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A Numerical-Computational Technique for Solving Transformed Cauchy-Euler Equidimensional Equations of Homogeneous Type

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Abstract

This work considers the solution of Transformed-Cauchy-Euler differential equations via Differential Transform method (DTM). For illustration and application of the method's efficiency and reliability, two examples of order 2 and 3 are solved. The results agreed with the exact solution obtained via Laplace transform method.

Keywords: Cauchy-Euler differential equations, DTM, Laplace Transform method, Series solution

1. Introduction

A Cauchy-Euler equation, also known as Euler's equation named after Augustine Louis Cauchy (1789-1857), is a linear homogeneous ordinary differential equation with variable coefficients. This type of equation commonly appears as second order equation in physics and engineering applications such as Laplace's equation in polar coordinates and many other aspects of applied sciences [1]. One of the characteristic features of Cauchy-Euler equation is that the order of each derivative in the equation equals the power of the independent variable, say, x in the corresponding coefficient. This type of equations can be reduced to linear equation with constant coefficients by using appropriate substitution [1-2].
Various numerical methods have been proposed to solve ordinary differential equations. The Differential transformation method (DTM) adopted in this work was first proposed by Zhou [3] to solve both linear and nonlinear initial value problems in electric circuits analysis. Other applications of Differential Transform method can be found in [4-10]. It is also very useful when dealing with integro-differential equations encountered in finance [11].
In this work, a numerical-computational method is applied to solve Cauchy-Euler Equidimensional Equations. To illustrate this, second and third order Cauchy-Euler differential equations of the transformed type are solved.

2. The Cauchy-Euler Equidimensional Equation

A Cauchy-Euler differential equation is a differential equation of the form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_1 x y^1 + a_\circ y = 0 \tag{1}$$

where $a_0, a_1, ..., a_n$ are constants and $y^{(m)}$ represents an mth derivative of the unknown function of $y(x)$. Considering a second order differential equation of the form:

$$x^2 \zeta''(x) + ax\zeta'(t) + b\zeta = 0 \tag{2}$$

The solution can be obtained through change of variables by applying Euler's formula:

$$x = e^t, \ (\Rightarrow \ t = \ln x \ \& \ xdt = dx) \tag{3}$$

$$\ni \ \zeta'(x) = x^{-1}\zeta'(t) \ \text{ and} \tag{4}$$

$$\zeta''(x) = x^{-2}(\zeta''(t) - \zeta'(t)) \tag{5}$$

substituting the results in (3-5) in (2) yields a second order differential equation with constant coefficients of the form:

$$x^2x^{-2}(\zeta''(t)-\zeta'(t))+axx^{-1}\zeta'(t)+b\zeta=0 \tag{6}$$

This on simplification gives:

$$\zeta''(t)-\zeta'(t)+a\zeta'(t)+b\zeta=0 \tag{7}$$

$$\zeta''(t)+(a-1)\zeta'(t)+b\zeta=0 \tag{8}$$

Equations (7-8) above can be solved using the characteristic polynomial:

$$\varphi^2+(a-1)\varphi+b=0 \tag{9}$$

With roots φ_1 and φ_2 which give the general solution depending on the type of roots it has (or w.r.t. (3) resp.)

Case1: $$\zeta(t)=a_1e^{\varphi_1 t}+a_2e^{\varphi_2 t} \quad (\text{or } \zeta(x)=a_1x^{\varphi_1}+a_2x^{\varphi_2}) \tag{10}$$

Case2: $$\zeta(t)=a_1e^{\varphi_1 t}+a_2te^{\varphi_2 t} \quad (\text{ or } \zeta(x)=a_1x^{\varphi_1}+a_1(\ln x)x^{\varphi_1}) \tag{11}$$

Case (3) $$\begin{gathered}\zeta(t)=e^{\varphi_1 t}(a_1\cos\varphi_2 t+a_2\sin\varphi_2 t)\\ \left(\text{or } \zeta(x)=x^{\varphi_1}(a_1\cos(\varphi_2\ln x)+a_2\sin(\varphi_2\ln x))\right)\end{gathered} \tag{12}$$

3. Analysis of Differential Transformation Method

In illustrating this method, we consider an arbitrary one dimensional function $w(x)$ whose differential transform is defined as:

$$W(k)=\frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{x=0} \tag{13}$$

In the equation above, $w(x)$ is the original function and $W(k)$ is the transformed function. The Taylor series about a point $x=0$ is given as

$$w(k)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{x=0} \tag{14}$$

The inverse differential transform is expressed as:

$$w(x)=\sum_{k=0}^{\infty}W(k)x^k \tag{15}$$

The following theorems are deduced from (13) – (15). The proofs of these are found in [8] and in other related research papers.

i. If $w(x)=v_1(x)\pm v_2(x),$ then $W(k)=V_1(k)\pm V_2(k)$
ii. If $w(x)=\alpha v(x),$ then $W(k)=\alpha V(k),$ where α is a constant
iii. If $w(x)=\dfrac{dv(x)}{dx},$ then $W(k)=(k+1)V(k+1)$
iv. If $w(x)=\dfrac{d^2v(x)}{dx^2},$ then $W(k)=(k+1)(k+2)V(k+2)$

If $w(x)=x^n,$ then $W(k)=\delta(k-n)=\begin{cases}1\text{ , for } k=n\\ 0\text{ , for } k\neq n\end{cases}$, where δ is the kronecker delta.

4. Numerical Results

In this subsection, we apply differential transform method to solve some examples of Cauchy-Euler differential equations of the transformed type.

Problem 1 [2]: Consider the following second order Cauchy-Euler differential equation:

$$3x^2y''(x)-4xy'(x)+2y=0,\quad y(1)=2,\ y'(1)=1 \tag{16}$$

Using the above substitution, (16) is transformed to a second order differential equation with constant coefficients:

$$3y''(t)-7y'(t)+2y=0, y(0)=2, y'(0)=1 \tag{17}$$

The theoretical solution of the equation is:

$$y(t)=\frac{9}{5}e^{\frac{1}{3}t}+\frac{1}{5}e^{2t},\ (x=e^t) \tag{18}$$

Taking the differential transform of (17), we obtain:

$$3(k+1)(k+2)Y(k+2)=7(k+1)Y(k+1)+2Y(k) \tag{19}$$

$$Y(0)=2,\ Y(1)=1 \tag{20}$$

Therefore, the recurrence relation in (19-20) gives the following:

$$Y(2)=\frac{1}{2},\ \text{for}\ \ k=0,$$

$$Y(3)=\frac{5}{18},\ \text{for}\ k=1,$$

$$Y(4)=\frac{29}{216},\ \text{for}\ k=2$$

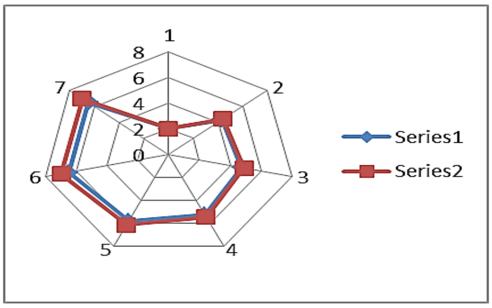
$$Y(5)=\frac{79}{2430},\ \text{for}\ k=3$$

Hence,

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=2+t+\frac{1}{2}t^2+\frac{5}{18}t^3+\frac{29}{216}t^4+\frac{79}{2430}t^5+\cdots\ \ ,(x=e^t) \tag{21}$$

Table I: *Tabular Solutions of problem1*

t	DTM	EXACT	ABS ERR
0.0	2.000000	2.000000	0.000000
1.1	4.323622	4.402158	0.078536
1.2	4.759256	4.889812	0.130556
1.3	5.259388	5.468930	0.209542
1.4	5.832763	6.159201	0.326438
1.5	6.488957	6.984657	0.495700
1.5	6.488957	6.984657	0.495700



Remark 1: In table **I**, a numerical comparison between the exact solution and the approximate solution (up to the term containing t^5 via the DTM) along with the corresponding absolute errors (ABS ERR) is presented. Also, graphical look of such solutions is in figure1, with series1 & series2 for the DTM & the exact solutions respectively.

Problem 2 [2]: Consider the following third order Cauchy-Euler differential equation:

$$x^3y'''(x)+10x^2y''(x)-20xy'+20=0, y(1)=0, y'(1)=-1, y''(1)=1 \tag{22}$$

Equation (22) is transformed to a second order differential equation with constant coefficient:

$$y'''(t)+7y''(t)-28y+20=0, y(0)=0, y'(0)=-1, y''(0)=1 \tag{23}$$

The theoretical solution of (22) is:

$$y(t) = -\frac{3}{4}e^{2t} + \frac{1}{44}e^{-10t} + \frac{8}{11}e^{t}\ ,\ (x = e^{t}) \tag{24}$$

The differential transform of (23) is:

$$\begin{aligned}(k+1)(k+2)(k+3)Y(k+3) = [&28(k+1)Y(k+1) - 20Y(k)\\ &-7(k+1)(k+2)Y(k+2)]\end{aligned} \tag{25}$$

with $Y(0) = 0, Y(1) = -1, Y(2) = \frac{1}{2}$

Therefore, the recurrence relation in (25) gives the following:

$$Y(3) = -\frac{35}{6},\quad \text{for}\ \ k = 0,$$

$$Y(4) = -\frac{293}{24},\ \text{for}\ k = 1,$$

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^{k} = -t + \frac{1}{2}t^{2} - \frac{35}{6}t^{3} + \frac{293}{24}t^{4} + \cdots \tag{26}$$

Table II: *Tabular Solutions of problem2*

t	EXACT	DTM	ABS ERR
0	0	0	0
0.001	-0.001	-0.00099951	4.99E-07
0.002	-0.00200004	-0.00199805	1.99E-06
0.003	-0.00300013	-0.00299566	4.47E-06
0.004	-0.0040003	-0.00399237	7.93E-06
0.005	-0.00500058	-0.00498822	1.24E-05
0.006	-0.006001	-0.00598324	1.78E-05
0.007	-0.00700158	-0.00697747	2.41E-05
0.008	-0.00800235	-0.00797094	3.14E-05

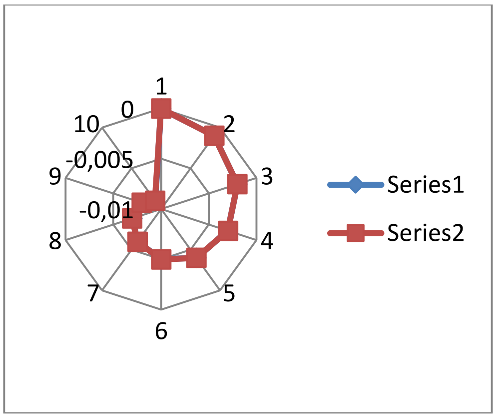


Figure2: *Solution graphs of problem2*

Remark 1: In table **II**, a numerical comparison between the exact solution and the approximate solution (up to the term containing t^5 via the DTM) along with the corresponding absolute errors (ABS ERR) is presented. Also, graphical look of such solutions is in figure**2,** with series1 & series2 for the DTM & the exact solutions respectively.

5. Concluding Remarks

In this paper, Differential transform method has been applied successfully to solve transformed Cauchy-Euler differential equations, whose results can be linked easily with the original Cauchy-Euler's differential equations. The results obtained are in strong agreement with the exact solution. Differential transform method is easy to apply to any kind of differential equation. It does not require any form of linearization, perturbation or initial guess, and hence, it is recommended as an efficient and a reliable numerical-computational method for higher order Cauchy-Euler differential equations in applied sciences.

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