$½\,\frac{M}{\mu}=¼\,\pi R^3\left(\frac{1-\nu}{1-2\nu}\right)\delta_B \qquad \text{or} \qquad \frac{R\,M}{I}=(\lambda+2\mu)\delta_B \tag{3-3d}$$

where λ and μ are the Lamé coefficients of the material

$$2\mu=E/(1+\nu), \quad \lambda=2\mu\nu/(1-2\nu)$$

and S and I are the area and the moment of inertia of the cross section

$$S=\iint d\bar{x}\,d\bar{y}$$

$$I=\iint \bar{x}^2 d\bar{x}\,d\bar{y}=¼\,R^2S \quad \text{for a circular cylinder.}$$

(*b*) **Infinitely Large Aspect Ratio.** In this limit configuration, the classical beam theory can be applied through Saint-Venant's principle and the following resultant forces and moments are obtained

$$\frac{T}{S}=E\,\delta_T \tag{3-4a}$$

$$\frac{R\,C}{I}=4\mu\,\delta_C \tag{3-4b}$$

$$\frac{R\,M}{I}=E\left(\delta_B+\frac{6}{L}\,\delta_S\right) \tag{3-4c}$$

$$\frac{P}{S}=\frac{E}{4}\,\delta_S\,\frac{12}{L^2} \tag{3-4d}$$

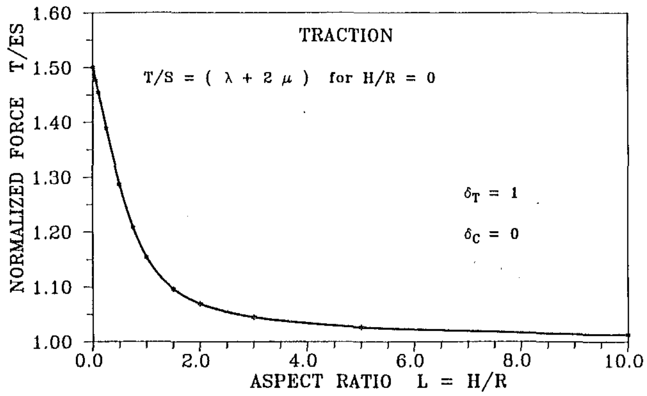

Fig. 3 Normalized traction force *T/ES* obtained for different aspects ratios $L = H/R$; Poisson's ratio is taken to be 1/3 and the axial displacement coefficient δ_T equals one

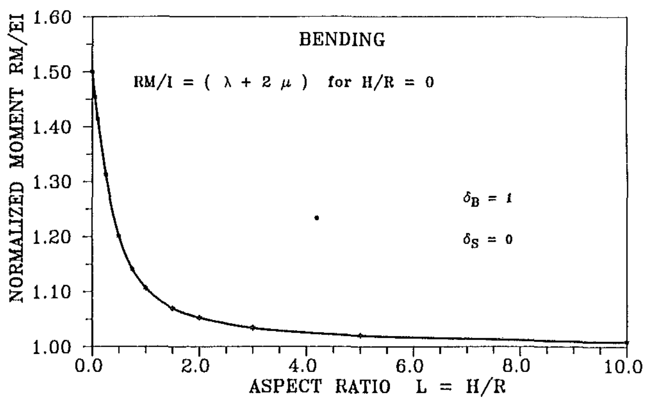

Fig. 4 Normalized bending moment *RM/EI* obtained for different aspect ratios $L = H/R$; Poisson's ratio is taken to be 1/3 and the bending displacement coefficient δ_B equals one

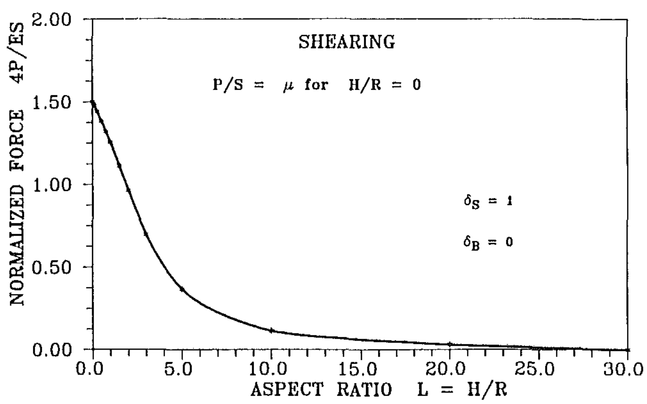

Fig. 5 Normalized shear force *4P/ES* obtained for different aspect ratios $L = H/R$; Poisson's ratio is taken to be 1/3 and the shearing displacement coefficient δ_S equals one

The equations associated with δ_S were derived by considering the superposition of the beam solutions for a concentrated load P and moment M such that the rotation of the end is zero.

4 Analytical Solution for Any Aspect Ratio

For a nonzero and finite aspect ratio $L = H/R$, the determination of the homogeneous solution requires in fact the determination of all the participating factors a_{mk}. This is done as in Gerhardt and Cheng (1981) and Robert and Keer (1987)

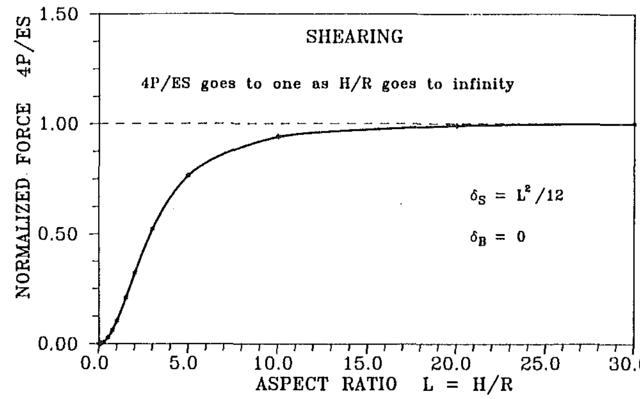

Fig. 6 Normalized shear force *4P/ES* obtained for different aspect ratios $L = H/R$; Poisson's ratio is taken to be 1/3 and the shearing displacement coefficient δ_S equals $L^2/12$

with a Galerkin-type method: Those equations are obtained with the prescribed displacements on the right and left interfaces ($z = 0$ and $z = L$) and can be written in the most general case as

$$\int_0^1 [(u_m - u_m^*)\chi_{mj}^1 + (v_m - v_m^*)\chi_{mj}^2 + (w_m - w_m^*)\chi_{mj}^3] r dr$$

$$= \sum_{k=1}^{N} a_{mk} e^{-\lambda_{mk} z} \int_0^1 [u_{mk}\chi_{mj}^1 + v_{mk}\chi_{mj}^2 + w_{mk}\chi_{mj}^3] r dr \qquad (4.1)$$

at $z = 0$ and $z = L$, for $m = 0$ or 1 and $j = 0, 1, 2 \cdots, N$; where u_m, v_m and w_m are the mth harmonics of the prescribed displacements at the flat ends. The optimum weighting functions χ_{mj}^1, χ_{mj}^2 and χ_{mj}^3 are found in Gerhardt and Cheng (1981) and Robert and Keer (1987) and are given next.

(*a*) **Torsion Problem.** The exact solution for the uniform torsion problem is by definition the homogeneous solution and, therefore, the participating factors a_{0_k} are all equal to zero. The simplicity of the solution can be explained by the fact that no Poisson's effect is involved during a uniform torsion. Thus, in this case, the stiffness does not depend upon the aspect ratio of the cylinder.

(*b*) **Traction Problem.** For $j = 0$

$$\chi_{00}^1 = \chi_{00}^2 = 0 \quad \text{and} \quad \chi_{00}^3 = 1$$

For $j \neq 0$

$$\begin{cases} \chi_{0j}^1 = [\lambda_j^3 J_{0_j} + (1-2\nu)\lambda_j^2 J_{1_j}] J_1(\lambda_j r) - \lambda_j^3 J_{1_j} r J_0(\lambda_j r) \\ \chi_{0j}^2 = 0 \\ \chi_{0j}^3 = [-\lambda_j^3 J_{0_j} + (3-2\nu)\lambda_j^2 J_{1_j}] J_0(\lambda_j r) - \lambda_j^3 J_{1_j} r J_1(\lambda_j r) \end{cases} \qquad (4\text{-}2)$$

where $J_{0_j} = J_0(\lambda_j)$; $J_{1_j} = J_1(\lambda_j)$ and λ_j is a nonzero eigenvalue for axially symmetric case $m = 0$.

(*c*) **Bending and Shearing Problem.** For $j = 0$, we have two different sets of weighting functions

$$\begin{cases} \chi_{10}^{1\prime} = 0 \\ \chi_{10}^{2\prime} = 0 \\ \chi_{10}^{3\prime} = r \end{cases} \qquad (4\text{-}3a)$$

and

$$\begin{cases} \chi_{10}^2 = -(3+2\nu)(1-r^2) \\ \chi_{10}^2 = (3+2\nu) + (-1+2\nu)r^2 \\ \chi_{10}^3 = 0 \end{cases} \qquad (4\text{-}3b)$$

For $j \neq 0$, the weighting functions are given by

$$\begin{cases} \chi^1_{1_j} + \chi^2_{1_j} & = A^1 \gamma_j J_{1_j} J_{2_j} r J(\gamma_j r) \\ & \quad + [B^1_j \gamma J^2_{1_j} + 2C^1 J_{1_j} J_{2_j} + D^1 \gamma_j J^2_{2_j}] J_2(\gamma_j r) \\ \chi^2_{1_j} - \chi^2_{1_j} & = A^2 \gamma_j J_{1_j} J_{2_j} r J_1(\gamma_j r) \\ & \quad + [B^2 \gamma_j J^2_{1_j} + 2C^2 J_{1_j} J_{2_j} + D^2 \gamma_j J^2_{2_j}] J_0(\gamma_j r) \\ \chi^3_{1_j} & = A^3 \gamma_j J_{1_j} J_{2_j} r J_2(\gamma_j r) \\ & \quad + [B^3 \gamma_j J^2_{1_j} + 2C^3 J_{1_j} J_{2j} + D^3 \gamma_j J^2_{2_j}] J_1(\gamma_j r) \end{cases} \tag{4.4}$$

where $J_{1_j} = J_1(\gamma_j)$; $J_{2_j} = J_2(\gamma_j)$; γ_j is a nonzero eigenvalue for the asymmetric case $m = 1$ and

$$\begin{array}{lll} A^1 = 2(1-\nu) & A^2 = 2(1-\nu) & A^3 = 2(1-\nu) \\ B^1 = -5+6\nu & B^2 = -1+2\nu & B^3 = 2(1-\nu) \\ C^1 = -2\nu^2 - \nu + 2 & C^2 = 2\nu^2 - 5\nu + 2 & C^3 = (1-\nu)(-5+2\nu) \\ D^1 = -3+4\nu & D^2 = -3+4\nu & D^3 = 0 \end{array}$$

The aforementioned weighting functions have been found in such a way that the truncated system of linear equations (4.1) is the more diagonally dominant possible.

A closed form for the integral $\int_0^1 [u_{mk}\, \chi^1_{mj} + v_{mk}\, \chi^2_{mj} + w_{mk}\, \chi^3_{mj}]\, rdr$ can be found in Robert and Keer (1987) for both $m = 0$ and $m = 1$. The Galerkin system (4.1) has been solved for each of the following cases

$\delta_T = 1$; $\delta_C = \delta_B = \delta_S = 0$ (Traction only; see Fig. 3)

$\delta_B = 1$; $\delta_T = \delta_C = \delta_S = 0$ (Bending only; see Fig. 4)

$\delta_S = 1$; $\delta_T = \delta_C = \delta_B = 0$ (Shear only; see Fig. 5)

$\delta_S = \frac{L^2}{12}$; $\delta_T = \delta_C = \delta_B = 0$ (Shear only; see Fig. 6)

with a number of terms in the eigenfunction expansion corresponding to 40 pairs and 40 triplets for $m = 0$ and $m = 1$, respectively. Only the case, when Poisson's ratio ν equals 1/3, has been studied, and the aspect ratio $L = H/R$ was taken between 0.05 and 30. Figures 3–6 illustrate the results and the convergence to the two limit cases studied previously is clear.

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