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AN EFFICIENT SOLUTION OF THE ELASTIC MECHANISM NON-LINEAR DIFFERENTIAL EQUATION OF MOTION

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ABSTRACT

A general form of non-linear differential motion equations for the elastic mechanism is presented. Based on the analysis of the structure property of the equations, a numerical solution method which combines the State Space Method with Iterative Method is put forward. The solution efficiency and precision are improved greatly through analyzing the solution method of the matrix exponent function and the equations with belt shape and side block coefficient matrices. Finally, as an example, the dynamic properties of a mold oscillation mechanism with 4-eccentric axes for continuous casting machine are analyzed. The results illustrate the correctness and feasibility of the method.

Keywords: state space method, closed-form numerical algorithm, elastic dynamic analysis, oscillation mechanism

INTRODUCTION

Based on the Kineto-ElastoDynamics (KED) theory, the motion differential equation for a general elastic mechanism can be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{Q} \tag{1}$$

Where

$\boldsymbol{M}$ ---- mass matrix of the system

$\boldsymbol{C}$ ---- damping matrix of the system

$\boldsymbol{K}$ ---- stiffness matrix of the system

$\boldsymbol{Q}$ ---- external loading matrix

$\boldsymbol{U}$, $\dot{\boldsymbol{U}}$, $\ddot{\boldsymbol{U}}$ ----elastic displacement, velocity and acceleration matrix respectively.

If the geometrical non-linearity and the coupling item of the rigid and elastic body motion are considered, the equation would be expressed as follows [1]:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + (\boldsymbol{K}_l + \boldsymbol{K}_G + \boldsymbol{K}_H + \boldsymbol{K}_q)\boldsymbol{U} = \boldsymbol{Q} - \boldsymbol{M}_C\dot{\boldsymbol{U}} - \boldsymbol{M}_a\boldsymbol{U} \tag{2}$$

Where

$\boldsymbol{K}_l$ ---- linear part of the stiffness matrix

$\boldsymbol{K}_G$, $\boldsymbol{K}_H$, $\boldsymbol{K}_q$ ---- non-linear parts of the stiffness matrix which are connected with the generalized coordinates $\boldsymbol{U}$ of the system

$\boldsymbol{M}_C$, $\boldsymbol{M}_a$ ---- non-symmetry matrix formed due to the coupling of the rigid and elastic body motion of the system

Generally, the mass, damping, stiffness and external loading matrices of the system may all be composed of linear and non-linear parts, so the general differential equation of an elastic mechanism system would be expressed as follows:

$$(\boldsymbol{M}_l + \boldsymbol{M}_N)\ddot{\boldsymbol{U}} + (\boldsymbol{C}_l + \boldsymbol{C}_N)\dot{\boldsymbol{U}} + (\boldsymbol{K}_l + \boldsymbol{K}_N)\boldsymbol{U} = \boldsymbol{Q}_l + \boldsymbol{Q}_N \tag{3}$$

The coefficient matrixes of Eq.(3) are functions of the mechanism's positions, so Eq.(3) is a coupling, variable coefficient, non-linear differential equation. In order to analyze the elastic dynamic properties of the mechanism, an efficient solution method for Eq.(3) must be found.

At present the most commonly used solution methods for variable coefficient linear differential Eq.(1) are vibration modal superposition method, Fourier series method and direct integration method respectively[2-3]. Vibration modal superposition method, which can be divided into real and complex vibration modal superposition method, is a traditional way for solving vibration problem. The real modal method is used while the system damping meets a certain condition, for general high damping system the complex modal method is adopted. Vibration modal superposition method is wildly used because it can provide natural frequencies and vibration modals of the system. But it has to solve eigenvalue problems, so the calculation workload would be very heavy if the system degrees of freedom are higher. For a periodical operating machine if only the steady circulation properties are needed, Fourier series method which does not require solving eigenvalue problems and has no limit to the form of the damp matrix would be introduced to solve the response of the system. The shortcoming of this method is the solution difficulties of the high dimensional linear equations. Direct integration methods mainly include step by step integration method and state space method. Step by step integration method does not need to solve vibration modals and frequencies so the damping matrix form is not limited. But it would bring larger error and sometimes the calculation would be unsteady. The most commonly used step by step integration methods are linear acceleration, Wilson-θ and Newmark-β methods [2] respectively. State space method, which can be applied to solve differential equation of motion with any form of damping matrix, is another efficient, steady, high precision direct integration method.

The responses of a planar elastic linkage mechanism compose of steady and transient response. At present there are two kinds of numerical method of solving steady response of high-speed elastic Linkage: open-form method and closed-form method. Midha.A[4] presents a closed-form numerical method that combines vibration modal superposition method with Duharm integration method. Bagci.C[5] first uses state space method to solve the elastic mechanism differential equation of motion, however, which is an open-form numerical method.

For non-linear differential equation of motion for elastic mechanism, most researchers[1,6] simplify it as a linear equation because it is difficult to solve non-linear differential equation for elastic system.

In this paper another method that combines state space method with iterative procedure is presented. The advantage of this method is that it does not need to solve eigenvalue problems and vibration modals of the system, so the method has no limit to the form of the damping matrix, and has high efficiency, better stability and higher precision.

1 STATE SPACE METHOD AND THE CORRESPONDING CLOSED-FORM NUMERICAL METHOD

It is supposed that the degrees of freedom of Eq.(1) is Nu. Then the state space model for Eq.(1) can be expressed as:

$$\dot{\boldsymbol{Z}} = \boldsymbol{F}\boldsymbol{Z} + \boldsymbol{G}\boldsymbol{Q} \tag{4}$$

Where

$$\boldsymbol{Z} = \begin{Bmatrix} \boldsymbol{U} \\ \dot{\boldsymbol{U}} \end{Bmatrix} = \{Z_1, Z_2, Z_3, \cdots Z_{Nu}, Z_{Nu+1}, \cdots, Z_{2Nu}\}^T$$

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1} \end{bmatrix}$$

$\boldsymbol{F}$ is a $2Nu \times 2Nu$ dimension coefficient matrix, $\boldsymbol{G}$ is a $2Nu \times Nu$ dimension control matrix, $\boldsymbol{I}$ is a Nu dimension unit matrix, $\boldsymbol{0}$ is a Nu dimension zero matrix. The general solution of Eq.(4) is

$$\boldsymbol{Z}(t) = \boldsymbol{Z}(0)e^{\boldsymbol{F}t} + \int_0^t e^{\boldsymbol{F}(t-\tau)}\boldsymbol{G}\boldsymbol{Q}(\tau)d\tau \tag{5}$$

Where $e^{\boldsymbol{F}t}$ is a matrix exponential function, which can be expressed as follows:

$$e^{\boldsymbol{F}t} = \sum_{k=0}^{\infty}\frac{\boldsymbol{F}^k t^k}{k!} = \boldsymbol{I} + \boldsymbol{F}t + \frac{1}{2}\boldsymbol{F}^2t^2 + \frac{1}{3!}\boldsymbol{F}^3t^3 + \cdots + \frac{1}{k!}\boldsymbol{F}^k t^k + \cdots \tag{6}$$

We suppose that Δt is such a short time interval that $Q(t)$ can be considered as a constant. That is

$$\boldsymbol{Q}(k\Delta t + \tau) = \boldsymbol{Q}(k\Delta t) \tag{7}$$

Where k=1, 2, 3, …, $\tau = 0 \sim \Delta t$. So the numerical solution of Eq.(7) is

$$\boldsymbol{Z}\left[(k+1)\Delta t\right] = \boldsymbol{A}\boldsymbol{Z}\left[k\Delta t\right] + \boldsymbol{B}\boldsymbol{Q}(k\Delta t) \tag{8}$$

Where

$$\boldsymbol{A} = e^{\boldsymbol{F}\Delta t} = \sum_{k=0}^{\infty}\frac{\boldsymbol{F}^k \Delta t^k}{k!} \tag{9}$$

$$\boldsymbol{B} = \int_0^{\Delta t} e^{\boldsymbol{F}t}\boldsymbol{G}dt = \Delta t\left[\sum_{k=0}^{\infty}\frac{\boldsymbol{F}^k\Delta t^k}{(k+1)!}\right]\boldsymbol{G} \tag{10}$$

The movement period of the mechanism is divided into n equal time intervals, and the time nodes are marked as t_0, t_1, … , t_{n-1}. The time between t_k and t_{k+1} is called the kth time element. The connection of the time nodes and time elements is shown as follows

$$\underbrace{t_0 \quad t_1}_{element_0} \quad \cdots \quad \underbrace{t_k \quad t_{k+1}}_{element_k} \quad \cdots \quad \underbrace{t_{n-2} \quad t_{n-1}}_{element_{n-2}}$$

In the kth time element the values of $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{Q}$ are equal to the values at the time t_k which are expressed as $\boldsymbol{F}_k$, $\boldsymbol{G}_k$ and $\boldsymbol{Q}_k$. The corresponding matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are expressed as $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$. Therefore

$$\boldsymbol{A}_k = \sum_{i=0}^{\infty}\frac{\boldsymbol{F}_k^i\Delta t^i}{i!}$$

$$\boldsymbol{B}_k = \Delta t\left[\sum_{i=0}^{\infty}\frac{\boldsymbol{F}_k^i\Delta t^i}{(i+1)!}\right]\boldsymbol{G}$$

If $\boldsymbol{Z}_k$ and $\boldsymbol{Z}_{k+1}$ are used to express the state variables at the time t_k and t_{k+1}, equation.(8) would be

$$\boldsymbol{Z}_{k+1} = \boldsymbol{A}_k\boldsymbol{Z}_k + \boldsymbol{B}_k\boldsymbol{Q}_k \quad (k = 0, 1, 2 \ldots)$$

Introduce $\boldsymbol{C}_k = -\boldsymbol{B}_k\boldsymbol{Q}_k$, then $\boldsymbol{C}_k = \boldsymbol{A}_k\boldsymbol{Z}_k - \boldsymbol{Z}_{k+1}$ (k = 0, 1, 2 …), that is

$$\begin{cases} \boldsymbol{C}_0 = \boldsymbol{A}_0\boldsymbol{Z}_0 - \boldsymbol{Z}_1 \\ \boldsymbol{C}_1 = \boldsymbol{A}_1\boldsymbol{Z}_1 - \boldsymbol{Z}_2 \\ \cdots\cdots\cdots\cdots \\ \boldsymbol{C}_{n-2} = \boldsymbol{A}_{n-2}\boldsymbol{Z}_{n-2} - \boldsymbol{Z}_{n-1} \\ \boldsymbol{C}_{n-1} = \boldsymbol{A}_{n-1}\boldsymbol{Z}_{n-1} - \boldsymbol{Z}_n \end{cases} \tag{11}$$

Considering the movement periodicity of the mechanism, we can obtain the periodical condition $\boldsymbol{Z}_n = \boldsymbol{Z}_0$. Then a closed calculation method based on state space method would be obtained through assembling Eq.(11) into a $2Nu \times n$ linear algebraic equations.

$$\begin{bmatrix} \boldsymbol{A}_0 & -\boldsymbol{I} & & & & \\ & \boldsymbol{A}_1 & -\boldsymbol{I} & & & \\ & & \boldsymbol{A}_2 & -\boldsymbol{I} & & \\ & & & \ddots & \ddots & \\ & & & & \boldsymbol{A}_{n-2} & -\boldsymbol{I} \\ -\boldsymbol{I} & & & & & \boldsymbol{A}_{n-1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{Z}_0 \\ \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \cdots \\ \boldsymbol{Z}_{n-2} \\ \boldsymbol{Z}_{n-1} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{C}_0 \\ \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \cdots \\ \boldsymbol{C}_{n-2} \\ \boldsymbol{C}_{n-1} \end{Bmatrix} \tag{12}$$

There are two main factors that influence the solution efficiency of the above differential equation: (1) the calculation efficiency of the matrix exponent function, (2) the computing efficiency of the equations with belt shape coefficient and side block matrices.

2 MATRIX EXPONENT FUNCTION

According to the expression of matrix exponent function, Taylor series method is a solution method which is easy to be thought of. But it is found from actual calculations that the convergence speed of the Taylor series method is slower and the solution efficiency is lower. In order to achieve higher calculation precision longer time would be spent especially for high dimensional matrix. After analyzing and comparing several solution methods for matrix exponent function, the authors have found that the variable scale square method which uses cross Pade approach[7-9] could improve solution efficiency greatly. The method is shown as follows:

The convergence index $\delta > 0$ is given arbitrarily, suppose A is a $n \times n$ dimension real matrix, then its matrix exponent function $\boldsymbol{F} = e^{A}$ could be obtained according to the following calculation steps[10]:

(1) Suppose $j = \max\{0, 1 + \text{floor}(\log_2(\|\boldsymbol{A}\|_\infty))\}$, $\boldsymbol{A} = \boldsymbol{A}/2^j$, $\boldsymbol{D} = \boldsymbol{I}$, $\boldsymbol{N} = \boldsymbol{I}$, $\boldsymbol{X} = \boldsymbol{I}$, where $\boldsymbol{I}$ is a $n \times n$ dimension unit matrix.

(2) suppose q is a least non-negative integer which meet the following condition

$$2^{3-2q} \frac{(q!)^2}{(2q)!(2q+1)!} \le \delta$$

(3) for $k = 1, 2, \cdots, q$, suppose $c = \dfrac{(2q-k)!q!}{(2q)!k!(q-k)!}$, then $\boldsymbol{X}$, $\boldsymbol{N}$, $\boldsymbol{D}$ are calculated according to the following equations.

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X}$$
$$\boldsymbol{N} = \boldsymbol{N} + c\boldsymbol{X}$$
$$\boldsymbol{D} = \boldsymbol{D} + (-1)^k c\boldsymbol{X}$$

(4) Then $\boldsymbol{F} = \boldsymbol{N}/\boldsymbol{D}$

(5) for $k = 1, 2, \ldots, j$, $\boldsymbol{F} = \boldsymbol{F}^2$

Above calculation procedure needs about $\left(q + j + \frac{1}{3}\right) n^3$ operations. The calculation precision for different q is given in the following Tab.1

Table 1 The reachable calculation precision

q	1	2	3	4	5	6
δ	10^{-1}	10^{-4}	10^{-6}	10^{-9}	10^{-13}	10^{-16}

3 SOLUTION METHOD OF EQUATIONS WITH BELT SHAPE COEFFICIENT AND SIDE BLOCK MATRICES

Another factor to restrict the efficiency of the closed-form state space method is the solution efficiency of equations with belt shape coefficient and side block matrices. It is easy to find that the matrix is a sparser one and the nonzero elements of which arrange regularly in a belt area near the main diagonal elements and only one nonzero block locates in the left-bottom corner. Based on the above property of the matrix, here a new solution for solving these equations with belt shape coefficient and side block matrices is given as follows.

The Augmented matrix of Eq.(12) is

$$\left[\begin{array}{cccccc|c} \boldsymbol{A}_0 & -\boldsymbol{I} & & & & & \boldsymbol{C}_0 \\ & \boldsymbol{A}_1 & -\boldsymbol{I} & & & & \boldsymbol{C}_1 \\ & & \boldsymbol{A}_2 & -\boldsymbol{I} & & & \boldsymbol{C}_2 \\ & & & \ddots & \ddots & & \cdots \\ & & & & \boldsymbol{A}_{n-2} & -\boldsymbol{I} & \boldsymbol{C}_{n-2} \\ -\boldsymbol{I} & & & & & \boldsymbol{A}_{n-1} & \boldsymbol{C}_{n-1} \end{array}\right]$$

$\boldsymbol{A}_1$ left multiply the first row and add the result to the second row and keep the first row invariable, then

$$\left[\begin{array}{cccccc|c} \boldsymbol{A}_0 & -\boldsymbol{I} & & & & & \boldsymbol{C}_0 \\ \boldsymbol{A}_1\boldsymbol{A}_0 & & -\boldsymbol{I} & & & & \boldsymbol{C}_1 + \boldsymbol{A}_1\boldsymbol{C}_0 \\ & & \boldsymbol{A}_2 & -\boldsymbol{I} & & & \boldsymbol{C}_2 \\ & & & \ddots & \ddots & & \cdots \\ & & & & \boldsymbol{A}_{n-2} & -\boldsymbol{I} & \boldsymbol{C}_{n-2} \\ -\boldsymbol{I} & & & & & \boldsymbol{A}_{n-1} & \boldsymbol{C}_{n-1} \end{array}\right]$$

$\boldsymbol{A}_2$ left multiply the second row and add the result to the third row and keep the second row invariable, …… and n-1 step is applied according to the above method and suppose

$$\left.\begin{array}{ll} \boldsymbol{A}_k^* = \boldsymbol{A}_k \cdots \boldsymbol{A}_2\boldsymbol{A}_1\boldsymbol{A}_0 & (k = 0, 1 \ldots, n-2) \\ \boldsymbol{A}_{n-1}^* = \boldsymbol{A}_{n-1} \cdots \boldsymbol{A}_2\boldsymbol{A}_1\boldsymbol{A}_0 - \boldsymbol{I} & \\ \boldsymbol{C}_k^* = \boldsymbol{C}_k + \boldsymbol{A}_k\left(\boldsymbol{C}_{k-1} + \boldsymbol{A}_{k-1}\left(\cdots\left(\boldsymbol{C}_1 + \boldsymbol{A}_1\boldsymbol{C}_0\right)\right)\right) & (k = 0, 1 \ldots, n-1) \end{array}\right\} \quad (13)$$

So Eq.(12) can be expressed as

$$\begin{bmatrix} \boldsymbol{A}_0^* & -\boldsymbol{I} & & & & \\ \boldsymbol{A}_1^* & & -\boldsymbol{I} & & & \\ \boldsymbol{A}_2^* & & & -\boldsymbol{I} & & \\ \cdots & & & & \ddots & \\ \boldsymbol{A}_{n-2}^* & & & & & -\boldsymbol{I} \\ \boldsymbol{A}_{n-1}^* & & & & & \end{bmatrix} \begin{Bmatrix} \boldsymbol{Z}_0 \\ \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \cdots \\ \boldsymbol{Z}_{n-2} \\ \boldsymbol{Z}_{n-1} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{C}_0^* \\ \boldsymbol{C}_1^* \\ \boldsymbol{C}_2^* \\ \cdots \\ \boldsymbol{C}_{n-2}^* \\ \boldsymbol{C}_{n-1}^* \end{Bmatrix} \quad (15)$$

Therefore $\boldsymbol{A}_{n-1}^*\boldsymbol{Z}_0 = \boldsymbol{C}_{n-1}^*$, that is $\boldsymbol{Z}_0 = \left(\boldsymbol{A}_{n-1}^*\right)^{-1}\boldsymbol{C}_{n-1}^*$. The solution of Eq.(12) can be obtained:

$$\left.\begin{array}{ll} \boldsymbol{Z}_0 = \left(\boldsymbol{A}_{n-1}^*\right)^{-1}\boldsymbol{C}_{n-1}^* & \\ \boldsymbol{Z}_k = \boldsymbol{A}_{k-1}\boldsymbol{Z}_1 - \boldsymbol{C}_{k-1} & (k = 1, 2, 3, \cdots, n-1) \end{array}\right\} \quad (16)$$

4 SOLUTION FOR NON-LINEAR DIFFERENTIAL EQUATION OF MOTION BY COMBINING THE STATE SPACE METHOD WITH ITERATIVE PRODUCES

The mass, damping, stiffness and external loading matrices of Eq.(3) are all composed of two parts. The first is linear part, which is denoted as $\boldsymbol{M}_l$, $\boldsymbol{C}_l$, $\boldsymbol{K}_l$ and $\boldsymbol{Q}_l$ respectively, The second is non-linear part, which is denoted as $\boldsymbol{M}_N$, $\boldsymbol{C}_N$, $\boldsymbol{K}_N$ and $\boldsymbol{Q}_N$ respectively. If we only consider the linear part and ignore the non-linear part then Eq.(3) would be simplified to

$$\boldsymbol{M}_l\ddot{\boldsymbol{U}} + \boldsymbol{C}_l\dot{\boldsymbol{U}} + \boldsymbol{K}_l\boldsymbol{U} = \boldsymbol{Q}_l \quad (17)$$

First of all the movement period of the mechanism is divided into several equal time intervals. Differential equations at every time element can be deduced, i.e.

$$\boldsymbol{M}_l^k\ddot{\boldsymbol{U}} + \boldsymbol{C}_l^k\dot{\boldsymbol{U}} + \boldsymbol{K}_l^k\boldsymbol{U} = \boldsymbol{Q}_l^k \quad (18)$$

Where i is ith time node. If time elements are divided so short that the coefficient matrix of Eq.(18) can be considered as a constant one, so Eq.(18) can be regarded as a constant coefficient different equation.

$$\dot{\boldsymbol{Z}}^k = \boldsymbol{F}^k \boldsymbol{Z}^k + \boldsymbol{G}^k \boldsymbol{Q}_l^k$$

Where

$$\boldsymbol{F}^k = \left[\begin{array}{c|c} \boldsymbol{0} & \boldsymbol{I} \\ \hline -(\boldsymbol{M}_l^k)^{-1}\boldsymbol{K}_l^k & -(\boldsymbol{M}_l^k)^{-1}\boldsymbol{C}_l^k \end{array}\right]$$

$$\boldsymbol{G}^k = \left[\begin{array}{c} \boldsymbol{0} \\ \hline (\boldsymbol{M}_l^k)^{-1} \end{array}\right]$$

The numerical solution of the above equation is shown as Eq.(8). Considering the movement periodic condition $\boldsymbol{Z}_0 = \boldsymbol{Z}_n$, a large-scale linear algebraic equations would be assembled based on Eq.(12)

$$\begin{bmatrix} \boldsymbol{A}_0 & -\boldsymbol{I} & & & & \\ & \boldsymbol{A}_1 & -\boldsymbol{I} & & & \\ & & \boldsymbol{A}_2 & -\boldsymbol{I} & & \\ & & & \ddots & \ddots & \\ & & & & \boldsymbol{A}_{n-2} & -\boldsymbol{I} \\ -\boldsymbol{I} & & & & & \boldsymbol{A}_{n-1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{Z}_0 \\ \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \cdots \\ \boldsymbol{Z}_{n-2} \\ \boldsymbol{Z}_{n-1} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{C}_0 \\ \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \cdots \\ \boldsymbol{C}_{n-2} \\ \boldsymbol{C}_{n-1} \end{Bmatrix} \tag{19}$$

Where n is the number of the time elements. $2Nu \times n$ dimension state variable matrix $\boldsymbol{Z}^0$ of the system, which is the steady response while ignoring the non-linear part of Eq.(3), can be obtained through solving Eq.(19). $\boldsymbol{M}_N^1$, $\boldsymbol{C}_N^1$, $\boldsymbol{K}_N^1$ and $\boldsymbol{Q}_N^1$ which are the first approximation of $\boldsymbol{M}_N$, $\boldsymbol{C}_N$, $\boldsymbol{K}_N$ and $\boldsymbol{Q}_N$ may be calculated. Then suppose $\boldsymbol{M}^1 = \boldsymbol{M}_l + \boldsymbol{M}_N^1$, $\boldsymbol{C}^1 = \boldsymbol{C}_l + \boldsymbol{C}_N^1$, $\boldsymbol{K}^1 = \boldsymbol{K}_l + \boldsymbol{K}_N^1$ and $\boldsymbol{Q}^1 = \boldsymbol{Q}_l + \boldsymbol{Q}_N^1$, a new differential equation of the system is given as follows

$$\boldsymbol{M}^1 \ddot{\boldsymbol{U}} + \boldsymbol{C}^1 \dot{\boldsymbol{U}} + \boldsymbol{K}^1 \boldsymbol{U} = \boldsymbol{Q}^1 \tag{20}$$

Repeating the above calculation process, then $2Nu \times n$ dimension state variable matrix $\boldsymbol{Z}^1$ of the system is obtained. Here we should estimate whether $\boldsymbol{Z}^1$ meets the convergence condition or not. If $\boldsymbol{Z}^1$ meets the following convergence condition Eq.(21)

$$\frac{\left\|\boldsymbol{U}^{i+1} - \boldsymbol{U}^i\right\|}{\left\|\boldsymbol{U}^{i+1}\right\|} \times 100\% \le \varepsilon \tag{21}$$

Where $\boldsymbol{U}^i$ and $\boldsymbol{U}^{i+1}$ which are derived from $\boldsymbol{Z}^i$ and $\boldsymbol{Z}^{i+1}$ are the generalized coordinate matrix of the system. ε is small enough positive number. $\|\boldsymbol{U}\|$ is Frobenius pattern and its definition is shown as follows:

$$\|\boldsymbol{U}\| = \left(\sum_{j=1}^{Nu}\sum_{k=1}^{n} a_{jk}^2\right)^{\frac{1}{2}} \tag{22}$$

Where a_{jk} is the j th row and k th column element of the matrix $\boldsymbol{U}$.

Then we would take $\boldsymbol{Z}^1$ as the solution of Eq.(3). Otherwise $\boldsymbol{M}_N^2$, $\boldsymbol{C}_N^2$, $\boldsymbol{K}_N^2$ and $\boldsymbol{Q}_N^2$ which are the second approximations of $\boldsymbol{M}_N$, $\boldsymbol{C}_N$, $\boldsymbol{K}_N$ and $\boldsymbol{Q}_N$ should be calculated, and suppose $\boldsymbol{M}^2 = \boldsymbol{M}_l + \boldsymbol{M}_N^2$, $\boldsymbol{C}^2 = \boldsymbol{C}_l + \boldsymbol{C}_N^2$, $\boldsymbol{K}^2 = \boldsymbol{K}_l + \boldsymbol{K}_N^2$ and $\boldsymbol{Q}^2 = \boldsymbol{Q}_l + \boldsymbol{Q}_N^2$, therefore

$$\boldsymbol{M}^2 \ddot{\boldsymbol{U}} + \boldsymbol{C}^2 \dot{\boldsymbol{U}} + \boldsymbol{K}^2 \boldsymbol{U} = \boldsymbol{Q}^2 \tag{23}$$

Repeat the above process and suppose the ith calculated result $\boldsymbol{Z}^i$ meets the convergence condition Eq.(21), then $\boldsymbol{Z}^i$ is the solution of the non-linear differential equation.

5 A CASE STUDY EXAMPLE

As an example, the method is applied to analyze the elastic vibration property of an oscillation mechanism with 4-eccentric axes, which is widely used in modern continuous casting machine. The law of the dynamic displacement, stress and natural frequency that are changing with rotating speed and structure size of the machine must be researched in order to design it reasonably and decide the work conditions of the machine.

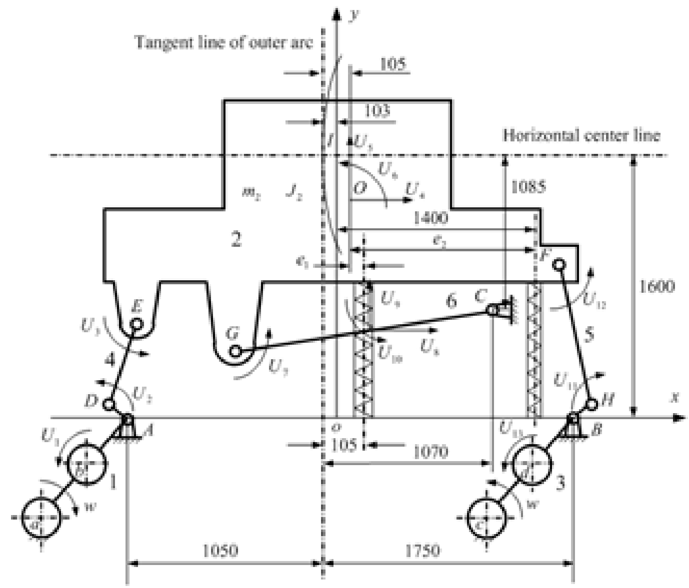


Figure 1 Substructure and Generalized Coordinate

First of all the dynamic analysis model of the machine should be established. It is a complex mechanism system combined of several components, and its applied loadings include driving force of electromotor, inertia force, pull billet force whose direction changes with the movement direction of the machine, gravity and nonlinear elastic force of the balance spring. In addition, the geometrical non-linearity of guiding leaf springs is also considered here. Under the influences of these forces and factors, complex elastic oscillation on the components of the machine would be brought about.

In order to study the dynamic properties of this elastic mechanism, a mechanical model illustrated in fig.1 is established [11]. The machine is divided into six substructures according to its components: outer arc transmission and eccentric axis 1, inner arc transmission and eccentric axis 3, oscillation table and crystallizing mould 2, outer arc linkage 4, inner arc linkage 5 and guiding leaf springs 6.

Generalized coordinates, which are expressed as U_1~U_{13}, are selected in term of the connection relationship among every substructure of the machine to analyze the dynamic properties of the mechanism. U_1 and U_{13} are the elastic rotation angles of the transmission axis 1 and 2 respectively; U_2 and U_3 are the elastic rotation angles of node D and E at the outer arc linkage 4; U_{11} and U_{12} are the elastic rotation angles of node H and F at the inner arc linkage 5; U_4, U_5 and U_6 are elastic displacements at direction x and y and elastic rotation angles at the center of mass of oscillation table respectively; U_7 is the elastic rotation angle at node G of the guiding leaf spring; U_8, U_9 and

U_{10} are elastic displacements at direction x and y and elastic rotation angles at the middle point of the guiding leaf springs.

Finally the motion differential equation is established for each substructure and the general motion differential equation is built through assembling all the six substructures according to its connection relationship [11].

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} + \boldsymbol{Q} \tag{24}$$

Where

$\boldsymbol{M}$ ---- mass matrix of the system

$\boldsymbol{C}$ ---- damping matrix of the system

$\boldsymbol{K}$ ---- stiffness matrix of the system

$\boldsymbol{F}$ ---- inertia force matrix of the rigid body

$\boldsymbol{Q}$ ---- nonlinear external loading matrix

$\boldsymbol{U}$, $\dot{\boldsymbol{U}}$, $\ddot{\boldsymbol{U}}$ ----elastic displacement, velocity and acceleration matrix respectively.

Eq.(25) is a coupling, variable coefficient, non-linear differential equation that is similar to the Eq.(3). And the dynamic property of the machine under the following case listed in table.2 is analyzed using the calculation method presented in this paper. The results using modal superposition method are also obtained. It is found that the contrastive results of nature frequency of the machine shown in Tab.(3) are similar, which approves the high precision of the presented method.

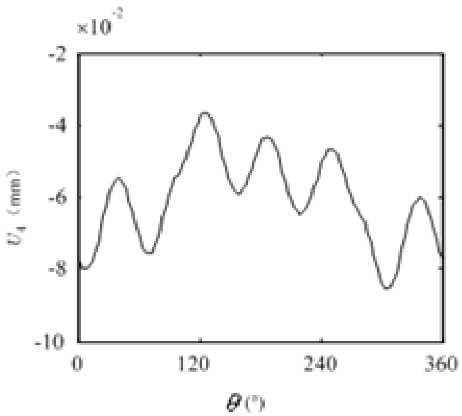
The elastic oscillation property curves for each generalized coordinate under various structure parameters and operating conditions are obtained. Figure.(2) shows parts of elastic displacement results. Figure.(3) and Fig.(4) show maximal dynamic stress of substructure 4 and 5 respectively.

Table 2 Crank length and amplitude of vibration (mm)

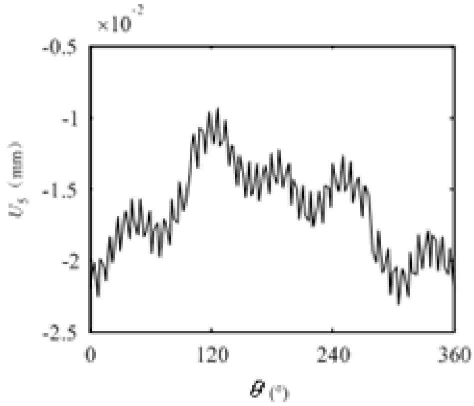
	Group 1	Group 2	Group 3	Group 4
amplitude of vibration (outer arc)	±3	±3.5	±4	±5
crank length (inner)	2.445	2.853	3.260	4.075
crank length (outer)	3.366	3.928	4.488	5.610

Table 3 Nature frequencies of the machine (Hz)

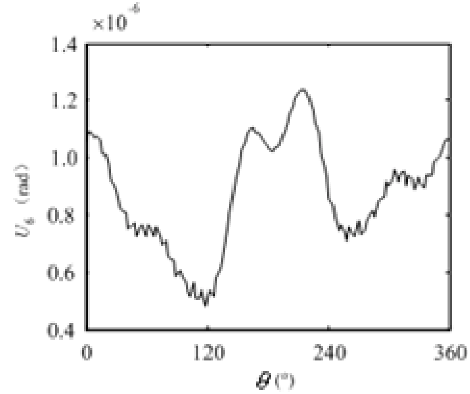
Calculation method	Group 1	Group 2	Group 3	Group 4
presented method	17.33	17.50	17.83	18.33
modal superposition method	17.35	17.55	17.85	18.50



(a) The 4th generalized coordinate (U_4)

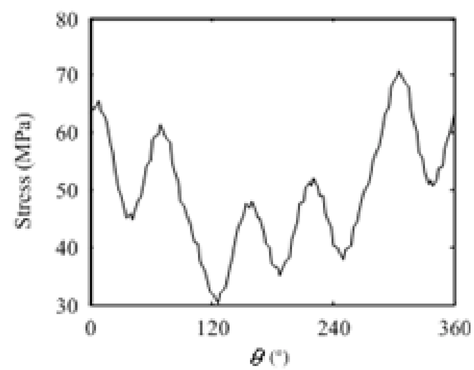


(b) The 5th generalized coordinate (U_5)

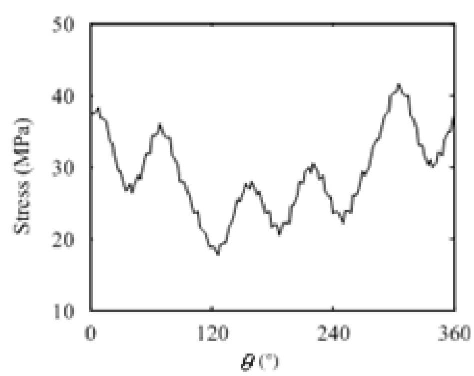


(c) The 6th generalized coordinate (U_6)

Figure 2 Parts of elastic displacement results

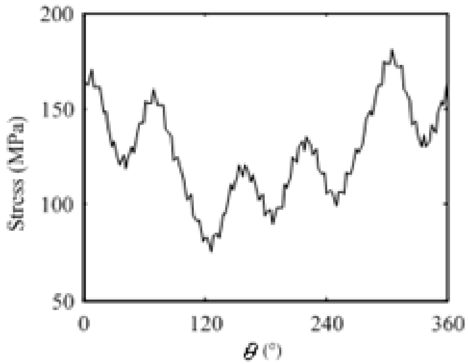


(a) Maximal dynamic stress of point *D*

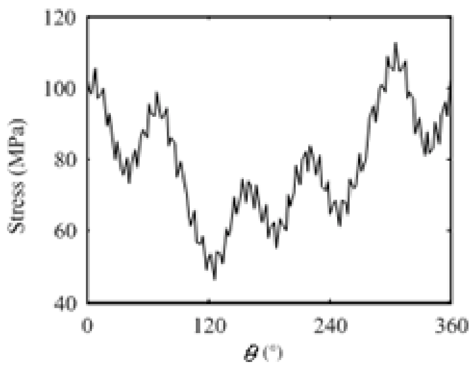


(b) Maximal dynamic stress of point *E*

Figure 3 Stress results of substructure 4



(a) Maximal dynamic stress of point *H*



(b) Maximal dynamic stress of point *F*

Figure 4 Stress results of substructure 5

6 CONCLUSION

A general form of non-linear differential motion equations for the elastic mechanism is presented, which includes the inertia, stiffness, and damping nonlinearities due to the coupling effects of rigid-body and elastic motion, and the geometrical nonlinearity. A solution method combining the state space method with iterative procedure is put forward. The solution efficiency and precision is improved greatly through analyzing and improving the solution method of the matrix exponent function and the equation groups with belt shape coefficient and side block matrices. As an example the dynamic property of an oscillation mechanism with 4-eccentric axes for continuous casting machine is analyzed, the nature frequencies of the machine, elastic displacements and dynamic stresses of the machine's parts are obtained. The numerical calculation results illustrate the correctness and feasibility of the presented method.

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