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Step Responses of Liquid Lines With Frequency-Dependent Effects of Viscosity

Universal step-response plots are given from theoretical considerations for rigid cylindrical lines containing a compressible Newtonian laminar-flow liquid. The pressure and flow step inputs and pressure and flow outputs for semi-infinite lines can, with the principle of superposition, be used to estimate the responses of a network of lines, terminations, and so on, for any transient input. Where possible, analytic expressions were found for these step responses, but in a certain region of each curve complex numerical routines based on the analytic frequency response were necessary. Analytical expressions are based on propagation and characteristic impedance operators published earlier by one of the authors.

Introduction

The primary purpose of this paper is to present complete plots of the step responses of liquid-filled lines as an aid to designers of fluid systems. The secondary purpose is to place these results in context by briefly reviewing other available design aids. The tertiary purpose is to very briefly indicate the analytic techniques which were used, and which are detailed elsewhere for anyone who might wish to check the results or solve related problems.

The effective wall-shear resistance to fluid flow in a pipe increases monotonically with the frequency of the excitation. This well-known fact can be understood in terms of the velocity profiles. For laminar flow at low frequency the profile is parabolic, whereas at high frequency it approaches a slug-flow distribution with a thin boundary-layer. Since *resistance* is defined as the ratio of axial pressure-gradient changes to flow rate changes, a thin boundary-layer implies a high resistance.

Iberall [1][1] and Nichols [2] derived the correct frequency-dependent propagation function and characteristic impedance function, which adequately describe the phenomenon, assuming small disturbances in a laminar flow in a rigid tube with either a Newtonian liquid or Stokesian perfect gas. The primary or longitudinal mode of motion alone was considered and corresponding limitations stated, but for pipes with a large length-to-diameter ratio the higher modes are rarely of interest.

Simultaneous with Nichols one of the present authors derived the following Laplacian propagation and characteristic impedance *operators* [3], using identical assumptions:

$$\Gamma(s) = T_0 s\left[1 - \frac{2J_1(ja\sqrt{s/\nu_0})}{ja\sqrt{s/\nu_0}\,J_0\,(ja\sqrt{s/\nu_0})}\right]^{-1/2} \qquad (1)$$

$$z_c(s) = Z_{c0}\Gamma(s) \qquad (2)$$

Substitution of the imaginary frequency $j\omega$, for the Laplacian operator s, gives identically Iberall's and Nichols' results, whereas the reverse substitutions are not evident. These results, which are only for liquids,[2] contain the *nominal* delay time T_0 for a wave to travel from the upstream station to the downstream station at the unimpeded wave speed

$$c_0 = \sqrt{\beta/\rho} \qquad (3)$$

where β and ρ are the effective bulk modulus[3] and density of the fluid, respectively; the Bessel functions J_0 and J_1; the internal radius of the pipe a; the kinematic viscosity of the fluid ν_0; and the nominal value of the surge impedance[4]

[1] Numbers in brackets designate References at end of paper.

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[2] Dispersion in air is considerably greater than in a liquid as a result of heat transfer; for results see references [1–5].

[3] The entire analysis can be applied with small error to nonrigid pipes by appropriately decreasing the effective bulk modulus β.

[4] Some authors use volume flow rather than mass flow and consequently introduce the density ρ into the numerator of equation (4).

Nomenclature

a = radius of pipe, in.
b = coefficients of solution to problem, Fig. 13
c_0 = isentropic speed of sound, in/sec
e = 2.718...
F = integrand for step response
g_k = coefficients of low-frequency series approximations of Γ and Z_c
H = frequency-dependent transfer function, lb/in.² or lb sec/in.
J_0, J_1 = Bessel functions of first kind
j = unit imaginary number
L = distance between upstream and downstream stations, in.
p = pressure, lb/in.²
p_u, p_d = upstream and downstream pressures, respectively, lb/in.²
q = pressure response, Fig. 13, lb/in.²
r = pressure response, Fig. 13, lb/in.²
s = Laplace operator, sec⁻¹
T_0 = nominal delay time between stations for waves traveling at speed c_0, sec
t = time from generation of step, sec
w_u, w_d = upstream and downstream flows, respectively, lb sec/in.
Z_c = characteristic or surge impedance of line, 1/in-sec
Z_{c0} = nominal value of characteristic impedance at high frequencies, 1/in-sec
β = bulk modulus of fluid, lb/in.²
Γ = propagation function or operator
ϕ = step response (pressure or flow)
ξ = complex number
ν_0 = kinematic viscosity of liquid, in²/sec
ρ = density of liquid, lb sec²/in.⁴
τ = nondimensional time; see equation (13)
τ_0 = nondimensional delay time; see equation (6)
Ω = nondimensional frequency; see equation (5)
ω = frequency, sec⁻¹

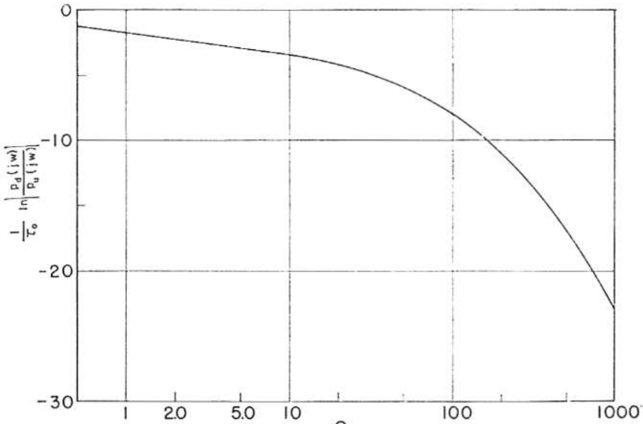

Fig. 1 Attenuation of pressure or flow sinusoidal disturbances

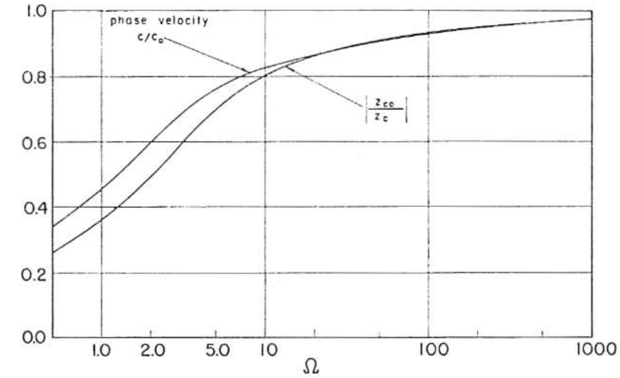

Fig. 2 Phase velocity and magnitude of characteristic impedance

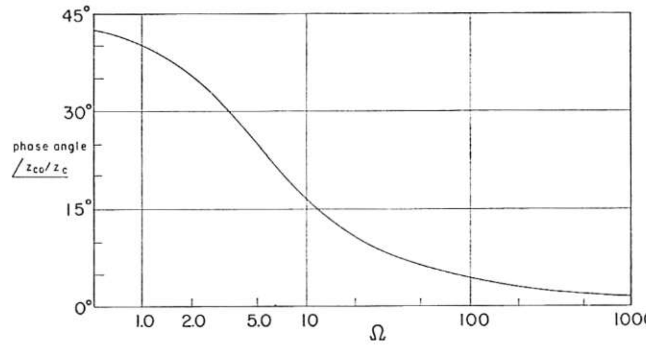

Fig. 3 Phase of characteristic impedance

$$Z_{c0} = \frac{c_0}{\pi a^2} \tag{4}$$

For completeness the frequency-response results for liquids, as calculated by the authors, are plotted in Figs. 1–3. Nondimensionalizations include the frequency

$$\Omega = \frac{a^2\omega}{\nu_0} \tag{5}$$

and the delay time

$$\tau_0 = \frac{\nu_0 T_0}{a^2} \tag{6}$$

The ordinate of Fig. 1 is proportional to the log amplitude ratio of the downstream to upstream pressures or flows, which is related to the propagation function $\Gamma(j\omega)$ by

$$\ln\left|\frac{p_d\,(j\omega)}{p_u\,(j\omega)}\right| = -\ \mathrm{Re}\ \Gamma(j\omega) \tag{7}$$

This applies for waves in one direction only, which implies perhaps a semi-infinite line, or at least no reflection of waves. The phase relationship, given in Fig. 2, is also related to the propagation function:

$$\text{phase angle}\ \angle \frac{p_d(j\omega)}{p_u(j\omega)} = \mathrm{Im}\ \Gamma\ (j\omega) \tag{8}$$

Also given in Figs. 2 and 3 are the magnitude and phase angle of the *characteristic or surge impedance* Z_c which represents the actual impedance seen by a pressure or flow source.

The advantage of the Laplacian forms $\Gamma(s)$ and $Z_c(s)$ is the ability to calculate *transient* responses from them. The initial portions of the step and impulse responses, for both liquids and air, were given in the same paper [3]. More complete results are also available [4, 5]. These calculations were based on high-frequency approximations to equations (1) and (2), as shown in Appendix A. Unfortunately, these series become inexact when

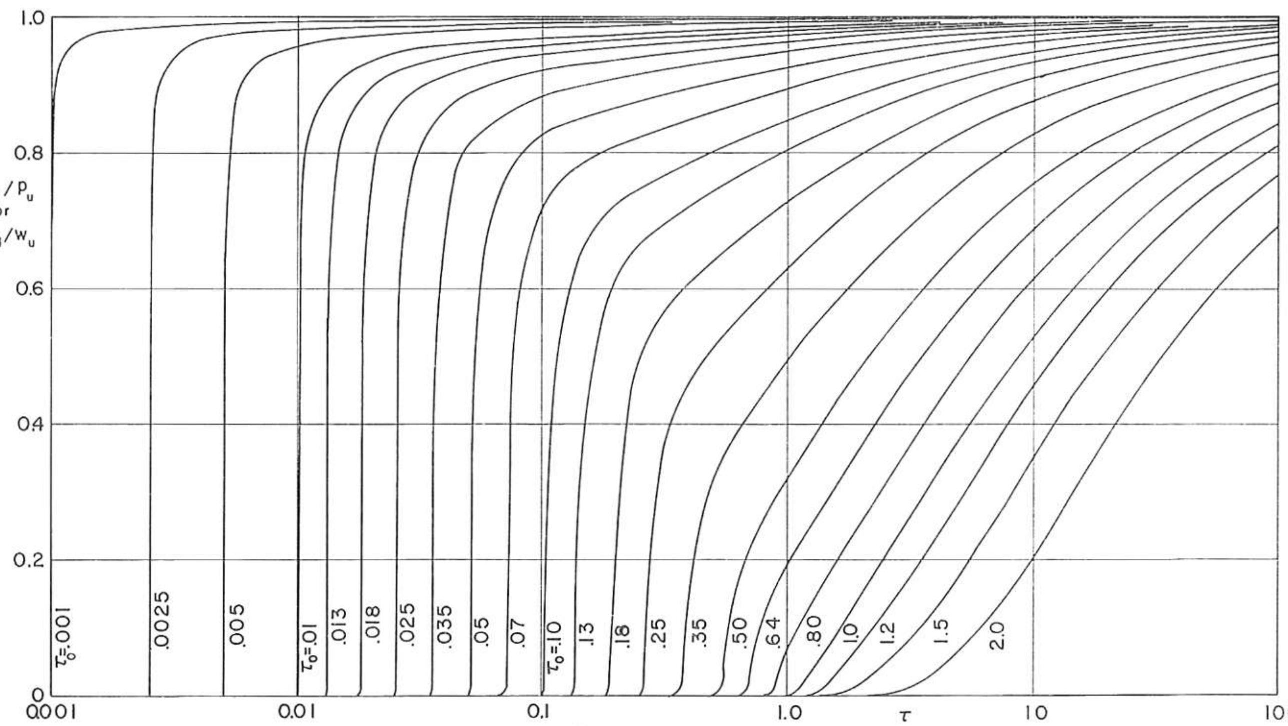

Fig. 4 Pressure (flow) responses to step of pressure (flow)

extended further, and significant regions of the step responses remain uncalculated.

The next step was to utilize exact low-frequency approximations for $\Gamma(s)$ and $Z_c(s)$, as also given in Appendix A. This allowed a large portion of the tails of the step responses to be calculated, but serious convergence limitations left a significant gap between the calculated "heads" and "tails" of the responses, corresponding to a middle-frequency range for which neither approach was useful. As a last resort it was suggested [4] that the known and easily calculable frequency-response results be converted to the step response in this difficult region.

Implementation of this idea was based on results given by Leonhard [6]. The response to a step input is

$$\phi(t) = \int_0^\infty \frac{2}{\pi} \frac{\mathrm{Re}H(j\omega)}{\omega} \sin \omega t \, d\omega \tag{9}$$

where $H(j\omega)$ is the appropriate frequency-dependent transfer function. The results herein for the "gap" region were calculated numerically, using a digital computer, by replacing $d\omega$ with appropriate discrete $\Delta\omega$, summing over an appropriate range, and so on. Details of what proved to be a very complex and tricky business are discussed briefly in Appendix A, and more thoroughly in a thesis by one of the authors [7]. Interestingly, the "gap" region is the only one in which this numerical approach was feasible.

Results

The results as given herein are step responses for uniform *semi-infinite* lines; that is, waves are traveling in one direction only, with no reflections from terminations, and so on. Of course, most physical applications of these results do in fact involve reflections, but the basic linearity allows use of the principle of superposition to account for these effects. Moreover, any arbitrary disturbance can be subdivided into a series of steps with any desired degree of refinement. (Steps, incidentally, are usually more convenient than impulses, since a period of constant pressure or flow can be represented by a *single* step.)

Four basic types of responses are of interest:

(*a*) The pressure response downstream to an upstream step of pressure.

(*b*) The flow response downstream to an upstream step of flow.

(*c*) The flow response to a step of pressure.

(*d*) The pressure response to a step of flow.

Types (*a*) and (*b*) have *identical* responses, given by the inverse transform of

$$\frac{1}{s} e^{-\Gamma(s)} \tag{10}$$

Types (*c*) and (*d*) are given by, respectively,

$$\frac{1}{sZ_c(s)} e^{-\Gamma(s)} \tag{11}$$

$$\frac{Z_c(s)}{s} e^{-\Gamma(s)} \tag{12}$$

All responses are given as a function of the nondimensional time

$$\tau \equiv \frac{\nu_0 t}{a^2} \tag{13}$$

Types (*a*) and (*b*) are given in Fig. 4. For short distances downstream, the response is nearly a step, with a slight rounding. Further downstream, an extraordinarily long tail becomes very

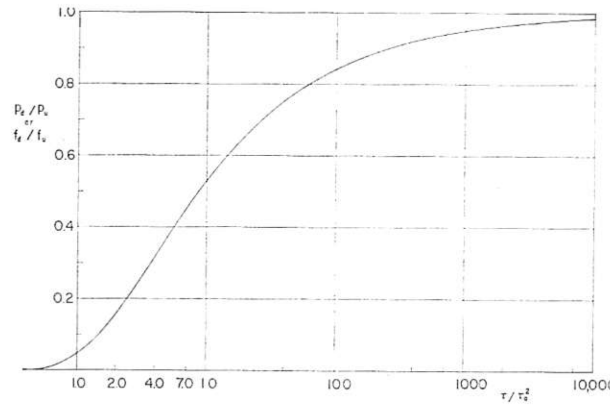

Fig. 5 Pressure (flow) responses to step of pressure (flow), asymptotic results as $\tau_0 \to \infty$ (generally valid when $\tau_0 > 2$)

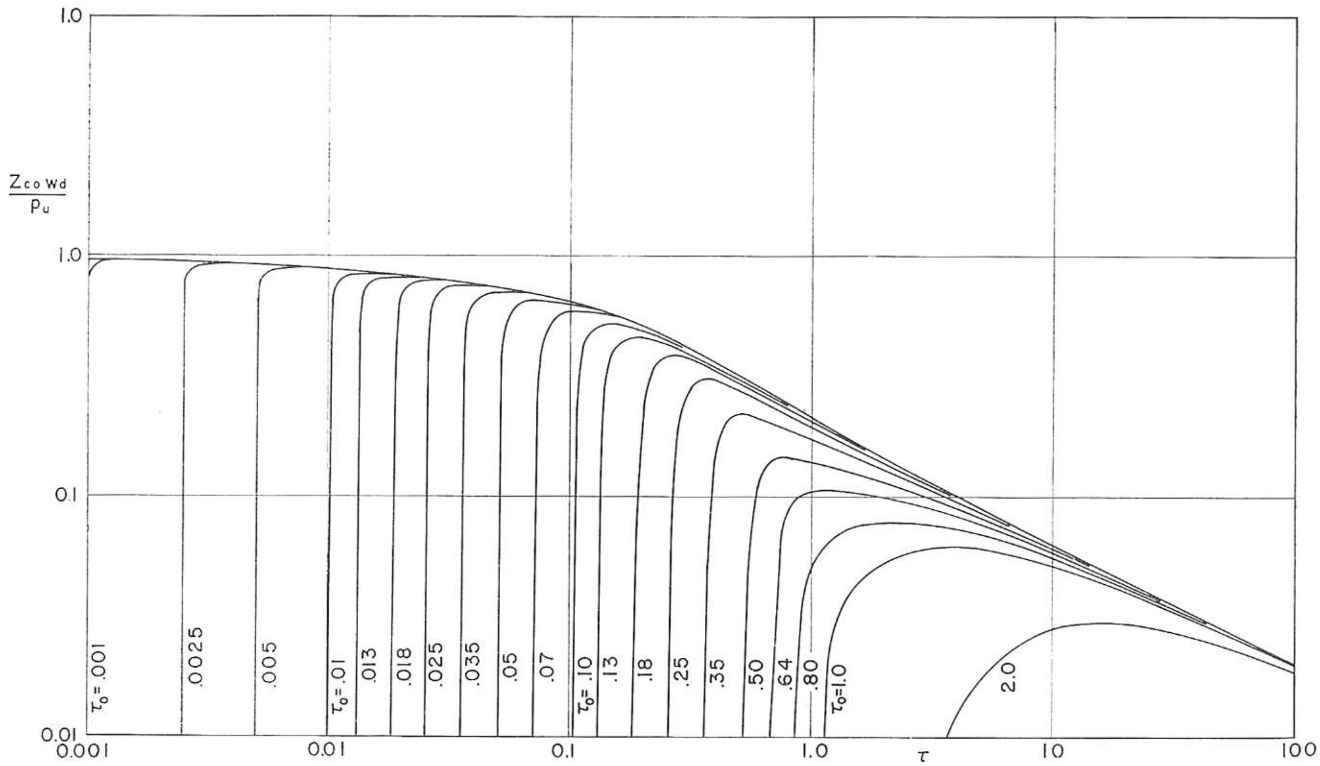

Fig. 6 Flow responses to step of pressure

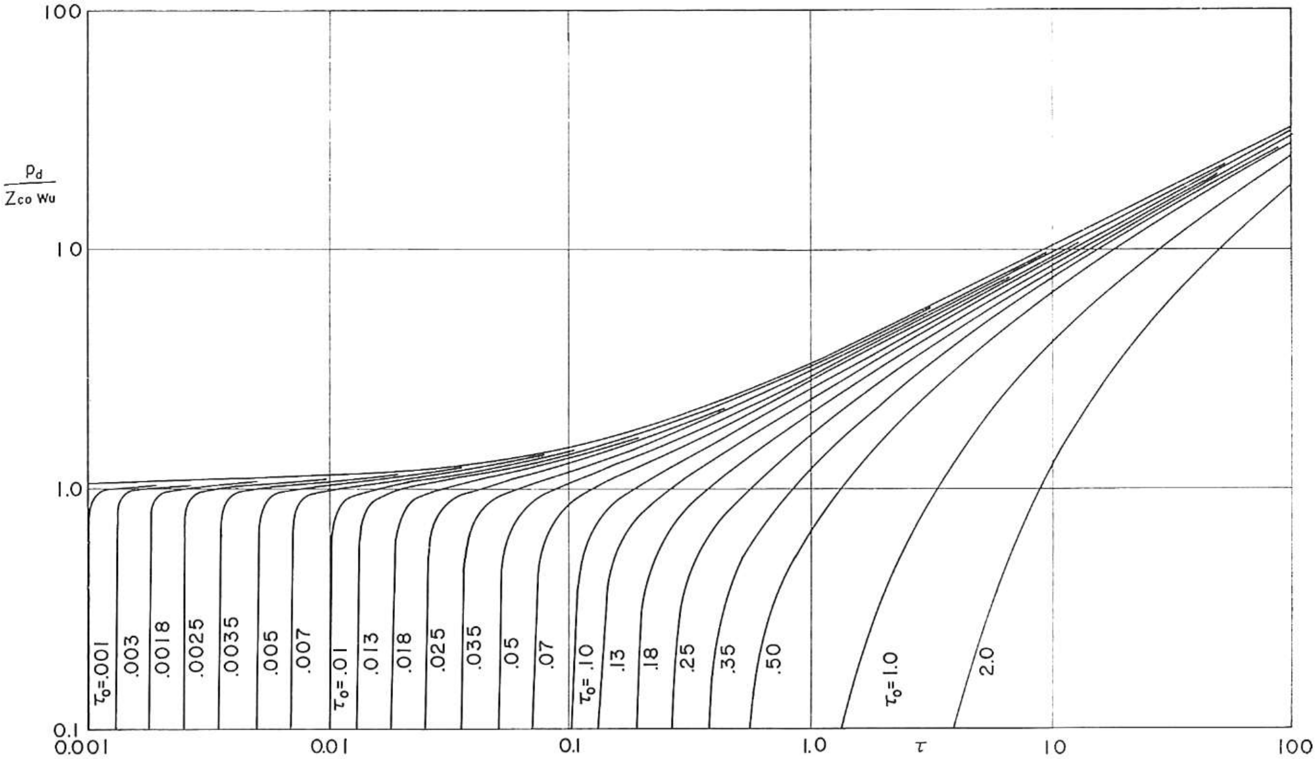

Fig. 7 Pressure responses to step of flow

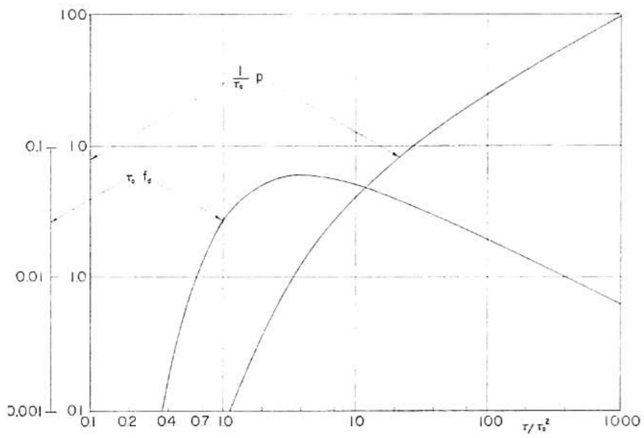

Fig. 8 Asymptotic responses, pressure for flow step and flow for pressure step, as $\tau_0 \rightarrow \infty$ (generally valid when $\tau_0 > 2$)

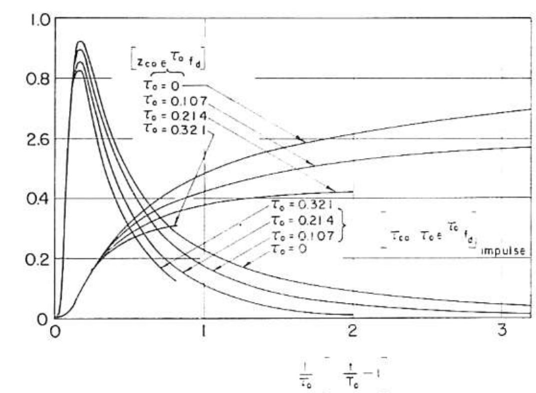

Fig. 10 Front portions of step and impulse responses, flow for pressure

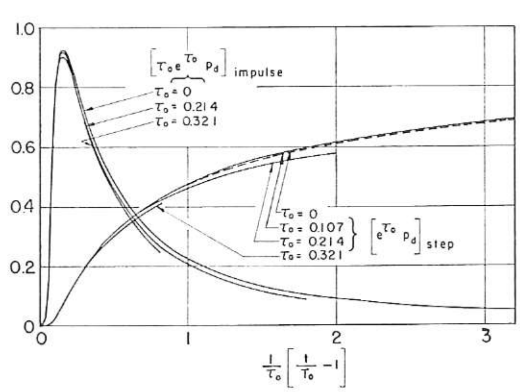

Fig. 9 Front portions of step and impulse responses, pressure for pressure or flow for flow

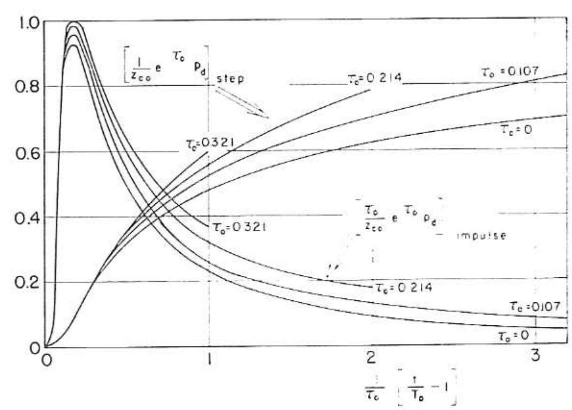

Fig. 11 Front portions of step and impulse responses, pressure for flow

significant as the front end of the response is smoothed, attenuated, and delayed in excess of the nominal speed-of-sound delay. Ultimately, the response approaches the limiting *shape* (without a limiting *position* on the plot) of a diffusive R-C line. This limit is shown in more detail in Fig. 5, where a new abscissa results in a unique curve.

Types (*c*) and (*d*) are plotted in Figs. 6 and 7, respectively. For a step in pressure the flow at all stations along the line must ultimately decay to nothing, since an infinitely long line has an infinite resistance to steady flow. Similarly, for a step in flow the pressure at all stations must increase until the pipe bursts. Note that the *driving-point responses*, for which $\Gamma = 0$, are included. The limiting cases of the diffusive R-C line, nearly valid for $\tau_0 > 2$, are given in Fig. 8. In all cases the accuracy is at least within $\pm$ 1 percent.

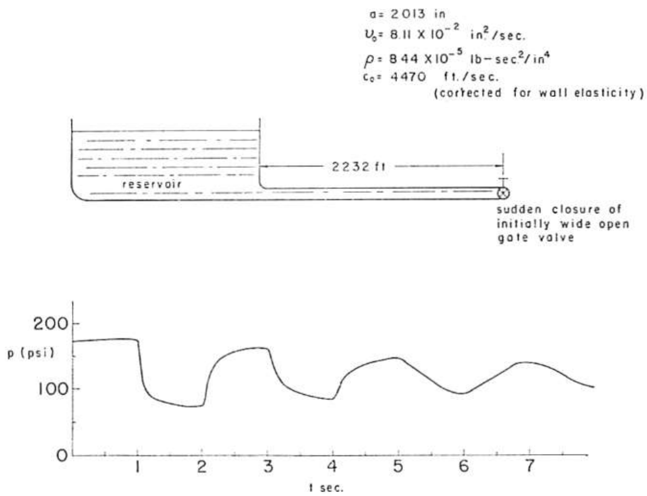

Fig. 12 Water-hammer problem

The semilog and log-log coordinates of these plots, as well as the respective nondimensionalizations, were chosen simply to fit all the results into a reasonable space. Unfortunately, this tends to obscure the physical meaning of the results. For example, the rapid deterioration of the front end of the responses is obscured and the great length of the tails foreshortened. Some of the plots given earlier by one of the authors [3, 4, 5], showing in more detail the front portion of the step responses and their derivatives, or impulse responses, are reproduced in Figs. 9–11. Note that the width of a pulse is very nearly proportional to the square of the distance it has traveled.

Applications Involving Reflections

Two examples are given in the following in which the principle of superposition is used to account for reflected waves. These are of course very special problems and are intended merely to be suggestive of possible approaches to other cases.

The first problem, shown in Fig. 12, is the classical water-hammer rapid-shutdown of a valve in a line leading from a reservoir. The pressure response to the flow step is plotted. Since the waves are totally reflected, in like sense at the blocked end and unlike sense at the open end, only the addition and subtraction is required of the pressure as seen at the imaginary distances down a semi-infinite line of $2L$, $4L$, $6L$, etc.

Reflection at a semiblocked termination, such as the nozzle in Fig. 13 which has the indicated nonlinear pressure-flow characteristic, is considerably more complex. In this problem the pressure at the right-hand end of the line is raised suddenly to a new fixed value. The resulting pressure at the left-hand end, assuming that the nozzle is blocked, is shown in part (*b*) of the figure. But of course the nozzle is not blocked; it has the pressure-flow characteristic of part (*d*). Thus, flow-step waves of appropriate amplitude are introduced into the line at the nozzle end to give the proper correction. The pressure at the nozzle resulting from a single such wave is given in part (*c*). All pressures and flows are

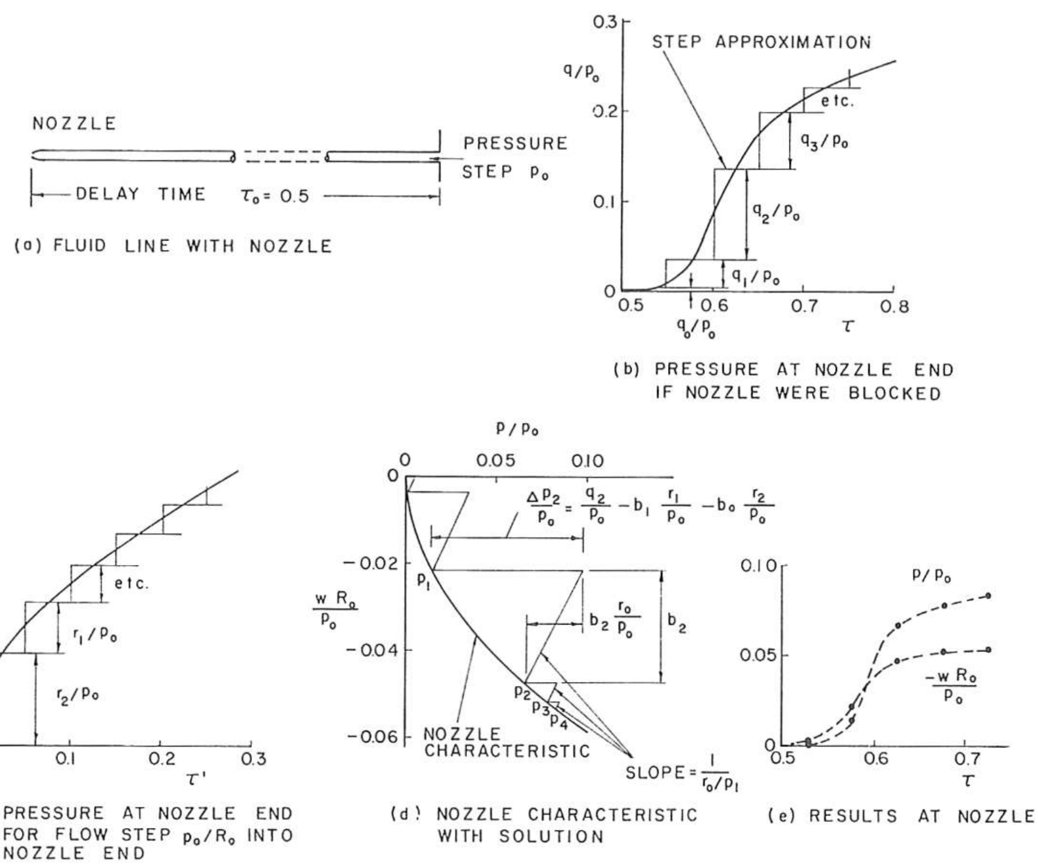

Fig. 13 Method of handling partial reflections

approximated by a series of steps. For the nth step the change of pressure is

$$P_n = q_n - (b_1 r_n + b_2 r_{n-1} + \ldots + b_{n-1} r_2 + b_n r_1) \quad (14)$$

All of the terms on the right side are known except the last; this is found graphically, as shown in part (d) for the second step. Incidentally, a bookkeeping table of the components of the response is easily and appropriately constructed, and if desired the whole process can be computerized.

Conclusions

Complete step responses for small-signal disturbances in Newtonian liquid-filled laminar-flow circular lines have been given. The results are rounded curves, due to the frequency dependency of the wall shear. Although the step responses have not been verified directly, the corresponding frequency responses closely match experimental data [8]. Moreover, since the three independent mathematical techniques which were employed match with each other, little question remains regarding the validity of the solutions.

Considerable basic work is yet to be either performed or reported. Complete step responses have not been found for air or other gases. N. B. Nichols is strongly encouraged to publish his work on the higher modes of motion of fluids in cylindrical containers. In collaboration with one of the authors, D. M. Auslander has written a thesis [9] in which predictions are made for the frequency responses for small disturbances in a grossly *turbulent* flow in both smooth and rough-walled tubes. The dispersion is much greater than in laminar flow. This work should be published more broadly, and experimental corroboration attempted. The present paper heuristically shows how the step-response results can be used to estimate closely the response of a complicated network to any transient. However, simpler and more powerful techniques for the analysis of systems of lines are possible and should have great utility. Adequate computational approaches to problems involving complex interactions between fluid and walls and large-amplitude disturbances[5] have yet to be developed except in very special cases.

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APPENDIX A

High and Low-Frequency Approximations

For $|\xi| > 3$, the asymptotic series

$$\frac{2J_1(j\xi)}{j\xi J_0(j\xi)} \approx \frac{2}{\xi} - \frac{1}{\xi^2} - \frac{1}{4\xi^3} \quad (15)$$

is quite good [10]. This gives the approximations

$$\Gamma(s) \approx T_0 s \left[1 + \left(\frac{\nu_0}{a^2 s}\right)^{1/2} + \frac{\nu_0}{a^2 s} + \frac{7}{8}\left(\frac{\nu_0}{a^2 s}\right)^{3/2}\right] \quad (16)$$

$$\frac{1}{Z_c(s)} \approx \frac{1}{Z_{c0}}\left[1 - \left(\frac{\nu_0}{a^2 s}\right)^{1/2} + \frac{1}{8}\left(\frac{\nu_0}{a^2 s}\right)^{3/2}\right] \quad (17)$$

for a liquid, which were used to find the front portion of the step responses [4, 5, 6]. Similar expressions are available for a perfect gas in which the Prandtl number represents the dispersive effect of heat transfer.

The traditional low-frequency exact expansion of the Bessel functions gives, for a liquid,

$$\Gamma(s) = \sqrt{8}\,\tau_0 \sum_{k=0}^{\infty} g_k \left(\frac{a^2 s}{\nu_0}\right)^{k+1/2} \quad (18)$$

$$Z_c(s) = Z_{c0} \sum_{k=0}^{\infty} g_k \left(\frac{a^2 s}{\nu_0}\right)^{k-1/2} \quad (19)$$

in which

$g_0 = 1$ | $g_4 = -3.649 \times 10^{-5}$
$g_1 = 8.333 \times 10^{-2}$ | $g_5 = 4.389 \times 10^{-6}$
$g_2 = -3.906 \times 10^{-3}$ | $g_6 = -5.668 \times 10^{-7}$
$g_3 = 3.400 \times 10^{-4}$

Corresponding series expressions for the three types of step responses have been found [7] but are too long to reproduce here. The values of g_k for air or other perfect gases have not been calculated.

APPENDIX B

Numerical Computation of the Mid-Region of the Step Responses Based on the Frequency Response

For purposes of numerical evaluation, equation (7) was placed in the form

$$\phi(t) = \Sigma F(\Omega)\Delta\Omega \quad (20)$$

The general shapes of the function $F(\Omega)$ for the three types of problems are shown in Fig. 14. After the first zero crossing these functions behave similarly, closely approximating the sum of a sinusoid with a short period and a sinusoid with a long period. The curves were divided into approximately 20 increments per cycle of the fast harmonic; trapezoidal approximations were then employed. Values of the accumulated area up to values of Ω where both harmonics pass through a maximum or a minimum were stored; these points give the closest estimates of the correct answer. Successive values were compared, and the solution truncated when the deviation from the mean of the last four values was less than 0.4 percent of this mean. Details are available [7].

A special problem arose for the pressure response to a flow step because the function $F_d(\Omega)$ is infinity for $\Omega = 0$. For very small values of Ω the following excellent approximation was utilized:

$$F_d(\Omega) \approx \frac{4\tau}{\pi} \frac{(1 - 4\tau_0\sqrt{\Omega} + 4\tau_0^2\Omega)}{\sqrt{\Omega}} \quad (21)$$

$$\Omega < 0.02 \text{ and } \Omega < \frac{0.001}{2\tau_0^2} \text{ and } \Omega < \frac{0.1}{\tau}$$

The portion of the step responses calculated with this numerical technique overlapped at one end the portion computed using the high-frequency Laplacian operators and at the other end the

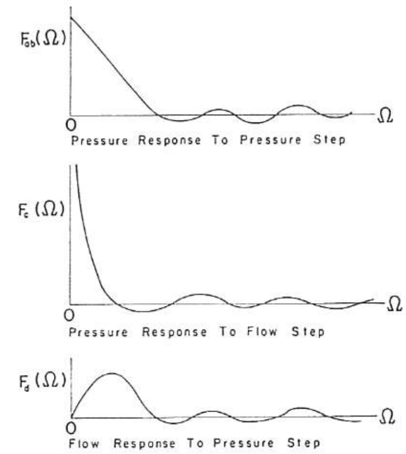

Fig. 14 General shapes of $F(\Omega)$ for step responses

[5] In laminar flow of a liquid, essentially any normally encountered disturbance is "small"; not so for a gas, however, or a liquid in turbulent flow.

portion computed using the low-frequency Laplacian operators. Agreement in the regions of overlap was always within at least 1 percent. This lends considerable confidence to the validity of the results, since the three solutions are quite independent except for reliance on the experimentally justified propagation and characteristic impedance operators.

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