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Mean-Square Response of Simple Mechanical Systems to Nonstationary Random Excitation

This paper concerns the mean-square response of a single-degree-of-freedom system to amplitude modulated random noise. The formulation is developed in terms of the frequency-response function of the system and generalized spectra of the nonstationary random excitation. Both the unit step and rectangular step functions are used for the amplitude modulation, and both white noise and noise with an exponentially decaying harmonic correlation function are considered. The time-varying mean-square response is shown not to exceed its stationary value for white noise. For correlated noise, however, it is shown that the system mean-square response may exceed its stationary value.

Introduction

Solutions to vibration problems which involve random excitation often require a prediction of the mean-square response of the structural system. The basic theory is known [1][1] for calculating the mean-square response of linear systems to both stationary and nonstationary random excitation. For nonstationary inputs, application of the theory gives rise to analytical expressions somewhat more taxing than for the stationary case. We consider here an application of the nonstationary theory to a problem fundamental to earthquake design and gust response predictions.

The nonstationary random excitation is of the form

$$f(t) = e(t)n(t) \qquad (1)$$

where $e(t)$ is a well-defined envelope function and $n(t)$ is Gaussian broadband stationary noise with zero mean. This excitation is a nonstationary process created by the multiplication of sample functions from a stationary process and the deterministic function $e(t)$. The mean-square response of a single-degree-of-freedom mechanical system already has been examined for a unit step envelope function with $n(t)$ assumed as white noise and the system assumed initially at rest [2]. For these same conditions, the mean-square response has been studied for noise correlated as an exponentially decaying harmonic function [3]. In both cases, the response was formulated in terms of the system impulse response and the autocorrelation function of the nonstationary input.

This paper considers an alternate formulation for the mean-square response of a single-degree-of-freedom mechanical system to nonstationary random excitation. It is developed in terms of the system frequency-response function and generalized spectral density function of the input excitation. Both white noise and noise with an exponentially decaying harmonic correlation function are considered. In addition, the white noise and correlated noise results are extended to include a rectangular step envelope function.

[1] Numbers in brackets designate References at end of paper.

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Problem Description

The equation of motion of a single-degree-of-freedom mechanical system is of the form

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{1}{m} f(t) \qquad (2)$$

Nomenclature

$a = \omega_d = \omega_n[1 - \zeta^2]^{1/2}$
$b = \zeta\omega_n$
c = viscous damping coefficient
$E[\ ]$ = expected value of [ ]
$E[y^2(t)]$ = time-varying mean-square response
$e(t)$ = envelope function
$F(\omega)$ = Fourier transform of input force
$F_e(\omega_2 - \omega)$ = frequency-shift Fourier transformation of $e(t_2)$
$f(t)$ = input force excitation
$G_n(\omega)$ = one-sided spectral density function of input noise
$H_0(\omega)$ = system frequency response function
$i = \sqrt{-1}$
$K(t, \omega)$ = modulation function due to unit step envelope
$K_s(t, \omega)$ = modulation function due to rectangular step envelope
k = linear spring constant
m = mass of system
$n(t)$ = input noise
$R_f(t_1, t_2)$ = nonstationary correlation function of input force excitation
$R_n(\tau)$ = autocorrelation function of input noise
$R_y(t_1, t_2)$ = nonstationary correlation function of response
$S_n(\omega)$ = two-sided spectral density function of input noise
$S_f(\omega_1, \omega_2)$ = generalized spectrum of input force excitation
S_0 = magnitude of white noise spectral density
$S_y(\omega_1, \omega_2)$ = generalized spectrum of response
$u(t)$ = unit step function
$Y(\omega)$ = Fourier transform of response
$y(t)$ = displacement response
α = decay coefficient of noise correlation function
$\delta(\omega)$ = delta function at ω equal to zero
ζ = system damping factor
ρ = frequency of noise correlation function
ω_d = system damped natural frequency
ω_n = system natural frequency
$\dot{(\ )} = \frac{d}{dt}(\ )$
$\leftrightarrow$ = a Fourier transform pair

where

$$2\zeta\omega_n = \frac{c}{m}$$

$$\omega_n^2 = \frac{k}{m} \tag{3}$$

$$\zeta = \frac{c}{c_c}$$

Let us assume the system is initially at rest and the input excitation is that of equation (1).

We are to determine the mean-square response $E[y^2(t)]$ when $e(t)$ is a unit step and $n(t)$ has the correlation functions

$$\begin{aligned} \bullet\quad & R_n(\tau) = 2\pi R_0\delta(\tau) \\ \bullet\quad & R_n(\tau) = R_0 e^{-\alpha|\tau|}\cos\rho\tau \end{aligned} \tag{4}$$

where τ is the time difference $t_2 - t_1$. The upper correlation function is that for white noise whereas the lower expression is for the correlated noise. In addition, the mean-square response is to be determined when $e(t)$ is a rectangular step and $n(t)$ has the correlation functions quoted above.

Response Formulation

The autocorrelation function of the system response to the nonstationary force input is given by

$$R_y(t_1, t_2) = E[y(t_1)y(t_2)] \tag{5}$$

where $E[\ ]$ is the expectation of $[\ ]$. Since

$$Y(\omega) = H_0(\omega)F(\omega) \tag{6}$$

where

$$H_0(\omega) = \frac{1}{m\omega_n^2}\cdot\frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + i2\zeta\dfrac{\omega}{\omega_n}}, \tag{7}$$

the response $y(t)$ may be expressed as

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_0(\omega)F(\omega)e^{i\omega t}\,d\omega \tag{8}$$

Upon substitution of equation (8) into (5),

$$R_y(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_y(\omega_1, \omega_2)e^{-i(\omega_1 t_1 - \omega_2 t_2)}\,d\omega_1 d\omega_2 \tag{9}$$

where

$$S_y(\omega_1, \omega_2) = H_0^*(\omega_1)H_0(\omega_2)S_f(\omega_1, \omega_2) \tag{10}$$

with the generalized spectrum of the input excitation defined by

$$S_f(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\,E\,[F^*(\omega_1)F(\omega_2)] \tag{11}$$

Now the mean-square response is

$$E[y^2(t)] = R_y[t, t] \tag{12}$$

so that from equation (9),

$$E[y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0^*(\omega_1)H_0(\omega_2)S_f(\omega_1, \omega_2)e^{-i(\omega_1 - \omega_2)t}\,d\omega_1 d\omega_2 \tag{13}$$

From the Weiner-Khintchine relations,

$$S_f(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} R_f(t_1, t_2)e^{i(\omega_1 t_1 - \omega_2 t_2)}\,dt_1 dt_2 \tag{14}$$

where the nonstationary correlation function of the modulated noise input is

$$R_f(t_1, t_2) = e(t_1)e(t_2)R_n(\tau) \tag{15}$$

Since

$$R_n(\tau) \leftrightarrow S_n(\omega), \tag{16}$$

the generalized spectrum of the input excitation becomes

$$S_f(\omega_1, \omega_2) = \int_{-\infty}^{\infty} S_n(\omega)F_e(\omega - \omega_1)F_e(\omega_2 - \omega)d\omega \tag{17}$$

where the envelope function transformations are

$$\begin{aligned} F_e(\omega - \omega_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e(t_1)e^{-i(\omega - \omega_1)t_1}\,dt_1 \\ F_e(\omega_2 - \omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e(t_2)e^{-i(\omega_2 - \omega)t_2}\,dt_2 \end{aligned} \tag{18}$$

These functions are noted to be conjugate pairs when $\omega_1 = \omega_2$. The substitution of equation (17) into equation (13) produces

$$E[y^2(t)] = \int_{-\infty}^{\infty} S_n(\omega)|I_2(t, \omega)|^2 d\omega \tag{19}$$

where

$$I_2(t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_0(\omega_2)F_e(\omega_2 - \omega)e^{i\omega_2 t}d\omega_2 \tag{20}$$

Equation (19) is the desired general formulation for inputs of amplitude modulated stationary noise.

Unit Step Envelope Function

For a unit step envelope function

$$e(t) = u(t) = 1,\ t \geq 0;\ \text{and zero elsewhere} \tag{21}$$

By the Fourier transformation [4]

$$e^{i\omega_0 t}\,u(t) \leftrightarrow \pi\delta(\omega - \omega_0) + \frac{1}{i(\omega - \omega_0)}, \tag{22}$$

the function $F_e(\omega_2 - \omega)$ becomes

$$F_e(\omega_2 - \omega) = \pi\delta(\omega_2 - \omega) + \frac{1}{i(\omega_2 - \omega)} \tag{23}$$

Upon substitution of this equation into equation (20) and evaluation of the resultant integral,

$$|I_2(t, \omega)|^2 = |H_0(\omega)|^2 K(t, \omega) \tag{24}$$

where

$$\begin{aligned} K(t, \omega) = 1 + A(t) + B(t)\left[\frac{b^2 - a^2 + \omega^2}{a^2}\right] \\ - 2C(t)\cos\omega t - 2D(t)\frac{\omega}{a}\sin\omega t \end{aligned} \tag{25}$$

with the time-dependent coefficients

$$\begin{aligned} A(t) &= e^{-2bt}\left[1 + \frac{b}{a}\sin 2at\right] \\ B(t) &= e^{-2bt}[\sin^2 at] \\ C(t) &= e^{-bt}\left[\cos at + \frac{b}{a}\sin at\right] \\ D(t) &= e^{-bt}[\sin at] \end{aligned} \tag{26}$$

Then, from equation (19), it follows that

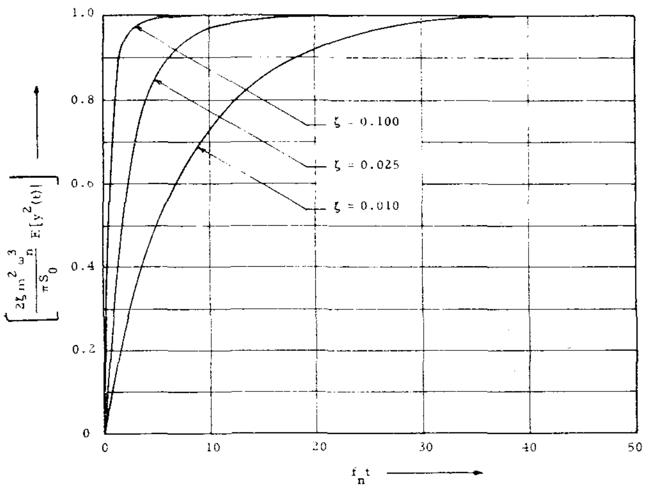

Fig. 1 Normalized mean-square response to white noise modulated by a unit step function

$$E[y^2(t)] = \int_{-\infty}^{\infty} |H_0(\omega)|^2 S_n(\omega) K(t, \omega) d\omega. \tag{27}$$

With $K(t, \omega) = 1$, this expression reduces to the mean-square formulation for stationary inputs.

White Noise Input. If the input noise is assumed white, then the spectral density function $S_n(\omega)$ reduces to the constant S_0 and the mean-square response becomes

$$E[y^2(t)] = S_0 \int_{-\infty}^{\infty} |H_0(\omega)|^2 K(t, \omega) d\omega \tag{28}$$

By residue theory and a bit of algebra,

$$E[y^2(t)] = \frac{\pi S_0}{2\zeta m^2 \omega_n^3} \times \left[1 - e^{-2bt}\left(1 + \frac{b}{a}\sin 2at + 2\left(\frac{b}{a}\right)^2 \sin^2 at\right)\right] \tag{29}$$

This result agrees with that derived in [1].

A normalized plot of equation (29) is shown as Fig. 1. It is a family of curves in ζ where $f_n t$ may be interpreted as the number of response cycles of the system. Since no curve exceeds unity, it is concluded that this time-varying mean-square response does not exceed the stationary mean-square response to white noise. One application of these curves is to establish stationarity criteria for the response of mechanical systems with one degree-of-freedom in a white noise environment.

Correlated Noise Input. If the input noise is assumed correlated as the damped harmonic of equation (4), then, by the transformation

$$S_n(\omega) \leftrightarrow R_n(\tau),$$

the two-sided spectral density becomes

$$S_n(\omega) = \frac{R_0}{\pi} \cdot \frac{\alpha(\alpha^2 + \rho^2 + \omega^2)}{(\omega^2 - s_3^2)(\omega^2 - s_4^2)} \tag{30}$$

where

$$s_3 = \rho + i\alpha$$

$$s_4 = -s_3^*$$

For white noise,

$$S_0 = \lim_{\alpha \to \infty} \alpha S_n(\omega) = \frac{R_0}{\pi} \tag{31}$$

This relationship is useful for checking purposes, as will be shown

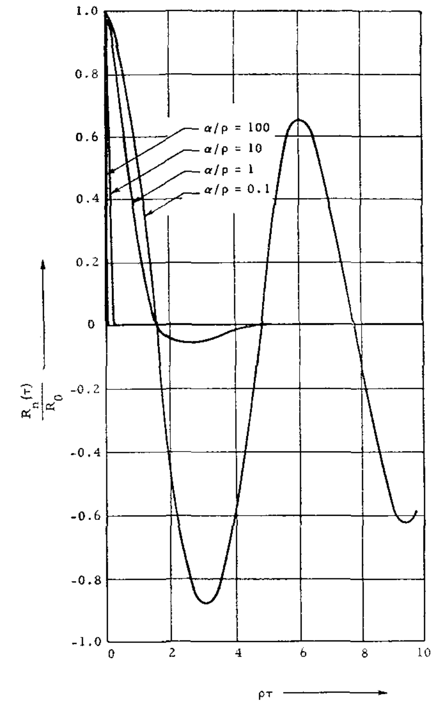

Fig. 2 Normalized autocorrelation functions of the correlated noise

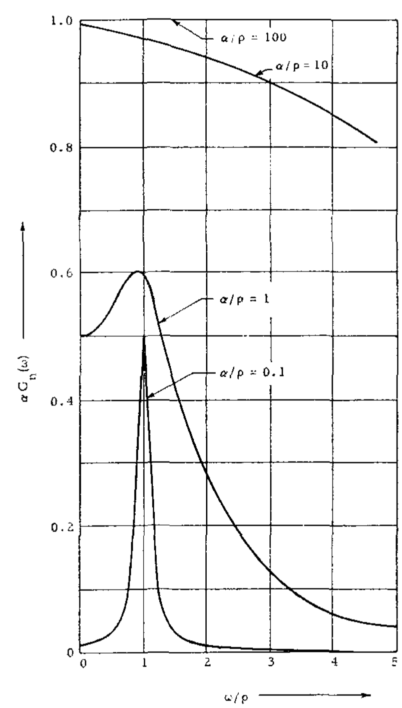

Fig. 3 Normalized spectral density functions of the correlated noise

subsequently. Normalized plots of $R_n(\tau)$ and the associated spectral density[2] $G_n(\omega)$ are shown as Figs. 2 and 3, respectively.

Upon substitution of equation (30) into equation (27) and evaluation of the resultant integral,

$$E[y^2(t)] = \frac{R_0}{m^2}[R_1T_1 - X_1T_2 + R_3T_3 - X_3T_4] \tag{32}$$

where the time-dependent terms are written as

[2] For convenience in plotting, the one-sided spectral density $G_n(\omega)$ is used. By folding $S_n(\omega)$ about $\omega = 0$, $G_n(\omega) = 2S_n(\omega)$.

$$T_1 = \frac{a}{2b}[1 - A(t)]$$

$$T_2 = -B(t)$$

$$T_3 = \left[1 + A(t) + \frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2} B(t) - 2[C(t) + \frac{\alpha}{a} D(t)]e^{-\alpha t}\cos\rho t - 2\frac{\rho}{a} D(t)e^{-\alpha t}\sin\rho t\right] \quad (33)$$

$$T_4 = 2\left[\frac{\rho\alpha}{a^2} B(t) - [C(t) + \frac{\alpha}{a} D(t)]e^{-\alpha t}\sin\rho t + \frac{\rho}{a} D(t)e^{-\alpha t}\cos\rho t\right].$$

The remaining coefficients R_1, R_3, X_1, and X_3 are given by

$$R_1 = \mathrm{Re}\left[\frac{(\rho^2 + \alpha^2 + s_1^2)}{s_1(s_1^2 - s_3^2)(s_1^2 - s_4^2)}\right]\cdot\frac{\alpha}{a^2}$$

$$R_3 = \mathrm{Re}\left[\frac{1}{(s_3^2 - s_1^2)(s_3^2 - s_2^2)}\right]$$

$$X_1 = \mathrm{Imag}\left[\frac{(\rho^2 + \alpha^2 + s_1^2)}{s_1(s_1^2 - s_3^2)(s_1^2 - s_4^2)}\right]\cdot\frac{\alpha}{a^2} \quad (34)$$

$$X_3 = \mathrm{Imag}\left[\frac{1}{(s_3^2 - s_1^2)(s_3^2 - s_2^2)}\right]$$

After much algebra, this mean-square response may be rewritten as

$$E[y^2(t)] = \frac{R_0}{m^2}[A_0 + e^{-2bt}(A_1 + A_2\sin 2at + A_3\sin^2 at) + e^{-(\alpha+b)t}(A_4\cos at\cos\rho t + A_5\sin at\cos\rho t) + A_6\cos at\sin\rho t + A_7\sin at\sin\rho t] \quad (35)$$

where

$$A_0 = \left[\frac{a}{2b}\right] R_1 + R_3$$

$$A_1 = -\left[\frac{a}{2b}\right] R_1 + R_3$$

$$A_2 = -\frac{R_1}{2} + \frac{b}{a} R_3$$

$$A_3 = \frac{1}{a^2}\left[a^2 X_1 + \frac{2(b^2 - a^2 + \rho^2 - \alpha^2)}{a^2} R_3 - \frac{4\rho\alpha}{a^2} X_3\right] \quad (36)$$

$$A_4 = -2R_3$$

$$A_5 = -\frac{2}{a}[(\alpha + b)R_3 + \rho X_3]$$

$$A_6 = 2X_3$$

$$A_7 = -\frac{2}{a}[\rho R_3 - (b + \alpha)X_3]$$

which is a form (perhaps) more suitable for interpretation. By the limiting operation,

$$\lim_{\alpha\to\infty} \alpha E[y^2(t)]$$

equations (32) and (35) both reduce to the white noise result given by equation (29).

Both of these expressions show the system mean-square response is dependent upon interrelationships which involve the system damping ζ, the system natural frequency f_n, the correlation function decay constant α, and the correlation function frequency ρ. For a relatively large number of response cycles, the exponential decay terms in equation (35) tend to zero and the mean-square response reduces to the stationary value

$$E[y^2(t)]_{t\to\infty} = \frac{R_0}{m^2}\left[\left(\frac{a}{2b}\right) R_1 + R_3\right] \quad (37)$$

Figs. 4–11 are normalized plots of equation (32). Figs. 4–9 depict the behavior of the system rms response plotted as a family of curves in a/ρ for specific values of Q and α/b. Figs. 10 and 11 show the mean-square response plotted as families of curves in a/b for $a/\rho = 1$ and $Q = 5$ and 50, respectively. The ratio a/ρ provides a relative comparison between the damped natural frequency of the system and the frequency of the noise correlation function. The ratio α/b provides a relative comparison between the exponential decay constant of the noise correlation function and the decay coefficient associated with the system response to a unit impulse function. The horizontal dashed lines are the stationary values.

Such plots show that the damping values influence the magnitude of the stationary value and affect how quickly stationarity is attained. The larger damping values result in lower stationary values and allow the response to become stationary in a lesser number of response cycles.

The ratios a/ρ and α/b both influence the rms response overshoot of the stationary value. For $\alpha/b = 10$, the largest value considered for this ratio, no overshoot occurs over the range $0.5 \leq a/\rho \leq 10$. This suggests that the response overshoot is relatively insensitive to the ratio α/p when $\alpha/b \geq 10$. Thus, for white noise (α becomes infinite in the limit), no overshoot is expected and such is verified by the plots in Fig. 1. For $\alpha/b = 0.1$ and $\alpha/b = 1$, overshoot generally is observed.

Figs. 10 and 11 indicate the effect on the system rms response of variations in α/b over the range $0.1 \leq \alpha/b \leq 10$ when $a = \rho$. Since the frequency of the noise correlation function equals the damped natural frequency of the system for these curves, the ratio $\alpha/b = 1$ implies a perfect overlap of the noise correlation function and the system unit impulse response. For this condition, the response buildup is the most rapid of the three curves, although the asymptotic (or stationary) value is intermediate to the values for $\alpha/b = 0.1$ and $\alpha/b = 10$.

Rectangular Step Envelope Function

For a rectangular step envelope function of duration t_0,

$$e(t) = u(t) - u(t - t_0) \quad (38)$$

so that

$$F_2(\omega_2 - \omega) = [1 - e^{-i(\omega_2-\omega)t_0}]\left[\pi\delta(\omega_2 - \omega) + \frac{1}{i(\omega_2 - \omega)}\right] \quad (39)$$

By substitution of this expression into equation (20) and evaluation of the resultant integral,

$$|I_2(t,\omega)|^2 = |H_0(\omega)|^2\Bigg\{K(t,\omega)u(t) + \Bigg(-K(t,\omega) + A(t) + A(t - t_0) + \left[\frac{b^2 - a^2 + \omega^2}{a^2}\right][B(t) + B(t - t_0)] - 2\left[C(t)C(t - t_0) + \left(\frac{\omega}{a}\right)^2 D(t)D(t - t_0)\right] \times \cos\omega t_0 + 2\frac{\omega}{a}[C(t)D(t - t_0) - C(t - t_0)D(t)]\sin\omega t_0\Bigg) \times u(t - t_0)\Bigg\} \quad (40)$$

From equation (19),

$$E[y^2(t)] = \int_{-\infty}^{\infty} |H_0(\omega)|^2 S_n(\omega)K(t,\omega)d\omega \qquad \text{for } 0 \leq t \leq t_0$$

$$E[y^2(t)] = \int_{-\infty}^{\infty} |H_0(\omega)|^2 S_n(\omega)K_s(t,\omega)d\omega \qquad \text{for } t \geq t_0 \quad (41)$$

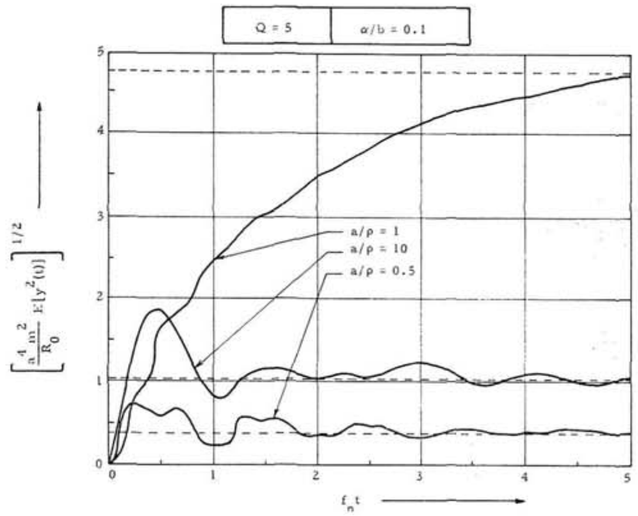

Fig. 4 Normalized rms response to the correlated noise modulated by a unit step function

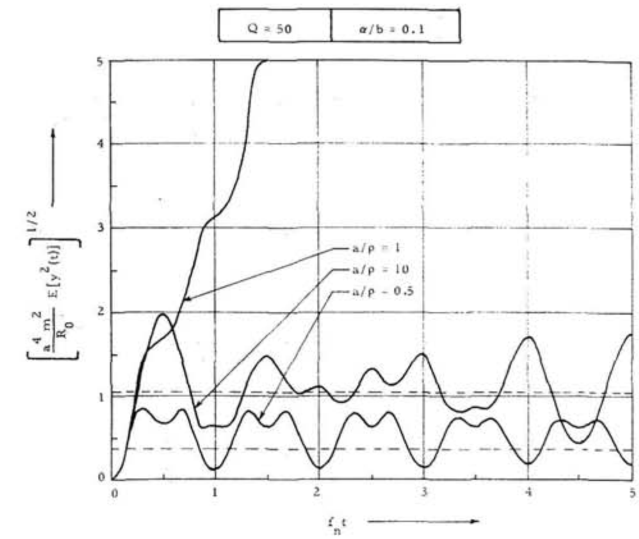

Fig. 7 Normalized rms response to the correlated noise modulated by a unit step function

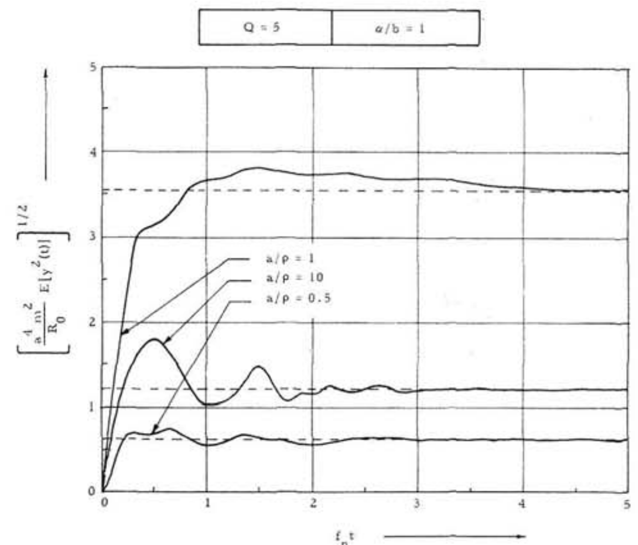

Fig. 5 Normalized rms response to the correlated noise modulated by a unit step function

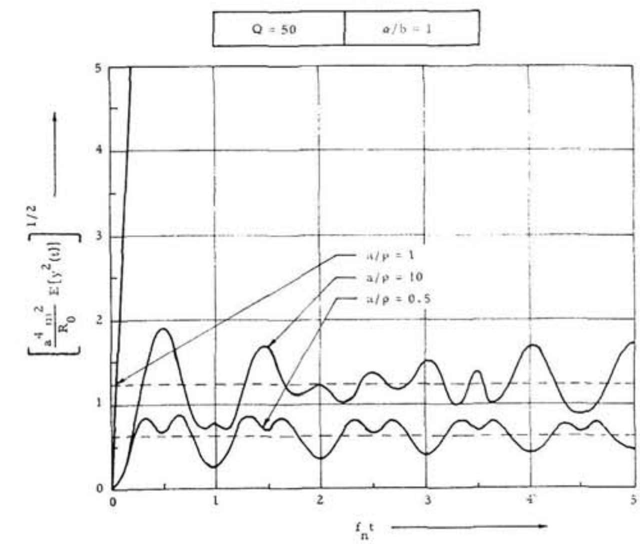

Fig. 8 Normalized rms response to the correlated noise modulated by a unit step function

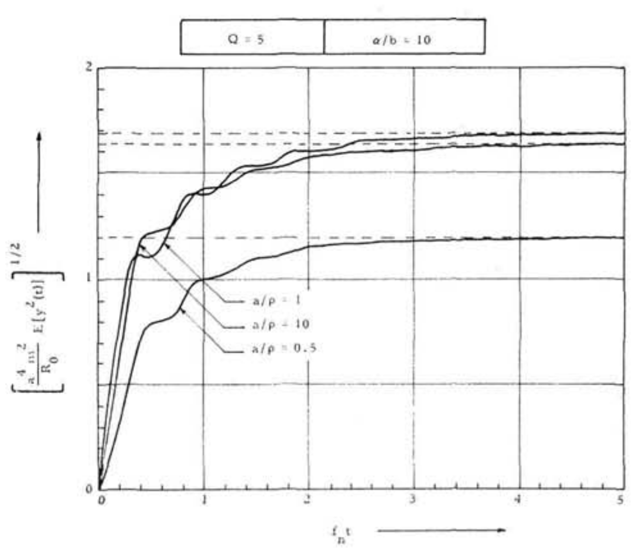

Fig. 6 Normalized rms response to the correlated noise modulated by a unit step function

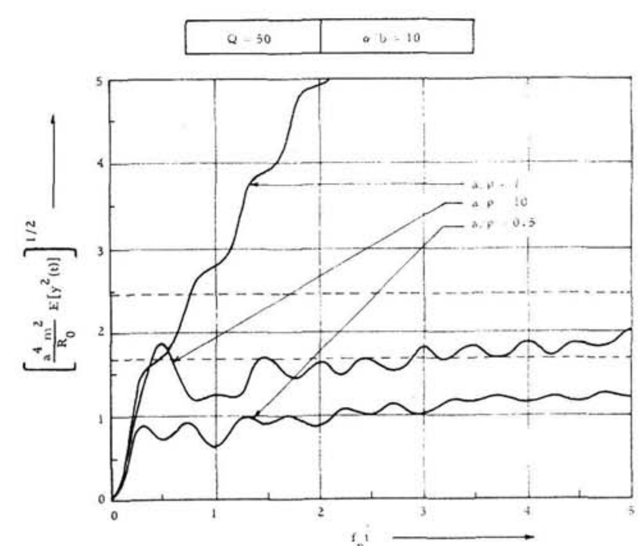

Fig. 9 Normalized rms response to the correlated noise modulated by a unit step function

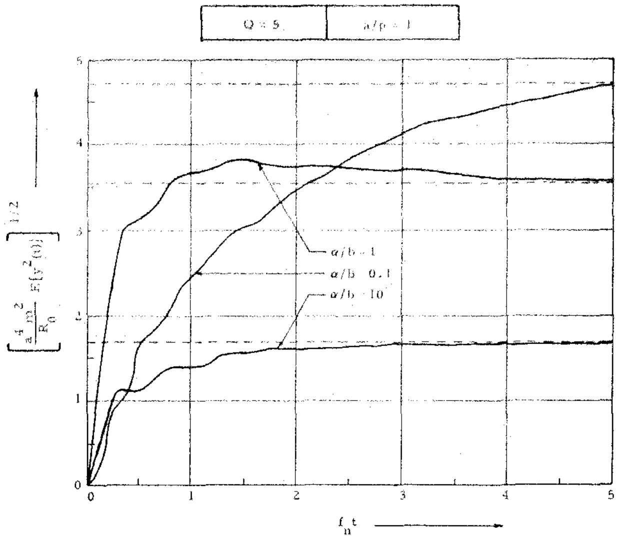

Fig. 10 Normalized rms response to the correlated noise modulated by a unit step function

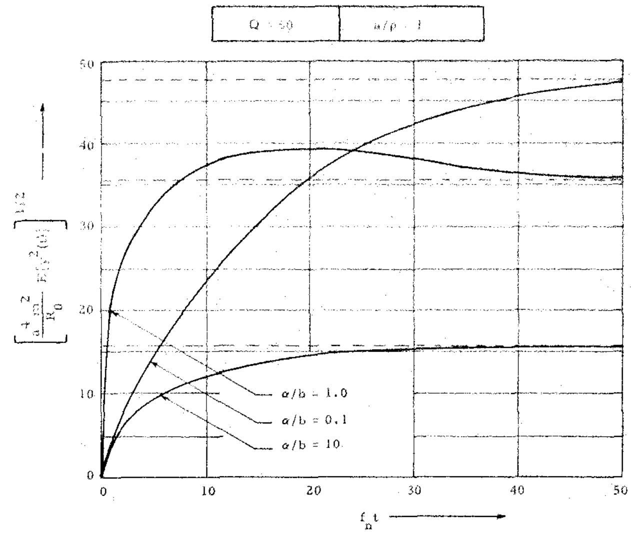

Fig. 11 Normalized rms response to the correlated noise modulated by a unit step function

where $K(t, \omega)$ is given by equation (25) and

$$K_s(t, \omega) = A(t) + A(t - t_0) + \frac{b^2 - a^2 + \omega^2}{a^2} \times [B(t) + B(t - t_0)] - 2\left[C(t)C(t - t_0) + \frac{\omega^2}{a^2} D(t)D(t - t_0)\right] \times \cos \omega t_0 + 2\,\frac{\omega}{a}[C(t)D(t - t_0) - C(t - t_0)D(t)] \sin \omega t_0 \quad (42)$$

White Noise Input. If the input noise is assumed white, then

$$E[y^2(t)] = S_0 \int_{-\infty}^{\infty} |H_0(\omega)|^2 K(t, \omega) d\omega \qquad \text{for } 0 \leq t \leq t_0$$

$$E[y^2(t)] = S_0 \int_{-\infty}^{\infty} |H_0(\omega)|^2 K_s(t, \omega) d\omega \qquad \text{for } t \geq t_0 \quad (43)$$

Since the first integral is precisely equation (28),

$$E[y^2(t)] = \frac{\pi S_0}{2\zeta m^2 \omega_n^3}\left[1 - e^{-2bt}\left(1 + \frac{b}{a} \sin 2\, at + 2\left(\frac{b}{a}\right)^2 \sin^2 at\right)\right] \qquad \text{for } 0 \leq t \leq t_0 \quad (44)$$

By residue theory and much algebra,

$$E[y^2(t)] = \frac{\pi S_0}{2\zeta m^2 \omega_n^3}\left\{A(t) + A(t - t_0) + 2\left(\frac{b}{a}\right)^2 \times [B(t) - B(t - t_0)] - 2\left[C(t)C(t_0) + \left(\frac{\omega_n}{a}\right)^2 D(t)D(t_0)\right] \times C(t - t_0) + 2\left(\frac{\omega_n}{a}\right)^2\left[2\,\frac{b}{a} D(t)D(t_0) - D(t)C(t_0) + C(t)D(t_0)\right] \times D(t - t_0)\right\} \qquad \text{for } t \geq t_0 \quad (45)$$

Correlated Noise Input. If the input noise is assumed correlated as the damped harmonic of equation (4), then $S_n(\omega)$ is given by equation (30). After some algebraically tedious integrations by residue theory, equation (19) yields

$$E[y^2(t)] = \frac{R_0}{m^2}[R_1T_1 - X_1T_2 + R_3T_3 - X_3T_4] \qquad \text{for } 0 \leq t \leq t_0$$

$$E[y^2(t)] = \frac{R_0}{m^2}[R_1T_a - X_1T_b + R_3T_c - X_3T_d] \qquad \text{for } t \geq t_0 \quad (46)$$

where

$$T_a = \frac{a}{2b}\left\{[A(t) + A(t - t_0)] - 2\left[\left(C(t) + \frac{b}{a} D(t)\right)C(t - t_0) - \left(\frac{b}{a} C(t) - \frac{a^2 - b^2}{a^2} D(t)\right)D(t - t_0)\right]C(t_0) + 2\left[\left(\frac{b}{a} C(t) + \frac{b^2 - a^2}{a^2} D(t)\right)C(t - t_0) - \left(\frac{b^2 - a^2}{a^2} C(t) - \frac{b(3a^2 - b^2)}{a^3} D(t)\right)D(t - t_0)\right]D(t_0)\right\} \quad (47a)$$

$$T_b = \frac{a}{b}\left\{\frac{b}{a}[B(t) + B(t - t_0)] + \left[D(t)C(t - t_0) - \left(C(t) + \frac{2b}{a} D(t)\right)D(t - t_0)\right]C(t_0) - \left[\left(C(t) + \frac{2b}{a} D(t)\right)C(t - t_0) - \left(\frac{2b}{a} C(t) + \frac{3b^2 - a^2}{a^2} D(t)\right)D(t - t_0)\right]D(t_0)\right\} \quad (47b)$$

$$T_c = A(t) + A(t - t_0) + \left(\frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}\right)[B(t) + B(t - t_0)] - 2\left[\left(C(t) + \frac{\alpha}{a} D(t)\right)C(t - t_0) - \left(\frac{\alpha}{a} C(t) - \frac{\rho^2 - \alpha^2}{a^2} D(t)\right) D(t - t_0)\right] e^{-\alpha t_0} \cos \rho t_0 - \frac{2\rho}{a}\left[D(t)C(t - t_0) - \left(C(t) + \frac{2\alpha}{a} D(t)\right)D(t - t_0)\right] \times e^{-\alpha t_0} \sin \rho t_0 \quad (47c)$$

$$T_d = 2\left\{\frac{\rho\alpha}{a^2}[B(t) + B(t - t_0)] - \left[\left(C(t) + \frac{\alpha}{a} D(t)\right) C(t - t_0)\right.\right. \quad (47d)$$

$$- \left(\frac{\alpha}{a} C(t) - \frac{\rho^2 - \alpha^2}{a^2} D(t)\right) D(t - t_0)\Bigg] e^{-\alpha t_0} \sin \rho t_0 \quad (47d) \; (Cont.)$$

$$+ \frac{\rho}{\alpha} \left[D(t)C(t - t_0) - \left(C(t) + \frac{2\alpha}{a} D(t) \right) D(t - t_0) \right] \times e^{-\alpha t_0} \cos \rho t_0 \Bigg\}$$

The buildup of the mean-square response $E[y^2(t)]$ over $0 \leq t \leq t_0$ is the same as that for a unit step envelope function and, consequently, additional plots are not necessary. Although cumbersome in appearance, the $E[y^2(t)]$ for $t > t_0$ reduces to the square of an exponentially decaying harmonic function with frequency $2\omega_d$. The decay exponent is governed by the product of the system damping and natural frequency.

Concluding Remarks

The mean-square response of a linear single-degree-of-freedom mechanical system to amplitude modulated noise is nonstationary. A general formulation is presented for calculating this response. It is developed in terms of the system frequency-response function and generalized spectrum of the input excitation. A unit step modulation and a rectangular step modulation are considered in conjunction with both correlated and perfectly uncorrelated noise of zero mean.

This time-varying response is dependent upon the system damping and natural frequency, the shape of the modulation function, and the parameters of the noise correlation function. For white noise modulated by a unit step function, the system mean-square response will not overshoot its stationary value. For correlated noise modulated in this same way, the system mean-square response may exceed its stationary value. These same comments are pertinent for the rectangular step modulation.

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