

# BOUNDEDNESS OF HIGHER-ORDER MARCINKIEWICZ-TYPE INTEGRALS

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Let  $A$  be a function with derivatives of order  $m$  and  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $0 < \beta < 1$ ,  $|\gamma| = m$ ). The authors in the paper proved that if  $\Omega \in L^s(S^{n-1})$  ( $s \geq n/(n - \beta)$ ) is homogeneous of degree zero and satisfies a vanishing condition, then both the higher-order Marcinkiewicz-type integral  $\mu_\Omega^A$  and its variation  $\tilde{\mu}_\Omega^A$  are bounded from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$  and from  $L^1(\mathbb{R}^n)$  to  $L^{n/(n-\beta), \infty}(\mathbb{R}^n)$ , where  $1 < p < n/\beta$  and  $1/q = 1/p - \beta/n$ . Furthermore, if  $\Omega$  satisfies some kind of  $L^s$ -Dini condition, then both  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  are bounded on Hardy spaces, and  $\mu_\Omega^A$  is also bounded from  $L^p(\mathbb{R}^n)$  to certain Triebel-Lizorkin space.

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## 1. Introduction

Suppose that  $S^{n-1}$  is the unit sphere of  $\mathbb{R}^n$  ( $n \geq 2$ ) equipped with the normalized Lebesgue measure  $d\sigma = d\sigma(x')$ . Let  $\Omega \in L^1(S^{n-1})$  be homogeneous of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1.1)$$

where  $x' = x/|x|$  for any  $x \neq 0$ . Then the Marcinkiewicz integral operator of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad (1.2)$$

where  $F_{\Omega,t}(f)(x) = (1/t) \int_{|x-y| \leq t} (\Omega(x-y)/|x-y|^{n-1}) f(y) dy$ .

And if we denote  $H$  as the Hilbert space  $H = \{h : \|h\|_H = (\int_0^\infty |h(t)|^2 (dt/t))^{1/2} < \infty\}$ , then  $\mu_\Omega(f)$  can be looked as the vector-valued function in  $H$ , that is

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t} \right)^{1/2} = \|F_{\Omega,t}(f)(x)\|_H. \quad (1.3)$$

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It is well known that the operator  $\mu_\Omega$  was defined first by Stein in [13], where the author proved that if  $\Omega$  is continuous and satisfies a  $\text{Lip}_\alpha$  ( $0 < \alpha \leq 1$ ) condition on  $S^{n-1}$ , then  $\mu_\Omega$  is an operator of type  $(p, p)$  for  $1 < p \leq 2$  and of weak type  $(1, 1)$ . Benedek et al. in [1] showed that if  $\Omega \in C^1(S^{n-1})$ , then  $\mu_\Omega$  is an operator of type  $(p, p)$  for  $1 < p < \infty$ . Recently, Ding et al. in [4] improved the results mentioned above. They gave the  $L^p(\mathbb{R}^n)$  ( $1 < p < \infty$ ) boundedness of  $\mu_\Omega$  for  $\Omega \in H^1(S^{n-1})$ , where  $H^1$  denotes the Hardy space on  $S^{n-1}$  (see [3] for the definition of  $H^1$ ).

On the other hand, let  $b \in L_{\text{loc}}(\mathbb{R}^n)$ , then the commutator of Marcinkiewicz integral is defined by

$$\mu_\Omega^b(f)(x) = \left( \int_0^\infty |F_{\Omega, b, t}(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad (1.4)$$

where

$$F_{\Omega, b, t}(f)(x) = \frac{1}{t} \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x) - b(y)) f(y) dy. \quad (1.5)$$

In 1990, Torchinsky and Wang [14] proved that if  $\Omega$  is continuous and satisfies a  $\text{Lip}_\alpha$  ( $0 < \alpha \leq 1$ ) condition, then for  $b \in \text{BMO}$ ,  $\mu_\Omega^b$  is bounded on  $L^p(\omega)$ , here  $\omega \in A_p$  ( $1 < p < \infty$ ).

For  $\beta > 0$ , the homogeneous Lipschitz space  $\dot{\Lambda}_\beta(\mathbb{R}^n)$  is the space of function  $f$  such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x, h \in \mathbb{R}^n, h \neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty, \quad (1.6)$$

where  $\Delta_h^k$  denotes the  $k$ th difference operator (see [12]).

When  $\Omega$  satisfies a  $\text{Lip}_\alpha$  ( $0 < \alpha \leq 1$ ) condition and  $b \in \dot{\Lambda}_\beta$  ( $0 < \beta < \min\{1/2, \alpha\}$ ), Liu [9] considered the  $(L^p, \dot{F}_p^{\beta, \infty})$  boundedness of  $\mu_\Omega^b$ , and Wang [15] showed that  $\mu_\Omega^b$  is also bounded from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ , for  $1 < p < n/\beta$  and  $1/q = 1/p - \beta/n$ . Later, in [11] we weakened the smoothness condition assumed on  $\Omega$  and got the same conclusions.

Moreover, let  $m \in \mathbb{N}$  and let  $A$  be a function on  $\mathbb{R}^n$ . We denote

$$\begin{aligned} R_{m+1}(A; x, y) &= A(x) - \sum_{|\gamma| \leq m} \frac{1}{\gamma!} D^\gamma A(y) (x-y)^\gamma, \\ Q_{m+1}(A; x, y) &= R_m(A; x, y) - \sum_{|\gamma|=m} \frac{1}{\gamma!} D^\gamma A(x) (x-y)^\gamma. \end{aligned} \quad (1.7)$$

Then the higher-order Marcinkiewicz-type integral and its variation are defined, respectively, by

$$\begin{aligned} \mu_\Omega^A(f)(x) &= \left( \int_0^\infty |F_{\Omega, t}^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \\ \tilde{\mu}_\Omega^A(f)(x) &= \left( \int_0^\infty |\tilde{F}_{\Omega, t}^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \end{aligned} \quad (1.8)$$

where

$$\begin{aligned} F_{\Omega,t}^A(f)(x) &= \frac{1}{t} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y) f(y) dy, \\ \tilde{F}_{\Omega,t}^A(f)(x) &= \frac{1}{t} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} Q_{m+1}(A;x,y) f(y) dy. \end{aligned} \quad (1.9)$$

When  $\Omega \in \text{Lip}_\alpha(S^{n-1})$  and  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $0 < \beta < \min\{1/2, \alpha\}$ ), Liu [8] considered the boundedness of  $\mu_\Omega^A$  and got the following results.

**THEOREM 1.1** [8]. *Let  $1 < p < \infty$ , let  $0 < \alpha \leq 1$ , let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^n$  and satisfy (1.1). If  $\Omega \in \text{Lip}_\alpha(S^{n-1})$  and  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $0 < \beta < \min\{1/2, \alpha\}$ ), then*

- (a)  $\mu_\Omega^A$  is bounded from  $L^p(\mathbb{R}^n)$  to  $\dot{F}_p^{\beta, \infty}(\mathbb{R}^n)$ ,
- (b)  $\mu_\Omega^A$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ , for  $1/p - 1/q = \beta/n$  and  $1/p > \beta/n$ .

It is well known that any weakness or removal of smoothness assumed on kernels is very interesting to the boundedness of singular integrals. Inspired by [9, 15, 11], we want to know whether the conditions assumed on  $\Omega$  in Theorem 1.1 can be weakened or removed. In fact, the answer is affirmative. And we will also study the boundedness of  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  on Hardy spaces. Let us now give a definition and formulate our results.

*Definition 1.2.* For  $\Omega \in L^s(S^{n-1})$  ( $s \geq 1$ ), the integral modulus  $\omega_s(\delta)$  of continuity of order  $s$  of  $\Omega$  is defined by

$$\omega_s(\delta) = \sup_{|\rho| \leq \delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^s d\delta(x') \right)^{1/s}, \quad (1.10)$$

where  $\rho$  is a rotation on  $S^{n-1}$ ,  $|\rho| = \|\rho - I\|$ . When  $\omega_s(\delta)$  satisfies

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty, \quad (1.11)$$

it is said that  $\Omega(x')$  satisfies the  $L^s$ -Dini condition.

**THEOREM 1.3.** *Let  $0 < \beta < 1$ , let  $1 < p < n/\beta$ , let  $1/q = 1/p - \beta/n$ , let  $s \geq n/(n - \beta)$ , and let  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $|\gamma| = m$ ). If  $\Omega \in L^s(S^{n-1})$  satisfies (1.1), then both  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  are bounded from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .*

**THEOREM 1.4.** *Let  $0 < \beta < 1$ , let  $s \geq n/(n - \beta)$ , and let  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $|\gamma| = m$ ). If  $\Omega \in L^s(S^{n-1})$  satisfies (1.1), then both  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  are bounded from  $L^1(\mathbb{R}^n)$  to  $L^{n/(n-\beta), \infty}(\mathbb{R}^n)$ .*

**THEOREM 1.5.** *Let  $1 \leq s' < p < \infty$ , let  $\Omega$  satisfy (1.1) and the condition*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\varepsilon}} d\delta < \infty \quad \text{for some } 0 < \varepsilon \leq 1. \quad (1.12)$$

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Then for  $D^\gamma \in \dot{\Lambda}_\beta$  ( $|\gamma| = m$ ,  $0 < \beta < \min\{1/2, \varepsilon\}$ ),

$$\|\mu_\Omega^A(f)\|_{\dot{F}_p^{\beta, \infty}} \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p}, \quad (1.13)$$

where  $1/s' + 1/s = 1$ .

**THEOREM 1.6.** *Let  $0 < \varepsilon \leq 1$ , let  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $|\gamma| = m$ ,  $0 < \beta \leq \min\{1/2, \varepsilon\}$ ), let  $n/(n + \beta) < p < 1$ , and let  $1/r = 1/p - \beta/n$ . If there exists some  $s \geq \max\{r, n/(n - \beta)\}$ , such that  $\Omega \in L^s(S^{n-1})$  satisfying (1.1) and (1.12), then both  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  are bounded from  $H^p(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$ .*

When  $p = 1$ , (1.12) can be replaced by (1.11) and we can take  $0 < \beta < 1$ .

**THEOREM 1.7.** *Let  $D^\gamma A \in \dot{\Lambda}_\beta$  ( $|\gamma| = m$ ,  $0 < \beta < 1$ ). If there exists some  $s \geq n/(n - \beta)$  such that  $\Omega \in L^s(S^{n-1})$  satisfying (1.1) and (1.11), then both  $\mu_\Omega^A$  and  $\tilde{\mu}_\Omega^A$  are bounded from  $H^1(\mathbb{R}^n)$  to  $L^{n/(n-\beta)}(\mathbb{R}^n)$ .*

*Remark 1.8.* When  $m = 0$ ,  $\mu_\Omega^A(f)$  is the commutator of Marcinkiewicz integral. So, our results in this paper are extensions of those in [9, 15, 11].

*Remark 1.9.* It is easy to see that if  $\Omega \in \text{Lip}_\alpha(S^{n-1})$  ( $0 < \alpha \leq 1$ ), then  $\Omega \in L^s(S^{n-1})$  for any  $s \geq 1$  and satisfies the  $L^s$ -Dini condition (1.12). In addition, (1.11) is weaker than (1.12) (see [6]). So, Theorems 1.3, 1.4, and 1.5 in the paper are substantial improvements of Theorem A. It should be pointed out that any smooth condition assumed on  $\Omega$  is not needed in Theorems 1.3 and 1.4.

## 2. Some basic notations and lemmas

**LEMMA 2.1** [7]. *Let  $A$  be a function with derivatives of order  $m$  in  $\dot{\Lambda}_\beta$  ( $0 < \beta < 1$ ), then there exists a constant  $C > 0$  such that*

$$|R_{m+1}(A; x, y)| \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) |x - y|^{m+\beta}; \quad (2.1)$$

$$|Q_{m+1}(A; x, y)| \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) |x - y|^{m+\beta}; \quad (2.2)$$

$$|R_{m+1}(A; x, y) - R_{m+1}(A; x, z)| \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \sum_{i=0}^m |x - z|^i |z - y|^{m-i+\beta}; \quad (2.3)$$

$$|R_{m+1}(A; x, y) - R_{m+1}(A; z, y)| \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \left( \sum_{i=1}^m |x - z|^i |z - y|^{m-i+\beta} + |x - z|^{m+\beta} \right); \quad (2.4)$$

$$\begin{aligned} & |Q_{m+1}(A; x, y) - Q_{m+1}(A; x, z)| \\ & \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \sum_{i=0}^{m-1} |x - z|^i |z - y|^{m-i} (|x - y|^\beta + |y - z|^\beta). \end{aligned} \quad (2.5)$$

LEMMA 2.2 [5]. Let  $0 < \alpha < n$ , let  $1 < p < n/\alpha$ , let  $1/q = 1/p - \alpha/n$ , and let  $s \geq n/(n - \alpha)$ . If  $\Omega \in L^s(S^{n-1})$ , then the fractional integral operator  $T_{\Omega,\alpha}$  defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy, \quad (2.6)$$

is bounded from  $L^p(\mathbb{R}^n)$  to  $L^q(\mathbb{R}^n)$ .

LEMMA 2.3 [2]. Let  $0 < \alpha < n$  and let  $s \geq n/(n - \alpha)$ . If  $\Omega \in L^s(S^{n-1})$ , then for any  $\lambda > 0$  and  $f \in L^1(\mathbb{R}^n)$ , there exists a constant  $C > 0$ , such that

$$|\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}f(x)| > \lambda\}| \leq C \left( \frac{\|f\|_{L^1}}{\lambda} \right)^{n/(n-\alpha)}. \quad (2.7)$$

Remark 2.4. Set

$$\bar{T}_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy. \quad (2.8)$$

It is easy to see that  $\bar{T}_{\Omega,\alpha}$  satisfies Lemmas 2.2 and 2.3.

LEMMA 2.5 [12]. For  $0 < \beta < 1$ ,  $1 < p < \infty$ ,

$$\|f\|_{F_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \right\|_{L^p}, \quad (2.9)$$

where  $f_Q = (1/|Q|) \int_Q f(x) dx$ .

LEMMA 2.6 [6]. Suppose that  $0 < \lambda < n$  and  $\Omega$  is homogeneous of degree zero and satisfies the  $L^s$ -Dini condition (1.11) for  $s > 1$ . If there exists a constant  $a_0 > 0$  such that  $|x| < a_0 R$ , then

$$\left( \int_{R < |y| < 2R} \left| \frac{\Omega(y-x)}{|y-x|^{n-\lambda}} - \frac{\Omega(y)}{|y|^{n-\lambda}} \right|^s dy \right)^{1/s} \leq CR^{n/s-(n-\lambda)} \left\{ \frac{|x|}{R} + \int_{|x|/2R}^{|x|/R} \frac{\omega_s(\delta)}{\delta} d\delta \right\}, \quad (2.10)$$

where the constant  $C > 0$  is independent of  $R$  and  $x$ .

### 3. Proofs of Theorems 1.3, 1.4, and 1.5

We first prove Theorems 1.3 and 1.4.

By Lemmas 2.2, 2.3 and Remark 2.4, we need only to show that there exists a constant  $C > 0$  such that

$$\begin{aligned} \mu_{\Omega}^A(f)(x) &\leq C \bar{T}_{\Omega,\beta}(f)(x), \\ \tilde{\mu}_{\Omega}^A(f)(x) &\leq C \bar{T}_{\Omega,\beta}(f)(x), \end{aligned} \quad (3.1)$$

for any  $x \in \mathbb{R}^n$ .

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In fact, for any fixed  $x \in \mathbb{R}^n$ , by the Minkowski inequality and (2.1), we have

$$\begin{aligned}
\mu_{\Omega}^A(f)(x) &= \left[ \int_0^{\infty} \left| \frac{1}{t} \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) f(y) dy \right|^2 \frac{dt}{t} \right]^{1/2} \\
&\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_{m+1}(A; x, y)| |f(y)| \left[ \int_{|x-y| \leq t} \frac{1}{t^3} dt \right]^{1/2} dy \\
&\leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n+m}} |R_{m+1}(A; x, y)| |f(y)| dy \\
&\leq C \left( \sum_{y=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)| dy \\
&\leq C \left( \sum_{y=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \tilde{T}_{\Omega, \beta}(f)(x).
\end{aligned} \tag{3.2}$$

Similarly, by the Minkowski inequality and (2.2),

$$\tilde{\mu}_{\Omega}^A(f)(x) \leq C \left( \sum_{y=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \tilde{T}_{\Omega, \beta}(f)(x). \tag{3.3}$$

So, we complete the proofs of Theorems 1.3 and 1.4. Let us now turn to prove Theorem 1.5.

Fix a cube  $Q(x_Q, l) \ni x$  with its center at  $x_Q$  and denote the half side length of  $Q$  by  $l$ . Let  $Q^* = 4\sqrt{n}Q$ , then for  $f \in L^p(\mathbb{R}^n)$ , we write  $f = f_1 + f_2$ , where  $f_1 = f\chi_{Q^*}$  and  $f_2 = f\chi_{(Q^*)^c}$ . It is obvious that there is an  $N \in \mathbb{N}$ , such that  $2^N \leq 4\sqrt{n} < 2^{N+1}$ .

Since by the definition of  $\mu_{\Omega}^A(f)$ , we have

$$\begin{aligned}
|\mu_{\Omega}^A(f)(y) - \mu_{\Omega}^A(f_2)(x_Q)| &= |||F_{\Omega, t}^A(f)(y)| - |F_{\Omega, t}^A(f_2)(x_Q)|| \\
&\leq |||F_{\Omega, t}^A(f_1)(y)|| + |||F_{\Omega, t}^A(f_2)(y) - F_{\Omega, t}^A(f_2)(x_Q)||.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q |\mu_{\Omega}^A(f)(y) - (\mu_{\Omega}^A(f))_Q| dy \\
&\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |\mu_{\Omega}^A(f)(y) - \mu_{\Omega}^A(f_2)(x_Q)| dy \\
&\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |\mu_{\Omega}^A(f_1)(y)| dy + \frac{2}{|Q|^{\beta/n}} \sup_{y \in Q} |||F_{\Omega, t}^A(f_2)(y) - F_{\Omega, t}^A(f_2)(x_Q)|| \\
&:= J_1 + J_2.
\end{aligned} \tag{3.5}$$

Choose  $1 < p_1 < n/\beta$  and  $1/q_1 = 1/p_1 - \beta/n$  such that  $1 < p_1 < p$ . Then by Hölder's inequality and the  $(L^{p_1}, L^{q_1})$  boundedness of  $\mu_\Omega^A$  (see Theorem 1.3), we have

$$\begin{aligned}
J_1 &\leq \frac{2}{|Q|^{1+\beta/n}} |Q|^{1-1/q_1} \|\mu_\Omega^A(f_1)\|_{L^{q_1}} \\
&\leq C|Q|^{-1/p_1} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|f_1\|_{L^{p_1}} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \left( \frac{1}{|Q|} \int_{Q^*} |f(y)|^{p_1} dy \right)^{1/p_1} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) M_{p_1}(f)(x).
\end{aligned} \tag{3.6}$$

Let us now estimate  $J_2$ .

Denote  $D = \|F_{\Omega,t}^A(f_2)(y) - F_{\Omega,t}^A(f_2)(x_Q)\|$ , then

$$\begin{aligned}
D &= \left[ \int_0^\infty \left| \left[ \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A; y, z) f_2(z) dz \right. \right. \right. \\
&\quad \left. \left. \left. - \int_{|x_Q-z|\leq t} \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A; x_Q, z) f_2(z) dz \right] \right|^2 \frac{dt}{t^3} \right]^{1/2} \\
&\leq \left[ \int_0^\infty \left| \int_{\{|y-z|\leq t, |x_Q-z|>t\}} \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A; y, z) f_2(z) dz \right|^2 \frac{dt}{t^3} \right]^{1/2} \\
&\quad + \left[ \int_0^\infty \left| \int_{\{|y-z|>t, |x_Q-z|\leq t\}} \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A; x_Q, z) f_2(z) dz \right|^2 \frac{dt}{t^3} \right]^{1/2} \\
&\quad + \left\{ \int_0^\infty \left| \int_{\{|y-z|\leq t, |x_Q-z|\leq t\}} \left[ \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A; y, z) \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A; x_Q, z) \right] f_2(z) dz \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\
&:= U + V + W.
\end{aligned} \tag{3.7}$$

Notice that  $|z - x_Q| \sim |y - z|$  for  $z \in (Q^*)^c$ . By the Minkowski inequality, (2.1), and Hölder's inequality, we have

$$\begin{aligned}
V &\leq C \int_{\mathbb{R}^n} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} |R_{m+1}(A; x_Q, z)| |f_2(z)| \left( \int_{|x_Q-z|}^{|y-z|} \frac{1}{t^3} dt \right)^{1/2} dz \\
&\leq C \int_{(Q^*)^c} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) |x_Q-z|^{m+\beta} |f(z)| \frac{t^{1/2}}{|x_Q-z|^{3/2}} dz
\end{aligned}$$

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$$\begin{aligned}
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} 2^{-k/2} (2^k l)^{-n+\beta} \int_{2^k l \leq |x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)| |f(z)| dz \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} 2^{-k/2} (2^k l)^{-n+\beta} \left( \int_{|x_Q - z| < 2^{k+1} l} |f(z)|^{s'} \right)^{1/s'} \\
&\quad \times \left( \int_{|x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)|^s \right)^{1/s} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} 2^{-k/2} (2^k l)^{-n+\beta+n/s'} M_{s'}(f)(x) \\
&\quad \times \left( \int_{|x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)|^s \right)^{1/s}. \tag{3.8}
\end{aligned}$$

Since  $\Omega \in L^s(S^{n-1})$ , it is easy to see that

$$\left[ \int_{|x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)|^s \right]^{1/s} \leq C (2^{k+1} l)^{n/s} \|\Omega\|_{L^s(S^{n-1})}. \tag{3.9}$$

Therefore, by  $0 < \beta < 1/2$ , we have

$$\begin{aligned}
V &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} 2^{-k/2} (2^k l)^{-n+\beta+n/s'+n/s} \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} 2^{-k(1/2-\beta)} |Q|^{\beta/n} \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.10}
\end{aligned}$$

In the same way, we have

$$U \leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.11}$$

Let us now estimate  $W$ .



Since,

$$\begin{aligned}
& \left| \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A; y, z) - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A; x_Q, z) \right| \\
& \leq \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right| \frac{1}{|y-z|^m} |R_{m+1}(A; y, z)| \\
& \quad + \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}} \left| \frac{1}{|y-z|^m} - \frac{1}{|x_Q-z|^m} \right| |R_{m+1}(A; y, z)| \\
& \quad + \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} |R_{m+1}(A; y, z) - R_{m+1}(A; x_Q, z)|.
\end{aligned} \tag{3.12}$$

By the Minkowski inequality and  $|y-z| \sim |x_Q-z|$  for any  $z \in (Q^*)^c$ , we have

$$\begin{aligned}
W & \leq \int_{\mathbb{R}^n} \left( \int_{\{|y-z| \leq t, |x_Q-z| \leq t\}} \frac{dt}{t^3} \right)^{1/2} \\
& \quad \times \left| \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A; y, z) - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A; x_Q, z) \right| |f_2(z)| dz \\
& \leq \int_{(Q^*)^c} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right| \frac{1}{|y-z|^m} |R_{m+1}(A; y, z)| |f(z)| \frac{1}{|x_Q-z|} dz \\
& \quad + \int_{(Q^*)^c} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}} \left| \frac{1}{|y-z|^m} - \frac{1}{|x_Q-z|^m} \right| |R_{m+1}(A; y, z)| |f(z)| \frac{1}{|x_Q-z|} dz \\
& \quad + \int_{(Q^*)^c} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} |R_{m+1}(A; y, z) - R_{m+1}(A; x_Q, z)| |f(z)| \frac{1}{|x_Q-z|} dz \\
& := W_1 + W_2 + W_3.
\end{aligned} \tag{3.13}$$

For  $W_1$ , using (2.1) and Hölder's inequality,

$$\begin{aligned}
W_1 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_{(Q^*)^c} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right| |y-z|^{\beta-1} |f(z)| dz \\
& \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{\beta-1} \\
& \quad \times \int_{2^k l \leq |z-x_Q| < 2^{k+1} l} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right| |f(z)| dz
\end{aligned}$$

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$$\begin{aligned}
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{\beta-1} \left[ \int_{|z-x_Q| \leq 2^{k+l}} |f(z)|^{s'} dz \right]^{1/s'} \\
&\quad \times \left[ \int_{2^k l \leq |z-x_Q| < 2^{k+1} l} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right|^s dz \right]^{1/s} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{\beta-1+n/s'} M_{s'}(f)(x) \\
&\quad \times \left[ \int_{2^k l \leq |z-x_Q| < 2^{k+1} l} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right|^s dz \right]^{1/s}. \tag{3.14}
\end{aligned}$$

However, by Lemma 2.6 and (1.12), we obtain

$$\begin{aligned}
&\left[ \int_{2^k l \leq |z-x_Q| < 2^{k+1} l} \left| \frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}} \right|^s dz \right]^{1/s} \\
&\leq C(2^k l)^{n/s-(n-1)} \left\{ \frac{|y-x_Q|}{2^k l} + \int_{|y-x_Q|/2^{k+1} l}^{|y-x_Q|/2^k l} \frac{\omega_s(\delta)}{\delta} d\delta \right\} \\
&\leq C(2^k l)^{n/s-(n-1)} \left\{ 2^{-k} + 2^{-k\varepsilon} \int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\varepsilon}} d\delta \right\} \\
&\leq C(2^k l)^{n/s-(n-1)} (2^{-k} + 2^{-k\varepsilon}). \tag{3.15}
\end{aligned}$$

Hence,

$$\begin{aligned}
W_1 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{\beta-1+n/s'} (2^k l)^{n/s-(n-1)} (2^{-k} + 2^{-k\varepsilon}) M_{s'}(f)(x) \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^{-k(1-\beta)} + 2^{-k(\varepsilon-\beta)}) |Q|^{\beta/n} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) M_{s'}(f)(x). \tag{3.16}
\end{aligned}$$

Since  $|y-z| \sim |x_Q-z|$ , for any  $z \in (Q^*)^c$ , it is similar to the estimate of  $V$ , and we have

$$\begin{aligned}
W_2 &\leq C \int_{(Q^*)^c} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}} \frac{l}{|y-z|^{m+1}} |R_{m+1}(A; y, z)| |f(z)| \frac{1}{|x_Q-z|} dz \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_{(Q^*)^c} l |\Omega(x_Q-z)| |x_Q-z|^{-n+\beta-1} |f(z)| dz
\end{aligned}$$

$$\begin{aligned}
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} l(2^k l)^{-n+\beta-1} \int_{2^k l \leq |z-x_Q| < 2^{k+1} l} |\Omega(x_Q - z)| |f(z)| dz \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} l(2^k l)^{-n+\beta-1+n} \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) M_{s'}(f)(x). \tag{3.17}
\end{aligned}$$

Let us now estimate  $W_3$ .

By (2.4), Hölder's inequality, and (3.9),

$$\begin{aligned}
W_3 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_{(Q^*)^c} \frac{|\Omega(x_Q - z)|}{|x_Q - z|^{n+m-1}} \\
&\quad \times \left( \sum_{i=1}^m |y - x_Q|^i |x_Q - z|^{m-i+\beta} + |y - x_Q|^{m+\beta} \right) \frac{|f(z)|}{|x_Q - z|} dz \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{-(n+m)} \left[ \sum_{i=1}^m l^i (2^{k+1} l)^{m-i+\beta} + l^{m+\beta} \right] \\
&\quad \times \int_{2^k l \leq |x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)| |f(z)| dz \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{-(n+m)} \left[ \sum_{i=1}^m l^i (2^{k+1} l)^{m-i+\beta} + l^{m+\beta} \right] \\
&\quad \times \left( \int_{|x_Q - z| < 2^{k+1} l} |\Omega(x_Q - z)|^s \right)^{1/s} \left( \int_{|x_Q - z| < 2^{k+1} l} |f(z)|^{s'} \right)^{1/s'} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^k l)^{-(n+m)} \left[ \sum_{i=1}^m l^i (2^{k+1} l)^{m-i+\beta} + l^{m+\beta} \right] \\
&\quad \times (2^{k+1} l)^n \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} \sum_{i=1}^m (2^{-k(i-\beta)} + 2^{-km}) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=N}^{\infty} (2^{-k(1-\beta)} + 2^{-km}) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x) \\
&\leq C |Q|^{\beta/n} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.18}
\end{aligned}$$

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Thus,

$$W \leq W_1 + W_2 + W_3 \leq C|Q|^{\beta/n} \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \quad (3.19)$$

Combining the estimates of  $U$ ,  $V$  with  $W$ , we have

$$D \leq C|Q|^{\beta/n} \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \quad (3.20)$$

So,

$$J_2 \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \quad (3.21)$$

Combining the estimates of  $J_1$  with  $J_2$ , we obtain

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q |\mu_\Omega^A(f)(y) - (\mu_\Omega^A(f))_Q| dy \\ & \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} [M_{p_1}(f)(x) + M_{s'}(f)(x)], \end{aligned} \quad (3.22)$$

where  $1 < p_1, s' < p$ . So, by Lemma 2.5 and the  $L^p(\mathbb{R}^n)$  boundedness of  $M_{p_1}$  and  $M_{s'}$ , we conclude that

$$\|\mu_\Omega^A(f)\|_{\dot{F}_p^{\beta,\infty}} \leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p}. \quad (3.23)$$

We complete the proof of Theorem 1.5.

### 4. Proofs of Theorems 1.6 and 1.7

First, let us introduce some notations related to Hardy spaces.

*Definition 4.1* [10]. Let  $0 < p \leq 1 \leq q \leq \infty$ , let  $p < q$ , and let  $s \geq s_0$ , where  $s_0 = [n(1/p - 1)]$ . A function  $a$  is said to be a  $(p, q, s)$  atom, if  $a \in L^q(\mathbb{R}^n)$  and satisfies the following conditions:

- (i)  $\text{supp } a \subset B$ ;
- (ii)  $\|a\|_{L^q} \leq |B|^{1/q-1/p}$ ;
- (iii)  $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ , for all  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , with  $0 \leq |\alpha| = \sum_{i=1}^n \alpha_i \leq s$ .

*Definition 4.2* [10]. Let  $0 < p \leq 1 \leq q$  and let  $p < q$ , then the atomic Hardy space  $H_a^{p,q,s}(\mathbb{R}^n)$  is defined by

$$H_a^{p,q,s}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \sum_j \lambda_j a_j, \text{ here } a_j \text{ is a } (p, q, s) \text{ atom and } \sum_j |\lambda_j|^p < \infty \right\}. \quad (4.1)$$

Then,

$$\|f\|_{H_a^{p,q,s}(\mathbb{R}^n)} = \inf \left\{ \left( \sum_j |\lambda_j|^p \right)^{1/p}, \text{ for all decompositions of } f = \sum_j \lambda_j a_j \right\}. \quad (4.2)$$

**LEMMA 4.3** [10]. *Let  $0 < p \leq 1 \leq q$  and let  $p < q$ , then*

$$H_a^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n), \quad \|f\|_{H_a^{p,q,s}(\mathbb{R}^n)} = \|f\|_{H^p(\mathbb{R}^n)}. \quad (4.3)$$

Let us now turn to prove Theorem 1.6.

First, we estimate  $\mu_\Omega^A(f)$ . Notice that, when  $n/(n+\beta) < p \leq 1$  and  $0 < \beta < 1$ ,  $s_0 = [n(1/p-1)] \leq [\beta] = 0$ .

By Lemma 4.3 and a standard argument, it is sufficient for us to show that there is a constant  $C > 0$  such that for any  $(p, \infty, 0)$  atom  $a$ ,  $\|\mu_\Omega^A(a)\|_{L^r} \leq C$ .

Take a  $(p, \infty, 0)$  atom  $a$  with  $\text{supp } a \subset B(x_0, l)$ . Then,

$$\begin{aligned} \|\mu_\Omega^A(a)\|_{L^r} &\leq \left[ \int_{2B} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} + \left[ \int_{(2B)^c} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} \\ &\leq \left[ \int_{2B} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} \\ &\quad + \left\{ \int_{(2B)^c} \left[ \int_0^{|x-x_0|+2l} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y)a(y)dy \right|^2 \frac{dt}{t^3} \right]^{r/2} dx \right\}^{1/r} \\ &\quad + \left\{ \int_{(2B)^c} \left[ \int_{|x-x_0|+2l}^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y)a(y)dy \right|^2 \frac{dt}{t^3} \right]^{r/2} dx \right\}^{1/r} \\ &:= I + II + III. \end{aligned} \quad (4.4)$$

Choose  $p_1$  and  $q_1$  satisfying  $1 < p_1 < n/\beta$  and  $1/q_1 = 1/p_1 - \beta/n$ . It is obvious that  $r < q_1$ . So, by Hölder's inequality and the  $(L^{p_1}, L^{q_1})$  boundedness of  $\mu_\Omega^A$  (see Theorem 1.3),

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we have

$$\begin{aligned}
 I &\leq C \|\mu_{\Omega}^A(a)\|_{L^{q_1}} |2B|^{1/r-1/q_1} \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \|a\|_{L^{p_1}} |2B|^{(1/r-1/q_1)} \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \|a\|_{L^{\infty}} |B|^{1/p_1} |2B|^{(1/r-1/q_1)} \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) |B|^{(1/p_1-1/p)} |2B|^{(1/r-1/q_1)} \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right).
 \end{aligned} \tag{4.5}$$

Since for any  $y \in B$ ,  $x \in (2B)^c$ , we have  $|x-y| \sim |x-x_0| \sim |x-x_0|+2l$ . By the Minkowski inequality, Hölder's inequality, (2.1), and (3.9),

$$\begin{aligned}
 II &\leq C \left\{ \int_{(2B)^c} \left[ \int_{\mathbb{R}^n} \left( \int_{|x-y|}^{|x-x_0|+2l} \frac{dt}{t^3} \right)^{1/2} \frac{|\Omega(x-y)| |a(y)|}{|x-y|^{n+m-1}} |R_{m+1}(A; x, y)| dy \right]^r dx \right\}^{1/r} \\
 &\leq C \left\{ \int_{(2B)^c} \left[ \int_B \frac{|l|^{1/2} |\Omega(x-y)| |a(y)|}{|x-y|^{n+m+1/2}} |R_{m+1}(A; x, y)| dy \right]^r dx \right\}^{1/r} \\
 &\leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|l|^{1/2} |\Omega(x-y)|}{|x-y|^{n+m+1/2}} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) |x-y|^{m+\beta} \right]^r dx \right\}^{1/r} |a(y)| dy \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \int_B \sum_{k=1}^{\infty} 2^{-k/2} (2^k l)^{-n+\beta} (2^{k+1} l)^{n(1/r-1/s)} \\
 &\quad \times \left[ \int_{|x-x_0| < 2^{k+1} l} |\Omega(x-y)|^s dx \right]^{1/s} |a(y)| dy \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \left( \sum_{k=1}^{\infty} 2^{-k(1/2-\beta)} 2^{-kn(1-1/r)} \right) \|\Omega\|_{L^s(S^{n-1})} l^{-n(1-1/p)} \|a\|_{L^{\infty}} |B| \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_{\beta}} \right) \|\Omega\|_{L^s(S^{n-1})}.
 \end{aligned} \tag{4.6}$$

Notice that for any  $y \in B$ , we have  $t \geq |x - x_0| + 2l \geq |x - x_0| + |y - x_0| \geq |x - y|$ . So, by the vanishing condition of  $a$ , we have

$$\begin{aligned}
& \left[ \int_{|x-x_0|+2l}^{\infty} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) a(y) dy \right|^2 \frac{dt}{t^3} \right]^{1/2} \\
&= \left\{ \int_{|x-x_0|+2l}^{\infty} \left| \int_B \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) a(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\
&= \left| \int_B \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) a(y) dy \right| \left( \int_{|x-x_0|+2l}^{\infty} \frac{dt}{t^3} \right)^{1/2} \\
&= \left| \int_B \left[ \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}} R_{m+1}(A; x, x_0) \right] \frac{a(y)}{|x-x_0|+2l} dy \right|. \tag{4.7}
\end{aligned}$$

On the other hand, it is similar to (3.12), and we have

$$\begin{aligned}
& \left| \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}} R_{m+1}(A; x, x_0) \right| \\
&\leq \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \frac{|R_{m+1}(A; x, y)|}{|x-y|^m} \\
&\quad + \left| \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \left| \frac{1}{|x-y|^m} - \frac{1}{|x-x_0|^m} \right| |R_{m+1}(A; x, y)| \\
&\quad + \left| \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}} \right| |R_{m+1}(A; x, y) - R_{m+1}(A; x, x_0)|. \tag{4.8}
\end{aligned}$$

So,

$$\begin{aligned}
III &\leq \left\{ \int_{(2B)^c} \left[ \int_B \left| \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A; x, y) - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}} R_{m+1}(A; x, x_0) \right| \right. \right. \\
&\quad \left. \left. \times \frac{|a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
&\leq \left\{ \int_{(2B)^c} \left[ \int_B \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \frac{|R_{m+1}(A; x, y)|}{|x-y|^m} \left( \frac{|a(y)|}{|x-x_0|+2l} \right) dy \right]^r dx \right\}^{1/r}
\end{aligned}$$

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$$\begin{aligned}
 & + \left\{ \int_{(2B)^c} \left[ \int_B \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x-x_0|^m} \right| \frac{|R_{m+1}(A;x,y)| |a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
 & + \left\{ \int_{(2B)^c} \left[ \int_B \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}} |R_{m+1}(A;x,y) - R_{m+1}(A;x,x_0)| \frac{|a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
 & := III_1 + III_2 + III_3. \tag{4.9}
 \end{aligned}$$

For  $III_1$ , by the Minkowski inequality, (2.1), Hölder's inequality, and (3.15), we have

$$\begin{aligned}
 III_1 & \leq C \int_B \left\{ \int_{(2B)^c} \left[ \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \right. \right. \\
 & \quad \left. \left. \times \frac{|x-y|^{m+\beta}}{|x-y|^m (|x-x_0|+2l)} \right]^r dx \right\}^{1/r} |a(y)| dy \\
 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{-1+\beta+n(1/r-1/s)} \\
 & \quad \times \left\{ \int_{2^k l \leq |x-x_0| < 2^{k+1} l} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right|^s dx \right\}^{1/s} |a(y)| dy \\
 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{\beta+n/r-n} (2^{-k} + 2^{-k\varepsilon}) |a(y)| dy \\
 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} 2^{-kn(1-1/r)} (2^{-k(1-\beta)} + 2^{-k(\varepsilon-\beta)}) l^{m(1/p-1)} \|a\|_{L^\infty} |B| \\
 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right). \tag{4.10}
 \end{aligned}$$

Since  $|x-y| \sim |x-x_0|$  for any  $x \in (2B)^c$ , by the Minkowski inequality, (2.1), Hölder's inequality, and (3.9),

$$\begin{aligned}
 III_2 & \leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}} \frac{l}{|x-x_0|^{m+1}} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \frac{|x-y|^{m+\beta}}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
 & \quad \times |a(y)| dy \\
 & \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} l (2^k l)^{-n-1+\beta} \times \left( \int_{2^k l \leq |x-x_0| < 2^{k+1} l} |\Omega(x-x_0)|^r dx \right)^{1/r} \\
 & \quad \times |a(y)| dy
 \end{aligned}$$



$$\begin{aligned}
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} l(2^k l)^{-n-1+\beta} (2^{k+1} l)^{n(1/r-1/s)} \\
&\quad \times \left( \int_{l \leq |x-x_0| < 2^{k+1} l} |\Omega(x-x_0)|^s dx \right)^{1/s} |a(y)| dy \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \left( \sum_{k=1}^{\infty} 2^{-k(1-\beta)} 2^{-kn(1-1/r)} \right) l^{-n(1-1/p)} \|a\|_{L^\infty} |B| \|\Omega\|_{L^s(S^{n-1})} \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \tag{4.11}
\end{aligned}$$

Moreover, by the Minkowski inequality, (2.3), and (3.9),

$$\begin{aligned}
III_3 &\leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}} |R_{m+1}(A; x, y) - R_{m+1}(A; x, x_0)| \frac{1}{|x-x_0|+2l} \right]^r dx \right\}^{1/r} \\
&\quad \times |a(y)| dy \\
&\leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m}} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \times \left( \sum_{i=0}^m |x-x_0|^i |x_0-y|^{m-i+\beta} \right) \right]^r dx \right\}^{1/r} \\
&\quad \times |a(y)| dy \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{-(n+m)} \left( \sum_{i=0}^m (2^{k+1} l)^i l^{m-i+\beta} \right) \\
&\quad \times \left( \int_{2^k l \leq |x-x_0| < 2^{k+1} l} |\Omega(x-x_0)|^r dx \right)^{1/r} |a(y)| dy \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} (2^k l)^{-(n+m)} \left( \sum_{i=0}^m (2^{k+1} l)^i l^{m-i+\beta} \right) \\
&\quad \times (2^{k+1} l)^{n/r} \|\Omega\|_{L^s(S^{n-1})} \|a\|_{L^\infty} |B| \\
&\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} \left( \sum_{i=0}^m 2^{-k[(m-i)+n(1-1/r)]} \right) l^{-n(1-1/p)} \|\Omega\|_{L^s(S^{n-1})} \|a\|_{L^\infty} |B|
\end{aligned}$$

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$$\begin{aligned}
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} 2^{-kn(1-1/r)} l^{-n(1-1/p)} \|a\|_{L^\infty} \|B\| \|\Omega\|_{L^s(S^{n-1})} \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}.
 \end{aligned} \tag{4.12}$$

So,

$$III \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \tag{4.13}$$

Combining the estimates of  $I$ ,  $II$  with  $III$ , we get

$$\|\mu_\Omega^A(a)\|_{L^r} \leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \tag{4.14}$$

Replacing  $\mu_\Omega^A(f)$  by  $\tilde{\mu}_\Omega^A(f)$  and using (2.2) and (2.5) instead of (2.1) and (2.3) in the above estimates, we can show that  $\tilde{\mu}_\Omega^A$  is also bounded from  $L^p(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$  for  $n/(n+\beta) < p < 1$  and  $1/r = 1/p - \beta/n$ .

In fact, we need only to check  $III_3$ , where  $R_{m+1}$  is replaced by  $Q_{m+1}$ :

$$\begin{aligned}
 III_3 &= C \left\{ \int_{(2B)^c} \left[ \int_B \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}} |Q_{m+1}(A;x,y) - Q_{m+1}(A;x,x_0)| \frac{|a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
 &\leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}} |Q_{m+1}(A;x,y) - Q_{m+1}(A;x,x_0)| \frac{1}{|x-x_0|+2l} \right]^r dx \right\}^{1/r} \\
 &\quad \times |a(y)| dy \\
 &\leq C \int_B \left\{ \int_{(2B)^c} \left[ \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m}} \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \right. \right. \\
 &\quad \left. \left. \times \left( \sum_{i=0}^{m-1} |x-x_0|^i |x_0-y|^{m-i} (|x-y|^\beta + |y-x_0|^\beta) \right) \right]^r dx \right\}^{1/r} |a(y)| dy \\
 &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{-(n+m)} \sum_{i=0}^{m-1} (2^{k+1} l)^i l^{m-i} \left( (2^{k+1} l)^\beta + l^\beta \right) \\
 &\quad \times \left( \int_{l \leq |x-x_0| < 2^{k+1} l} |\Omega(x-x_0)|^r dx \right)^{1/r} |a(y)| dy
 \end{aligned}$$

$$\begin{aligned}
&\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} (2^k l)^{-(n+m)+n/r} \sum_{i=0}^{m-1} (2^{k+1} l)^i l^{m-i} \\
&\quad \times \left( (2^{k+1} l)^\beta + l^\beta \right) \|\Omega\|_{L^s(S^{n-1})} \|a\|_{L^\infty} |B| \\
&\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} \sum_{i=0}^{m-1} [2^{-k(n+m-i-\beta-n/r)} + 2^{-k(n+m-i-n/r)}] \|\Omega\|_{L^s(S^{n-1})} \|a\|_{L^\infty} |B| \\
&\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \sum_{k=1}^{\infty} \left\{ 2^{-k[n(1-1/r)+(1-\beta)]} + 2^{-k[n(1-1/r)+1]} \right\} \|\Omega\|_{L^s(S^{n-1})} \\
&\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \tag{4.15}
\end{aligned}$$

Thus we complete the proof of Theorem 1.6.

Let us now prove Theorem 1.7. The main idea is the same as that of proving Theorem 1.6.

Let  $a$  be a  $(1, \infty, 0)$  atom with  $\text{supp } a \subset B(x_0, l)$  and  $r = n/(n - \beta)$ , then

$$\|\mu_\Omega^A(a)\|_{L^r} \leq I + II + III, \tag{4.16}$$

where  $I$ ,  $II$ , and  $III$  are the same as in the proof of Theorem 1.6.

In the same way as in the estimates of (4.5) and (4.6), when  $r = n/(n - \beta)$ , we have

$$\begin{aligned}
I &\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right), \\
II &\leq C \left( \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \tag{4.17}
\end{aligned}$$

As in the estimate of (4.9), we have

$$\begin{aligned}
III &\leq \left\{ \int_{(2B)^c} \left[ \int_B \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \frac{|R_{m+1}(A; x, y)| |a(y)|}{|x-y|^m (|x-x_0|+2l)} dy \right]^r dx \right\}^{1/r} \\
&\quad + \left\{ \int_{(2B)^c} \left[ \int_B \left| \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \frac{1}{|x-y|^m} - \frac{1}{|x-x_0|^m} \right| \frac{|R_{m+1}(A; x, y)| |a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
&\quad + \left\{ \int_{(2B)^c} \left[ \int_B \left| \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}} |R_{m+1}(A; x, y) - R_{m+1}(A; x, x_0)| \frac{|a(y)|}{|x-x_0|+2l} dy \right]^r dx \right\}^{1/r} \\
&:= E + F + G. \tag{4.18}
\end{aligned}$$

In the same way as in the estimates of (4.11) and (4.12), we have

$$\begin{aligned} F &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}, \\ G &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \quad (4.19)$$

So, it is sufficient for us to estimate  $E$ .

In fact, it is similar to the estimate of  $III_1$ , by Lemma 2.6,  $r = n/(n - \beta)$ , and (1.11), we have

$$\begin{aligned} E &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{-1+\beta+n(1/r-1/s)} \\ &\quad \times \left\{ \int_{2^k l \leq |x-x_0| < 2^{k+1} l} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right|^s dx \right\}^{1/s} |a(y)| dy \\ &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \sum_{k=1}^{\infty} (2^k l)^{-1+\beta+n(1/r-1/s)} (2^k l)^{n/s-n+1} \\ &\quad \times \left\{ \frac{|y-x_0|}{2^k l} + \int_{|y-x_0|/2^{k+1} l}^{|y-x_0|/2^k l} \frac{\omega_s(\delta)}{\delta} d\delta \right\} |a(y)| dy \\ &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \left\{ \sum_{k=1}^{\infty} 2^{-k} + \sum_{k=1}^{\infty} \int_{|y-x_0|/2^{k+1} l}^{|y-x_0|/2^k l} \frac{\omega_s(\delta)}{\delta} d\delta \right\} |a(y)| dy \\ &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \int_B \left\{ 1 + \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta \right\} |a(y)| dy \\ &\leq C \left( \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta} \right) \|a\|_{L^\infty} |B| \\ &\leq C \sum_{|y|=m} \|D^y A\|_{\dot{\Lambda}_\beta}. \end{aligned} \quad (4.20)$$

Thus, we get the estimate of  $\mu_\Omega^A(f)$  for  $f \in H^1(\mathbb{R}^n)$ . It is analogous to the argument for  $\tilde{\mu}_\Omega^A$  in the proof of Theorem 1.6, and we can get the desired result for  $\tilde{\mu}_\Omega^A$  by repeating the above estimates and using (4.15), when  $f \in H^1(\mathbb{R}^n)$ . So, we complete the proofs of Theorem 1.7.

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