BOUNDEDNESS OF HIGHER-ORDER MARCINKIEWICZ-TYPE INTEGRALS

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Let A be a function with derivatives of order m and $D^\gamma A \in \dot{\Lambda}_\beta$ $(0 < \beta < 1,\ |\gamma| = m)$. The authors in the paper proved that if $\Omega \in L^s(S^{n-1})$ $(s \geq n/(n-\beta))$ is homogeneous of degree zero and satisfies a vanishing condition, then both the higher-order Marcinkiewicz-type integral μ_Ω^A and its variation $\widetilde{\mu}_\Omega^A$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{n/(n-\beta),\infty}(\mathbb{R}^n)$, where $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Furthermore, if Ω satisfies some kind of L^s-Dini condition, then both μ_Ω^A and $\widetilde{\mu}_\Omega^A$ are bounded on Hardy spaces, and μ_Ω^A is also bounded from $L^p(\mathbb{R}^n)$ to certain Triebel-Lizorkin space.

1. Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \tag{1.2}$$

where $F_{\Omega,t}(f)(x) = (1/t)\int_{|x-y|\leq t}(\Omega(x-y)/|x-y|^{n-1})f(y)dy$.

And if we denote H as the Hilbert space $H = \{h : \|h\|_H = (\int_0^\infty |h(t)|^2(dt/t))^{1/2} < \infty\}$, then $\mu_\Omega(f)$ can be looked as the vector-valued function in H, that is

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left|F_{\Omega,t}(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2} = \|F_{\Omega,t}(f)(x)\|_H. \tag{1.3}$$

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It is well known that the operator μ_Ω was defined first by Stein in [13], where the author proved that if Ω is continuous and satisfies a Lip_α $(0<\alpha\le 1)$ condition on S^{n-1}, then μ_Ω is an operator of type (p,p) for $1<p\le 2$ and of weak type $(1,1)$. Benedek et al. in [1] showed that if $\Omega\in C^1(S^{n-1})$, then μ_Ω is an operator of type (p,p) for $1<p<\infty$. Recently, Ding et al. in [4] improved the results mentioned above. They gave the $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ boundedness of μ_Ω for $\Omega\in H^1(S^{n-1})$, where H^1 denotes the Hardy space on S^{n-1} (see [3] for the definition of H^1).

On the other hand, let $b\in L_{\text{loc}}(\mathbb{R}^n)$, then the commutator of Marcinkiewicz integral is defined by

$$\mu_\Omega^b(f)(x)=\left(\int_0^\infty \left|F_{\Omega,b,t}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2}, \tag{1.4}$$

where

$$F_{\Omega,b,t}(f)(x)=\frac{1}{t}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy. \tag{1.5}$$

In 1990, Torchinsky and Wang [14] proved that if Ω is continuous and satisfies a Lip_α $(0<\alpha\le 1)$ condition, then for $b\in\text{BMO}$, μ_Ω^b is bounded on $L^p(\omega)$, here $\omega\in A_p$ $(1<p<\infty)$.

For $\beta>0$, the homogeneous Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is the space of function f such that

$$\|f\|_{\dot{\Lambda}_\beta}=\sup_{x,h\in\mathbb{R}^n,h\neq 0}\frac{\left|\Delta_h^{[\beta]+1}f(x)\right|}{|h|^\beta}<\infty, \tag{1.6}$$

where Δ_h^k denotes the kth difference operator (see [12]).

When Ω satisfies a Lip_α $(0<\alpha\le 1)$ condition and $b\in\dot{\Lambda}_\beta$ $(0<\beta<\min\{1/2,\alpha\})$, Liu [9] considered the $(L^p,\dot{F}_p^{\beta,\infty})$ boundedness of μ_Ω^b, and Wang [15] showed that μ_Ω^b is also bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for $1<p<n/\beta$ and $1/q=1/p-\beta/n$. Later, in [11] we weakened the smoothness condition assumed on Ω and got the same conclusions.

Moreover, let $m\in\mathbb{N}$ and let A be a function on $\mathbb{R}^n$. We denote

$$\begin{aligned}R_{m+1}(A;x,y)&=A(x)-\sum_{|\gamma|\le m}\frac{1}{\gamma!}D^\gamma A(y)(x-y)^\gamma,\\ Q_{m+1}(A;x,y)&=R_m(A;x,y)-\sum_{|\gamma|=m}\frac{1}{\gamma!}D^\gamma A(x)(x-y)^\gamma.\end{aligned} \tag{1.7}$$

Then the higher-order Marcinkiewicz-type integral and its variation are defined, respectively, by

$$\begin{aligned}\mu_\Omega^A(f)(x)&=\left(\int_0^\infty \left|F_{\Omega,t}^A(f)(x)\right|^2\frac{dt}{t}\right)^{1/2},\\ \widetilde{\mu}_\Omega^A(f)(x)&=\left(\int_0^\infty \left|\widetilde{F}_{\Omega,t}^A(f)(x)\right|^2\frac{dt}{t}\right)^{1/2},\end{aligned} \tag{1.8}$$

where

$$F^A_{\Omega,t}(f)(x) = \frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y)f(y)dy,$$
$$\widetilde{F}^A_{\Omega,t}(f)(x) = \frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} Q_{m+1}(A;x,y)f(y)dy. \tag{1.9}$$

When $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ and $D^\gamma A \in \dot{\Lambda}_\beta$ $(0 < \beta < \min\{1/2,\alpha\})$, Liu [8] considered the boundedness of μ^A_Ω and got the following results.

Theorem 1.1 [8]. *Let $1 < p < \infty$, let $0 < \alpha \le 1$, let Ω be homogeneous of degree zero on $\mathbb{R}^n$ and satisfy (1.1). If $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ and $D^\gamma A \in \dot{\Lambda}_\beta$ $(0 < \beta < \min\{1/2,\alpha\})$, then*

(a) *μ^A_Ω is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}^{\beta,\infty}_p(\mathbb{R}^n)$,*
(b) *μ^A_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for $1/p - 1/q = \beta/n$ and $1/p > \beta/n$.*

It is well known that any weakness or removal of smoothness assumed on kernels is very interesting to the boundedness of singular integrals. Inspired by [9, 15, 11], we want to know whether the conditions assumed on Ω in Theorem 1.1 can be weakened or removed. In fact, the answer is affirmative. And we will also study the boundedness of μ^A_Ω and $\widetilde{\mu}^A_\Omega$ on Hardy spaces. Let us now give a definition and formulate our results.

Definition 1.2. For $\Omega \in L^s(S^{n-1})$ $(s \ge 1)$, the integral modulus $\omega_s(\delta)$ of continuity of order s of Ω is defined by

$$\omega_s(\delta) = \sup_{|\rho|\le\delta}\left(\int_{S^{n-1}} \left|\Omega(\rho x') - \Omega(x')\right|^s d\delta(x')\right)^{1/s}, \tag{1.10}$$

where ρ is a rotation on S^{n-1}, $|\rho| = \|\rho - I\|$. When $\omega_s(\delta)$ satisfies

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty, \tag{1.11}$$

it is said that $\Omega(x')$ satisfies the L^s-Dini condition.

Theorem 1.3. *Let $0 < \beta < 1$, let $1 < p < n/\beta$, let $1/q = 1/p - \beta/n$, let $s \ge n/(n-\beta)$, and let $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m)$. If $\Omega \in L^s(S^{n-1})$ satisfies (1.1), then both μ^A_Ω and $\widetilde{\mu}^A_\Omega$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 1.4. *Let $0 < \beta < 1$, let $s \ge n/(n-\beta)$, and let $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m)$. If $\Omega \in L^s(S^{n-1})$ satisfies (1.1), then both μ^A_Ω and $\widetilde{\mu}^A_\Omega$ are bounded from $L^1(\mathbb{R}^n)$ to $L^{n/(n-\beta),\infty}(\mathbb{R}^n)$.*

Theorem 1.5. *Let $1 \le s' < p < \infty$, let Ω satisfy (1.1) and the condition*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\varepsilon}} d\delta < \infty \quad \text{for some } 0 < \varepsilon \le 1. \tag{1.12}$$

Then for $D^\gamma \in \dot{\Lambda}_\beta$ $(|\gamma| = m,\ 0 < \beta < \min\{1/2, \varepsilon\})$,

$$||\mu_\Omega^A(f)||_{\dot{F}_p^{\beta,\infty}} \le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L^p}, \tag{1.13}$$

where $1/s' + 1/s = 1$.

Theorem 1.6. *Let* $0 < \varepsilon \le 1$, *let* $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m,\ 0 < \beta \le \min\{1/2, \varepsilon\})$, *let* $n/(n+\beta) < p < 1$, *and let* $1/r = 1/p - \beta/n$. *If there exists some* $s \ge \max\{r, n/(n-\beta)\}$, *such that* $\Omega \in L^s(S^{n-1})$ *satisfying (1.1) and (1.12), then both* μ_Ω^A *and* $\tilde{\mu}_\Omega^A$ *are bounded from* $H^p(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$.

When $p = 1$, (1.12) can be replaced by (1.11) and we can take $0 < \beta < 1$.

Theorem 1.7. *Let* $D^\gamma A \in \dot{\Lambda}_\beta$ $(|\gamma| = m,\ 0 < \beta < 1)$. *If there exists some* $s \ge n/(n-\beta)$ *such that* $\Omega \in L^s(S^{n-1})$ *satisfying (1.1) and (1.11), then both* μ_Ω^A *and* $\tilde{\mu}_\Omega^A$ *are bounded from* $H^1(\mathbb{R}^n)$ *to* $L^{n/(n-\beta)}(\mathbb{R}^n)$.

Remark 1.8. When $m = 0$, $\mu_\Omega^A(f)$ is the commutator of Marcinkiewicz integral. So, our results in this paper are extensions of those in [9, 15, 11].

Remark 1.9. It is easy to see that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$, then $\Omega \in L^s(S^{n-1})$ for any $s \ge 1$ and satisfies the L^s-Dini condition (1.12). In addition, (1.11) is weaker than (1.12) (see [6]). So, Theorems 1.3, 1.4, and 1.5 in the paper are substantial improvements of Theorem A. It should be pointed out that any smooth condition assumed on Ω is not needed in Theorems 1.3 and 1.4.

2. Some basic notations and lemmas

Lemma 2.1 [7]. *Let A be a function with derivatives of order m in* $\dot{\Lambda}_\beta$ $(0 < \beta < 1)$, *then there exists a constant* $C > 0$ *such that*

$$|R_{m+1}(A; x, y)| \le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) |x-y|^{m+\beta}; \tag{2.1}$$

$$|Q_{m+1}(A; x, y)| \le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) |x-y|^{m+\beta}; \tag{2.2}$$

$$|R_{m+1}(A; x, y) - R_{m+1}(A; x, z)| \le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) \sum_{i=0}^{m} |x-z|^i |z-y|^{m-i+\beta}; \tag{2.3}$$

$$|R_{m+1}(A; x, y) - R_{m+1}(A; z, y)| \le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) \left(\sum_{i=1}^{m} |x-z|^i |z-y|^{m-i+\beta} + |x-z|^{m+\beta}\right); \tag{2.4}$$

$$\begin{aligned} &|Q_{m+1}(A; x, y) - Q_{m+1}(A; x, z)| \\ &\quad\le C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right) \sum_{i=0}^{m-1} |x-z|^i |z-y|^{m-i} \left(|x-y|^\beta + |y-z|^\beta\right). \end{aligned} \tag{2.5}$$

Lemma 2.2 [5]. *Let $0<\alpha<n$, let $1<p<n/\alpha$, let $1/q=1/p-\alpha/n$, and let $s\geq n/(n-\alpha)$. If $\Omega\in L^s(S^{n-1})$, then the fractional integral operator $T_{\Omega,\alpha}$ defined by*

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy, \tag{2.6}$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Lemma 2.3 [2]. *Let $0<\alpha<n$ and let $s\geq n/(n-\alpha)$. If $\Omega\in L^s(S^{n-1})$, then for any $\lambda>0$ and $f\in L^1(\mathbb{R}^n)$, there exists a constant $C>0$, such that*

$$\left|\{x\in\mathbb{R}^n:\left|T_{\Omega,\alpha}f(x)\right|>\lambda\}\right|\leq C\left(\frac{\|f\|_{L^1}}{\lambda}\right)^{n/(n-\alpha)}. \tag{2.7}$$

Remark 2.4. Set

$$\bar{T}_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy. \tag{2.8}$$

It is easy to see that $\bar{T}_{\Omega,\alpha}$ satisfies Lemmas 2.2 and 2.3.

Lemma 2.5 [12]. *For $0<\beta<1$, $1<p<\infty$,*

$$\|f\|_{\dot{F}_p^{\beta,\infty}}\approx\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|f-f_Q|\right\|_{L^p}, \tag{2.9}$$

where $f_Q=(1/|Q|)\int_Q f(x)dx$.

Lemma 2.6 [6]. *Suppose that $0<\lambda<n$ and Ω is homogeneous of degree zero and satisfies the L^s-Dini condition (1.11) for $s>1$. If there exists a constant $a_0>0$ such that $|x|<a_0R$, then*

$$\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\lambda}}-\frac{\Omega(y)}{|y|^{n-\lambda}}\right|^s dy\right)^{1/s}\leq CR^{n/s-(n-\lambda)}\left\{\frac{|x|}{R}+\int_{|x|/2R}^{|x|/R}\frac{\omega_s(\delta)}{\delta}d\delta\right\}, \tag{2.10}$$

where the constant $C>0$ is independent of R and x.

3. Proofs of Theorems 1.3, 1.4, and 1.5

We first prove Theorems 1.3 and 1.4.

By Lemmas 2.2, 2.3 and Remark 2.4, we need only to show that there exists a constant $C>0$ such that

$$\begin{aligned}\mu_\Omega^A(f)(x)&\leq C\bar{T}_{\Omega,\beta}(f)(x),\\ \widetilde{\mu}_\Omega^A(f)(x)&\leq C\bar{T}_{\Omega,\beta}(f)(x),\end{aligned} \tag{3.1}$$

for any $x\in\mathbb{R}^n$.

In fact, for any fixed $x \in \mathbb{R}^n$, by the Minkowski inequality and (2.1), we have

$$
\begin{aligned}
\mu_\Omega^A(f)(x) &= \left[\int_0^\infty \left|\frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y) f(y)\,dy\right|^2 \frac{dt}{t}\right]^{1/2} \\
&\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_{m+1}(A;x,y)|\,|f(y)| \left[\int_{|x-y|\le t} \frac{1}{t^3}\,dt\right]^{1/2} dy \\
&\le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n+m}} |R_{m+1}(A;x,y)|\,|f(y)|\,dy \\
&\le C\left(\sum_{\gamma=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)|\,dy \\
&\le C\left(\sum_{\gamma=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \bar{T}_{\Omega,\beta}(f)(x).
\end{aligned} \tag{3.2}
$$

Similarly, by the Minkowski inequality and (2.2),

$$
\widetilde{\mu}_\Omega^A(f)(x) \le C\left(\sum_{\gamma=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) \bar{T}_{\Omega,\beta}(f)(x). \tag{3.3}
$$

So, we complete the proofs of Theorems 1.3 and 1.4. Let us now turn to prove Theorem 1.5.

Fix a cube $Q(x_Q,l) \ni x$ with its center at x_Q and denote the half side length of Q by l. Let $Q^* = 4\sqrt{n}Q$, then for $f \in L^p(\mathbb{R}^n)$, we write $f = f_1 + f_2$, where $f_1 = f\chi_{Q^*}$ and $f_2 = f\chi_{(Q^*)^c}$. It is obvious that there is an $N \in \mathbb{N}$, such that $2^N \le 4\sqrt{n} < 2^{N+1}$.

Since by the definition of $\mu_\Omega^A(f)$, we have

$$
\begin{aligned}
|\mu_\Omega^A(f)(y) - \mu_\Omega^A(f_2)(x_Q)| &= \big|\|F_{\Omega,t}^A(f)(y)\| - \|F_{\Omega,t}^A(f_2)(x_Q)\|\big| \\
&\le \|F_{\Omega,t}^A(f_1)(y)\| + \|F_{\Omega,t}^A(f_2)(y) - F_{\Omega,t}^A(f_2)(x_Q)\|.
\end{aligned} \tag{3.4}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q |\mu_\Omega^A(f)(y) - (\mu_\Omega^A(f))_Q|\,dy \\
&\quad\le \frac{2}{|Q|^{1+\beta/n}} \int_Q |\mu_\Omega^A(f)(y) - \mu_\Omega^A(f_2)(x_Q)|\,dy \\
&\quad\le \frac{2}{|Q|^{1+\beta/n}} \int_Q |\mu_\Omega^A(f_1)(y)|\,dy + \frac{2}{|Q|^{\beta/n}} \sup_{y\in Q} \|F_{\Omega,t}^A(f_2)(y) - F_{\Omega,t}^A(f_2)(x_Q)\| \\
&\quad:= J_1 + J_2.
\end{aligned} \tag{3.5}
$$

Choose $1 < p_1 < n/\beta$ and $1/q_1 = 1/p_1 - \beta/n$ such that $1 < p_1 < p$. Then by Hölder's inequality and the (L^{p_1}, L^{q_1}) boundedness of μ_Ω^A (see Theorem 1.3), we have

$$\begin{aligned} J_1 &\le \frac{2}{|Q|^{1+\beta/n}} |Q|^{1-1/q_1} \|\mu_\Omega^A(f_1)\|_{L^{q_1}} \\ &\le C|Q|^{-1/p_1} \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \|f_1\|_{L^{p_1}} \\ &\le C\left(\sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) \left(\frac{1}{|Q|} \int_{Q^*} |f(y)|^{p_1} dy \right)^{1/p_1} \\ &\le C\left(\sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) M_{p_1}(f)(x). \end{aligned} \tag{3.6}$$

Let us now estimate J_2.

Denote $D = \|F_{\Omega,t}^A(f_2)(y) - F_{\Omega,t}^A(f_2)(x_Q)\|$, then

$$\begin{aligned} D &= \Bigg[\int_0^\infty \Bigg| \Bigg[\int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A;y,z) f_2(z)dz \\ &\qquad - \int_{|x_Q-z|\le t} \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A;x_Q,z) f_2(z)dz \Bigg] \Bigg|^2 \frac{dt}{t^3} \Bigg]^{1/2} \\ &\le \left[\int_0^\infty \left| \int_{\{|y-z|\le t, |x_Q-z|>t\}} \frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A;y,z) f_2(z)dz \right|^2 \frac{dt}{t^3} \right]^{1/2} \\ &\quad + \left[\int_0^\infty \left| \int_{\{|y-z|>t, |x_Q-z|\le t\}} \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A;x_Q,z) f_2(z)dz \right|^2 \frac{dt}{t^3} \right]^{1/2} \\ &\quad + \Bigg\{ \int_0^\infty \Bigg| \int_{\{|y-z|\le t, |x_Q-z|\le t\}} \Bigg[\frac{\Omega(y-z)}{|y-z|^{n+m-1}} R_{m+1}(A;y,z) \\ &\qquad - \frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}} R_{m+1}(A;x_Q,z) \Bigg] f_2(z)dz \Bigg|^2 \frac{dt}{t^3} \Bigg\}^{1/2} \\ &:= U + V + W. \end{aligned} \tag{3.7}$$

Notice that $|z - x_Q| \sim |y - z|$ for $z \in (Q^*)^c$. By the Minkowski inequality, (2.1), and Hölder's inequality, we have

$$\begin{aligned} V &\le C \int_{\mathbb{R}^n} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} |R_{m+1}(A;x_Q,z)| |f_2(z)| \left(\int_{|x_Q-z|}^{|y-z|} \frac{1}{t^3} dt \right)^{1/2} dz \\ &\le C \int_{(Q^*)^c} \frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}} \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \right) |x_Q-z|^{m+\beta} |f(z)| \frac{l^{1/2}}{|x_Q-z|^{3/2}} dz \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}2^{-k/2}(2^k l)^{-n+\beta}\int_{2^k l\le|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)||f(z)|\,dz\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}2^{-k/2}(2^k l)^{-n+\beta}\left(\int_{|x_Q-z|<2^{k+1}l}|f(z)|^{s'}\right)^{1/s'}\\
&\quad\times\left(\int_{|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)|^{s}\right)^{1/s}\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}2^{-k/2}(2^k l)^{-n+\beta+n/s'}M_{s'}(f)(x)\\
&\quad\times\left(\int_{|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)|^{s}\right)^{1/s}.
\end{aligned}
\tag{3.8}
$$

Since $\Omega\in L^s(S^{n-1})$, it is easy to see that

$$
\left[\int_{|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)|^{s}\right]^{1/s}\le C(2^{k+1}l)^{n/s}\|\Omega\|_{L^s(S^{n-1})}.
\tag{3.9}
$$

Therefore, by $0<\beta<1/2$, we have

$$
\begin{aligned}
V&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}2^{-k/2}(2^k l)^{-n+\beta+n/s'+n/s}\|\Omega\|_{L^s(S^{n-1})}M_{s'}(f)(x)\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}2^{-k(1/2-\beta)}|Q|^{\beta/n}\|\Omega\|_{L^s(S^{n-1})}M_{s'}(f)(x)\\
&\le C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}M_{s'}(f)(x).
\end{aligned}
\tag{3.10}
$$

In the same way, we have

$$
U\le C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}M_{s'}(f)(x).
\tag{3.11}
$$

Let us now estimate W.

Since,

$$
\begin{aligned}
&\left|\frac{\Omega(y-z)}{|y-z|^{n+m-1}}R_{m+1}(A;y,z)-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}}R_{m+1}(A;x_Q,z)\right| \\
&\quad\le\left|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\right|\frac{1}{|y-z|^m}|R_{m+1}(A;y,z)| \\
&\qquad+\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}}\left|\frac{1}{|y-z|^m}-\frac{1}{|x_Q-z|^m}\right||R_{m+1}(A;y,z)| \\
&\qquad+\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}}|R_{m+1}(A;y,z)-R_{m+1}(A;x_Q,z)|.
\end{aligned}
\tag{3.12}
$$

By the Minkowski inequality and $|y-z|\sim|x_Q-z|$ for any $z\in(Q^*)^c$, we have

$$
\begin{aligned}
W &\le\int_{\mathbb{R}^n}\left(\int_{\{|y-z|\le t,|x_Q-z|\le t\}}\frac{dt}{t^3}\right)^{1/2} \\
&\qquad\times\left|\frac{\Omega(y-z)}{|y-z|^{n+m-1}}R_{m+1}(A;y,z)-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n+m-1}}R_{m+1}(A;x_Q,z)\right||f_2(z)|\,dz \\
&\le\int_{(Q^*)^c}\left|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\right|\frac{1}{|y-z|^m}|R_{m+1}(A;y,z)||f(z)|\frac{1}{|x_Q-z|}dz \\
&\quad+\int_{(Q^*)^c}\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}}\left|\frac{1}{|y-z|^m}-\frac{1}{|x_Q-z|^m}\right||R_{m+1}(A;y,z)||f(z)|\frac{1}{|x_Q-z|}dz \\
&\quad+\int_{(Q^*)^c}\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n+m-1}}|R_{m+1}(A;y,z)-R_{m+1}(A;x_Q,z)||f(z)|\frac{1}{|x_Q-z|}dz \\
&:=W_1+W_2+W_3.
\end{aligned}
\tag{3.13}
$$

For W_1, using (2.1) and Hölder's inequality,

$$
\begin{aligned}
W_1&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\int_{(Q^*)^c}\left|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\right||y-z|^{\beta-1}|f(z)|\,dz \\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\right)\sum_{k=N}^{\infty}(2^kl)^{\beta-1} \\
&\qquad\times\int_{2^kl\le|z-x_Q|<2^{k+1}l}\left|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\right||f(z)|\,dz
\end{aligned}
$$

$$\le C\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)\sum_{k=N}^{\infty}(2^k l)^{\beta-1}\bigg[\int_{|z-x_Q|\le 2^{k+1}l}|f(z)|^{s'}dz\bigg]^{1/s'}$$
$$\times\bigg[\int_{2^k l\le|z-x_Q|<2^{k+1}l}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\bigg|^s dz\bigg]^{1/s}$$
$$\le C\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)\sum_{k=N}^{\infty}(2^k l)^{\beta-1+n/s'}M_{s'}(f)(x)$$
$$\times\bigg[\int_{2^k l\le|z-x_Q|<2^{k+1}l}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\bigg|^s dz\bigg]^{1/s}. \tag{3.14}$$

However, by Lemma 2.6 and (1.12), we obtain

$$\bigg[\int_{2^k l\le|z-x_Q|<2^{k+1}l}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\bigg|^s dz\bigg]^{1/s}$$
$$\le C(2^k l)^{n/s-(n-1)}\bigg\{\frac{|y-x_Q|}{2^k l}+\int_{|y-x_Q|/2^{k+1}l}^{|y-x_Q|/2^k l}\frac{\omega_s(\delta)}{\delta}d\delta\bigg\}$$
$$\le C(2^k l)^{n/s-(n-1)}\bigg\{2^{-k}+2^{-k\varepsilon}\int_0^1\frac{\omega_s(\delta)}{\delta^{1+\varepsilon}}d\delta\bigg\}$$
$$\le C(2^k l)^{n/s-(n-1)}(2^{-k}+2^{-k\varepsilon}). \tag{3.15}$$

Hence,

$$W_1\le C\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)\sum_{k=N}^{\infty}(2^k l)^{\beta-1+n/s'}(2^k l)^{n/s-(n-1)}(2^{-k}+2^{-k\varepsilon})M_{s'}(f)(x)$$
$$\le C\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)\sum_{k=N}^{\infty}(2^{-k(1-\beta)}+2^{-k(\varepsilon-\beta)})|Q|^{\beta/n}M_{s'}(f)(x)$$
$$\le C|Q|^{\beta/n}\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)M_{s'}(f)(x). \tag{3.16}$$

Since $|y-z|\sim|x_Q-z|$, for any $z\in(Q^*)^c$, it is similar to the estimate of V, and we have

$$W_2\le C\int_{(Q^*)^c}\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}}\frac{l}{|y-z|^{m+1}}|R_{m+1}(A;y,z)||f(z)|\frac{1}{|x_Q-z|}dz$$
$$\le C\bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\bigg)\int_{(Q^*)^c}l|\Omega(x_Q-z)||x_Q-z|^{-n+\beta-1}|f(z)|dz$$

$$\leq C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}l(2^{k}l)^{-n+\beta-1}\int_{2^{k}l\leq|z-x_{Q}|<2^{k+1}l}|\Omega(x_{Q}-z)||f(z)|dz$$

$$\leq C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}l(2^{k}l)^{-n+\beta-1+n}\|\Omega\|_{L^{s}(S^{n-1})}M_{s'}(f)(x)$$

$$\leq C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)M_{s'}(f)(x). \tag{3.17}$$

Let us now estimate W_3.

By (2.4), Hölder's inequality, and (3.9),

$$\begin{aligned}
W_{3}\leq{}& C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\int_{(Q^{*})^{c}}\frac{|\Omega(x_{Q}-z)|}{|x_{Q}-z|^{n+m-1}}\\
&\times\left(\sum_{i=1}^{m}|y-x_{Q}|^{i}|x_{Q}-z|^{m-i+\beta}+|y-x_{Q}|^{m+\beta}\right)\frac{|f(z)|}{|x_{Q}-z|}dz\\
\leq{}& C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}(2^{k}l)^{-(n+m)}\left[\sum_{i=1}^{m}l^{i}(2^{k+1}l)^{m-i+\beta}+l^{m+\beta}\right]\\
&\times\int_{2^{k}l\leq|x_{Q}-z|<2^{k+1}l}|\Omega(x_{Q}-z)||f(z)|dz\\
\leq{}& C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}(2^{k}l)^{-(n+m)}\left[\sum_{i=1}^{m}l^{i}(2^{k+1}l)^{m-i+\beta}+l^{m+\beta}\right]\\
&\times\left(\int_{|x_{Q}-z|<2^{k+1}l}|\Omega(x_{Q}-z)|^{s}\right)^{1/s}\left(\int_{|x_{Q}-z|<2^{k+1}l}|f(z)|^{s'}\right)^{1/s'}\\
\leq{}& C\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}(2^{k}l)^{-(n+m)}\left[\sum_{i=1}^{m}l^{i}(2^{k+1}l)^{m-i+\beta}+l^{m+\beta}\right]\\
&\times(2^{k+1}l)^{n}\|\Omega\|_{L^{s}(S^{n-1})}M_{s'}(f)(x)\\
\leq{}& C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}\sum_{i=1}^{m}\left(2^{-k(i-\beta)}+2^{-km}\right)\|\Omega\|_{L^{s}(S^{n-1})}M_{s'}(f)(x)\\
\leq{}& C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\sum_{k=N}^{\infty}\left(2^{-k(1-\beta)}+2^{-km}\right)\|\Omega\|_{L^{s}(S^{n-1})}M_{s'}(f)(x)\\
\leq{}& C|Q|^{\beta/n}\left(\sum_{|\gamma|=m}||D^{\gamma}A||_{\dot{\Lambda}_{\beta}}\right)\|\Omega\|_{L^{s}(S^{n-1})}M_{s'}(f)(x).
\end{aligned} \tag{3.18}$$

Thus,

$$W \leq W_1 + W_2 + W_3 \leq C|Q|^{\beta/n}\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.19}$$

Combining the estimates of U, V with W, we have

$$D \leq C|Q|^{\beta/n}\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.20}$$

So,

$$J_2 \leq C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})} M_{s'}(f)(x). \tag{3.21}$$

Combining the estimates of J_1 with J_2, we obtain

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|\mu_\Omega^A(f)(y) - \left(\mu_\Omega^A(f)\right)_Q\right| dy \\
&\quad \leq C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}\left[M_{p_1}(f)(x) + M_{s'}(f)(x)\right],
\end{aligned} \tag{3.22}$$

where $1 < p_1, s' < p$. So, by Lemma 2.5 and the $L^p(\mathbb{R}^n)$ boundedness of M_{p_1} and $M_{s'}$, we conclude that

$$||\mu_\Omega^A(f)||_{\dot{F}_p^{\beta,\infty}} \leq C\left(\sum_{|\gamma|=m} ||D^\gamma A||_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}\|f\|_{L^p}. \tag{3.23}$$

We complete the proof of Theorem 1.5.

4. Proofs of Theorems 1.6 and 1.7

First, let us introduce some notations related to Hardy spaces.

Definition 4.1 [10]. Let $0 < p \leq 1 \leq q \leq \infty$, let $p < q$, and let $s \geq s_0$, where $s_0 = [n(1/p - 1)]$. A function a is said to be a (p,q,s) atom, if $a \in L^q(\mathbb{R}^n)$ and satisfies the following conditions:

(i) $\operatorname{supp} a \subset B$;

(ii) $\|a\|_{L^q} \leq |B|^{1/q-1/p}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$, for all $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, with $0 \leq |\alpha| = \sum_{i=1}^n \alpha_i \leq s$.

Definition 4.2 [10]. Let $0 < p \le 1 \le q$ and let $p < q$, then the atomic Hardy space $H_a^{p,q,s}(\mathbb{R}^n)$ is defined by

$$H_a^{p,q,s}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \sum_j \lambda_j a_j, \text{ here } a_j \text{ is a } (p,q,s) \text{ atom and } \sum_j |\lambda_j|^p < \infty \right\}. \tag{4.1}$$

Then,

$$\|f\|_{H_a^{p,q,s}(\mathbb{R}^n)} = \inf\left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p}, \text{ for all decompositions of } f = \sum_j \lambda_j a_j \right\}. \tag{4.2}$$

LEMMA 4.3 [10]. *Let $0 < p \le 1 \le q$ and let $p < q$, then*

$$H_a^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n), \qquad \|f\|_{H_a^{p,q,s}(\mathbb{R}^n)} = \|f\|_{H^p(\mathbb{R}^n)}. \tag{4.3}$$

Let us now turn to prove Theorem 1.6.

First, we estimate $\mu_\Omega^A(f)$. Notice that, when $n/(n+\beta) < p \le 1$ and $0 < \beta < 1$, $s_0 = [n(1/p - 1)] \le [\beta] = 0$.

By Lemma 4.3 and a standard argument, it is sufficient for us to show that there is a constant $C > 0$ such that for any $(p, \infty, 0)$ atom a, $\|\mu_\Omega^A(a)\|_{L^r} \le C$.

Take a $(p, \infty, 0)$ atom a with $\operatorname{supp} a \subset B(x_0, l)$. Then,

$$\begin{aligned}
||\mu_\Omega^A(a)||_{L^r} &\le \left[\int_{2B} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} + \left[\int_{(2B)^c} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} \\
&\le \left[\int_{2B} |\mu_\Omega^A(a)(x)|^r dx \right]^{1/r} \\
&\quad + \left\{ \int_{(2B)^c} \left[\int_0^{|x-x_0|+2l} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y)a(y)dy \right|^2 \frac{dt}{t^3} \right]^{r/2} dx \right\}^{1/r} \\
&\quad + \left\{ \int_{(2B)^c} \left[\int_{|x-x_0|+2l}^{\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_{m+1}(A;x,y)a(y)dy \right|^2 \frac{dt}{t^3} \right]^{r/2} dx \right\}^{1/r} \\
&:= I + II + III.
\end{aligned} \tag{4.4}$$

Choose p_1 and q_1 satisfying $1 < p_1 < n/\beta$ and $1/q_1 = 1/p_1 - \beta/n$. It is obvious that $r < q_1$. So, by Hölder's inequality and the (L^{p_1}, L^{q_1}) boundedness of μ_Ω^A (see Theorem 1.3), we have

$$\begin{aligned} I &\le C\|\mu_\Omega^A(a)\|_{L^{q_1}}|2B|^{1/r-1/q_1} \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\|a\|_{L^{p_1}}|2B|^{(1/r-1/q_1)} \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\|a\|_{L^\infty}|B|^{1/p_1}|2B|^{(1/r-1/q_1)} \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)|B|^{(1/p_1-1/p)}|2B|^{(1/r-1/q_1)} \le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big). \end{aligned} \tag{4.5}$$

Since for any $y \in B$, $x \in (2B)^c$, we have $|x-y| \sim |x-x_0| \sim |x-x_0|+2l$. By the Minkowski inequality, Hölder's inequality, (2.1), and (3.9),

$$\begin{aligned} II &\le C\left\{\int_{(2B)^c}\left[\int_{\mathbb{R}^n}\left(\int_{|x-y|}^{|x-x_0|+2l}\frac{dt}{t^3}\right)^{1/2}\frac{|\Omega(x-y)|\,|a(y)|}{|x-y|^{n+m-1}}|R_{m+1}(A;x,y)|\,dy\right]^r dx\right\}^{1/r} \\ &\le C\left\{\int_{(2B)^c}\left[\int_B\frac{|l|^{1/2}|\Omega(x-y)|\,|a(y)|}{|x-y|^{n+m+1/2}}|R_{m+1}(A;x,y)|\,dy\right]^r dx\right\}^{1/r} \\ &\le C\int_B\left\{\int_{(2B)^c}\left[\frac{|l|^{1/2}|\Omega(x-y)|}{|x-y|^{n+m+1/2}}\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)|x-y|^{m+\beta}\right]^r dx\right\}^{1/r}|a(y)|\,dy \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\int_B\sum_{k=1}^{\infty}2^{-k/2}(2^k l)^{-n+\beta}(2^{k+1}l)^{n(1/r-1/s)} \\ &\quad\times\left[\int_{|x-x_0|<2^{k+1}l}|\Omega(x-y)|^s dx\right]^{1/s}|a(y)|\,dy \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\Big(\sum_{k=1}^{\infty}2^{-k(1/2-\beta)}2^{-kn(1-1/r)}\Big)\|\Omega\|_{L^s(S^{n-1})}l^{-n(1-1/p)}\|a\|_{L^\infty}|B| \\ &\le C\Big(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Big)\|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \tag{4.6}$$

Notice that for any $y \in B$, we have $t \geq |x-x_0| + 2l \geq |x-x_0| + |y-x_0| \geq |x-y|$. So, by the vanishing condition of a, we have

$$\begin{aligned}
&\left[\int_{|x-x_0|+2l}^{\infty}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)a(y)dy\right|^2\frac{dt}{t^3}\right]^{1/2}\\
&\quad=\left\{\int_{|x-x_0|+2l}^{\infty}\left|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)a(y)dy\right|^2\frac{dt}{t^3}\right\}^{1/2}\\
&\quad=\left|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)a(y)dy\right|\left(\int_{|x-x_0|+2l}^{\infty}\frac{dt}{t^3}\right)^{1/2}\\
&\quad=\left|\int_{B}\left[\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)-\frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}}R_{m+1}(A;x,x_0)\right]\frac{a(y)}{|x-x_0|+2l}dy\right|.
\end{aligned}\tag{4.7}$$

On the other hand, it is similar to (3.12), and we have

$$\begin{aligned}
&\left|\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)-\frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}}R_{m+1}(A;x,x_0)\right|\\
&\quad\leq\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|\frac{|R_{m+1}(A;x,y)|}{|x-y|^m}\\
&\qquad+\frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x-x_0|^m}\right||R_{m+1}(A;x,y)|\\
&\qquad+\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}|R_{m+1}(A;x,y)-R_{m+1}(A;x,x_0)|.
\end{aligned}\tag{4.8}$$

So,

$$\begin{aligned}
III&\leq\left\{\int_{(2B)^c}\left[\int_{B}\left|\frac{\Omega(x-y)}{|x-y|^{n+m-1}}R_{m+1}(A;x,y)-\frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1}}R_{m+1}(A;x,x_0)\right|\right.\right.\\
&\qquad\left.\left.\times\frac{|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r}\\
&\leq\left\{\int_{(2B)^c}\left[\int_{B}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|\frac{|R_{m+1}(A;x,y)||a(y)|}{|x-y|^m(|x-x_0|+2l)}dy\right]^r dx\right\}^{1/r}
\end{aligned}$$

$$+\left\{\int_{(2B)^c}\left[\int_B \frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x-x_0|^m}\right|\frac{|R_{m+1}(A;x,y)|\,|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r}$$

$$+\left\{\int_{(2B)^c}\left[\int_B \frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}\,|R_{m+1}(A;x,y)-R_{m+1}(A;x,x_0)|\,\frac{|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r}$$

$$:= III_1+III_2+III_3. \tag{4.9}$$

For III_1, by the Minkowski inequality, (2.1), Hölder's inequality, and (3.15), we have

$$\begin{aligned}
III_1 &\le C\int_B\left\{\int_{(2B)^c}\left[\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\right.\right.\\
&\qquad\left.\left.\times\frac{|x-y|^{m+\beta}}{|x-y|^m(|x-x_0|+2l)}\right]^r dx\right\}^{1/r}|a(y)|\,dy\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\int_B\sum_{k=1}^{\infty}(2^kl)^{-1+\beta+n(1/r-1/s)}\\
&\qquad\times\left\{\int_{2^kl\le|x-x_0|<2^{k+1}l}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|^s dx\right\}^{1/s}|a(y)|\,dy\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\int_B\sum_{k=1}^{\infty}(2^kl)^{\beta+n/r-n}(2^{-k}+2^{-k\varepsilon})\,|a(y)|\,dy\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\sum_{k=1}^{\infty}2^{-kn(1-1/r)}(2^{-k(1-\beta)}+2^{-k(\varepsilon-\beta)})\,l^{n(1/p-1)}\|a\|_{L^\infty}|B|\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right).
\end{aligned}\tag{4.10}$$

Since $|x-y|\sim|x-x_0|$ for any $x\in(2B)^c$, by the Minkowski inequality, (2.1), Hölder's inequality, and (3.9),

$$\begin{aligned}
III_2 &\le C\int_B\left\{\int_{(2B)^c}\left[\frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}}\frac{l}{|x-x_0|^{m+1}}\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\frac{|x-y|^{m+\beta}}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r}\\
&\qquad\times|a(y)|\,dy\\
&\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\int_B\sum_{k=1}^{\infty}l(2^kl)^{-n-1+\beta}\times\left(\int_{2^kl\le|x-x_0|<2^{k+1}l}|\Omega(x-x_0)|^r dx\right)^{1/r}\\
&\qquad\times|a(y)|\,dy
\end{aligned}$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\int_B\sum_{k=1}^{\infty}l(2^kl)^{-n-1+\beta}(2^{k+1}l)^{n(1/r-1/s)}$$

$$\times\Bigg(\int_{l\leq|x-x_0|<2^{k+1}l}|\Omega(x-x_0)|^s dx\Bigg)^{1/s}|a(y)|\,dy$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\Bigg(\sum_{k=1}^{\infty}2^{-k(1-\beta)}2^{-kn(1-1/r)}\Bigg)l^{-n(1-1/p)}\|a\|_{L^\infty}|B|\,\|\Omega\|_{L^s(S^{n-1})}$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\|\Omega\|_{L^s(S^{n-1})}. \tag{4.11}$$

Moreover, by the Minkowski inequality, (2.3), and (3.9),

$$III_3\leq C\int_B\Bigg\{\int_{(2B)^c}\Bigg[\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}|R_{m+1}(A;x,y)-R_{m+1}(A;x,x_0)|\frac{1}{|x-x_0|+2l}\Bigg]^r dx\Bigg\}^{1/r}$$

$$\times|a(y)|\,dy$$

$$\leq C\int_B\Bigg\{\int_{(2B)^c}\Bigg[\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m}}\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\times\Bigg(\sum_{i=0}^{m}|x-x_0|^i|x_0-y|^{m-i+\beta}\Bigg)\Bigg]^r dx\Bigg\}^{1/r}$$

$$\times|a(y)|\,dy$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\int_B\sum_{k=1}^{\infty}(2^kl)^{-(n+m)}\Bigg(\sum_{i=0}^{m}(2^{k+1}l)^i l^{m-i+\beta}\Bigg)$$

$$\times\Bigg(\int_{2^kl\leq|x-x_0|<2^{k+1}l}|\Omega(x-x_0)|^r dx\Bigg)^{1/r}|a(y)|\,dy$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\sum_{k=1}^{\infty}(2^kl)^{-(n+m)}\Bigg(\sum_{i=0}^{m}(2^{k+1}l)^i l^{m-i+\beta}\Bigg)$$

$$\times(2^{k+1}l)^{n/r}\|\Omega\|_{L^s(S^{n-1})}\|a\|_{L^\infty}|B|$$

$$\leq C\Bigg(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\Bigg)\sum_{k=1}^{\infty}\Bigg(\sum_{i=0}^{m}2^{-k[(m-i)+n(1-1/r)]}\Bigg)l^{-n(1-1/p)}\|\Omega\|_{L^s(S^{n-1})}\|a\|_{L^\infty}|B|$$

$$\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\sum_{k=1}^{\infty}2^{-kn(1-1/r)}l^{-n(1-1/p)}\|a\|_{L^\infty}|B|\|\Omega\|_{L^s(S^{n-1})}$$

$$\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}. \tag{4.12}$$

So,

$$III\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}. \tag{4.13}$$

Combining the estimates of I, II with III, we get

$$\|\mu_\Omega^A(a)\|_{L^r}\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\|\Omega\|_{L^s(S^{n-1})}. \tag{4.14}$$

Replacing $\mu_\Omega^A(f)$ by $\widetilde{\mu}_\Omega^A(f)$ and using (2.2) and (2.5) instead of (2.1) and (2.3) in the above estimates, we can show that $\widetilde{\mu}_\Omega^A$ is also bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for $n/(n+\beta)<p<1$ and $1/r=1/p-\beta/n$.

In fact, we need only to check III_3, where R_{m+1} is replaced by Q_{m+1}:

$$III_3=C\left\{\int_{(2B)^c}\left[\int_B\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}|Q_{m+1}(A;x,y)-Q_{m+1}(A;x,x_0)|\frac{|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r}$$

$$\le C\int_B\left\{\int_{(2B)^c}\left[\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}|Q_{m+1}(A;x,y)-Q_{m+1}(A;x,x_0)|\frac{1}{|x-x_0|+2l}\right]^r dx\right\}^{1/r}$$
$$\times|a(y)|dy$$

$$\le C\int_B\left\{\int_{(2B)^c}\left[\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m}}\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\right.\right.$$
$$\left.\left.\times\left(\sum_{i=0}^{m-1}|x-x_0|^i|x_0-y|^{m-i}\left(|x-y|^\beta+|y-x_0|^\beta\right)\right)\right]^r dx\right\}^{1/r}|a(y)|dy$$

$$\le C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\right)\int_B\sum_{k=1}^{\infty}(2^kl)^{-(n+m)}\sum_{i=0}^{m-1}(2^{k+1}l)^i l^{m-i}\left((2^{k+1}l)^\beta+l^\beta\right)$$
$$\times\left(\int_{l\le|x-x_0|<2^{k+1}l}|\Omega(x-x_0)|^r dx\right)^{1/r}|a(y)|dy$$

$$
\begin{aligned}
&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg)\sum_{k=1}^{\infty}(2^k l)^{-(n+m)+n/r}\sum_{i=0}^{m-1}(2^{k+1}l)^i l^{m-i} \\
&\quad\times\Big((2^{k+1}l)^\beta+l^\beta\Big)\|\Omega\|_{L^s(S^{n-1})}\|a\|_{L^\infty}|B| \\
&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg)\sum_{k=1}^{\infty}\sum_{i=0}^{m-1}\big[2^{-k(n+m-i-\beta-n/r)}+2^{-k(n+m-i-n/r)}\big]\|\Omega\|_{L^s(S^{n-1})}\|a\|_{L^\infty}|B| \\
&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg)\sum_{k=1}^{\infty}\Big\{2^{-k[n(1-1/r)+(1-\beta)]}+2^{-k[n(1-1/r)+1]}\Big\}\|\Omega\|_{L^s(S^{n-1})} \\
&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg)\|\Omega\|_{L^s(S^{n-1})}. \qquad (4.15)
\end{aligned}
$$

Thus we complete the proof of Theorem 1.6.

Let us now prove Theorem 1.7. The main idea is the same as that of proving Theorem 1.6.

Let a be a $(1,\infty,0)$ atom with $\operatorname{supp} a\subset B(x_0,l)$ and $r=n/(n-\beta)$, then

$$
||\mu_\Omega^A(a)||_{L^r}\le I+II+III, \qquad (4.16)
$$

where I, II, and III are the same as in the proof of Theorem 1.6.

In the same way as in the estimates of (4.5) and (4.6), when $r=n/(n-\beta)$, we have

$$
\begin{aligned}
I&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg), \\
II&\le C\Bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot\Lambda_\beta}\Bigg)\|\Omega\|_{L^s(S^{n-1})}.
\end{aligned} \qquad (4.17)
$$

As in the estimate of (4.9), we have

$$
\begin{aligned}
III\le&\left\{\int_{(2B)^c}\left[\int_B\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|\frac{|R_{m+1}(A;x,y)|\,|a(y)|}{|x-y|^m(|x-x_0|+2l)}dy\right]^r dx\right\}^{1/r} \\
&+\left\{\int_{(2B)^c}\left[\int_B\frac{|\Omega(x-x_0)|}{|x-x_0|^{n-1}}\left|\frac{1}{|x-y|^m}-\frac{1}{|x-x_0|^m}\right|\frac{|R_{m+1}(A;x,y)|\,|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r} \\
&+\left\{\int_{(2B)^c}\left[\int_B\frac{|\Omega(x-x_0)|}{|x-x_0|^{n+m-1}}\,|R_{m+1}(A;x,y)-R_{m+1}(A;x,x_0)|\,\frac{|a(y)|}{|x-x_0|+2l}dy\right]^r dx\right\}^{1/r} \\
:=&\,E+F+G.
\end{aligned} \qquad (4.18)
$$

In the same way as in the estimates of (4.11) and (4.12), we have

$$
\begin{aligned}
F &\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\|\Omega\|_{L^s(S^{n-1})},\\
G &\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\|\Omega\|_{L^s(S^{n-1})}.
\end{aligned}
\tag{4.19}
$$

So, it is sufficient for us to estimate E.

In fact, it is similar to the estimate of III_1, by Lemma 2.6, $r = n/(n-\beta)$, and (1.11), we have

$$
\begin{aligned}
E &\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\int_B\sum_{k=1}^{\infty}(2^k l)^{-1+\beta+n(1/r-1/s)}\\
&\quad\times\left\{\int_{2^k l\le|x-x_0|<2^{k+1}l}\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\right|^s dx\right\}^{1/s}|a(y)|\,dy\\
&\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\int_B\sum_{k=1}^{\infty}(2^k l)^{-1+\beta+n(1/r-1/s)}(2^k l)^{n/s-n+1}\\
&\quad\times\left\{\frac{|y-x_0|}{2^k l}+\int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^k l}\frac{\omega_s(\delta)}{\delta}d\delta\right\}|a(y)|\,dy\\
&\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\int_B\left\{\sum_{k=1}^{\infty}2^{-k}+\sum_{k=1}^{\infty}\int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^k l}\frac{\omega_s(\delta)}{\delta}d\delta\right\}|a(y)|\,dy\\
&\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\int_B\left\{1+\int_0^1\frac{\omega_s(\delta)}{\delta}d\delta\right\}|a(y)|\,dy\\
&\le C\bigg(\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}\bigg)\|a\|_{L^\infty}|B|\\
&\le C\sum_{|\gamma|=m}||D^\gamma A||_{\dot{\Lambda}_\beta}.
\end{aligned}
\tag{4.20}
$$

Thus, we get the estimate of $\mu_\Omega^A(f)$ for $f\in H^1(\mathbb{R}^n)$. It is analogous to the argument for $\widetilde{\mu}_\Omega^A$ in the proof of Theorem 1.6, and we can get the desired result for $\widetilde{\mu}_\Omega^A$ by repeating the above estimates and using (4.15), when $f\in H^1(\mathbb{R}^n)$. So, we complete the proofs of Theorem 1.7.

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