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Naive Probability: Model-Based Estimates of Unique Events

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Abstract

We describe a dual-process theory of how individuals estimate the probabilities of unique events, such as Hillary Clinton becoming U.S. President. It postulates that uncertainty is a guide to improbability. In its computer implementation, an intuitive system 1 simulates evidence in mental models and forms analog non-numerical representations of the magnitude of degrees of belief. This system has minimal computational power and combines evidence using a small repertoire of primitive operations. It resolves the uncertainty of divergent evidence for single events, for conjunctions of events, and for inclusive disjunctions of events, by taking a primitive average of non-numerical probabilities. It computes conditional probabilities in a tractable way, treating the given event as evidence that may be relevant to the probability of the dependent event. A deliberative system 2 maps the resulting representations into numerical probabilities. With access to working memory, it carries out arithmetical operations in combining numerical estimates. Experiments corroborated the theory's predictions. Participants concurred in estimates of real possibilities. They violated the complete joint probability distribution in the predicted ways, when they made estimates about conjunctions: $P(A)$, $P(B)$, $P(A \textit{ and } B)$, disjunctions: $P(A)$, $P(B)$, $P(A \textit{ or } B \textit{ or both})$, and conditional probabilities $P(A)$, $P(B)$, $P(B|A)$. They were faster to estimate the probabilities of compound propositions when they had already estimated the probabilities of each of their components. We discuss the implications of these results for theories of probabilistic reasoning.

Keywords: Bayesianism; Frequentism; Mental models; Subjective probabilities

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1. Introduction

Probabilistic thinking is ubiquitous in both numerate and innumerate cultures. Aristotle wrote: "A probability is a thing that happens for the most part" (Aristotle, *Rhetoric*, Book I, 1357a35, see Barnes, 1984). His account, as Franklin (2001, p. 109) argues, dominated philosophical and legal thinking about probabilities during the subsequent two millennia until the invention of the probability calculus. It continues to describe probabilistic thinking in contemporary contexts: Physicians tend to offer prognoses using qualitative expressions, for example, "a high risk" and "a low likelihood" rather than quantitative ones (White, Engelberg, Wenrich, Lo, & Curtis, 2010). The calculus transformed the qualitative conception of probability into a numerical one. It depends on logic or set theory, as evinced in its axiomatizations (Scott & Krauss, 1966) and in Boole's (1854) dual interpretation of formulas in his algebra as statements about sets and about numerical probabilities. Theorists accordingly treat probabilities as estimates of truth values, and as a natural extension or generalization of two-valued logic (e.g., de Finetti, 1936/1995; Jeffrey, 1983; Ramsey, 1926/1990).

The applicability of the probability calculus to everyday reasoning is controversial, because no consensus exists about the proper interpretation of the calculus, and hence about the situations to which it applies (see Baron, 2008, ch. 5; Nickerson, 2004, ch. 1). Philosophers are divided on this issue (see Hájek, 2003). Likewise, some psychologists defend a frequency interpretation of probabilities, which therefore does not apply to unique events (e.g., Cosmides & Tooby, 1996; Gigerenzer, 1994), whereas others defend a subjective view of probabilities corresponding to degrees of belief, which therefore does apply to unique events (e.g., Chater, Tenenbaum, & Yuille, 2006; Tversky & Kahneman, 1983). Complex epistemological issues underlie even these so-called Bayesian views of probability, which also divide into objective interpretations (e.g., Williamson, 2010) and subjective interpretations (e.g., Jeffrey, 2004). We will not pursue these epistemological issues and simply note that naive individuals—those who have not mastered the probability calculus—appear to estimate probabilities in several ways. They use frequency data if they are available, they adduce relevant physical considerations such as that dice have six sides, and they use evidence to estimate the probabilities of unique events. They can make informal non-numerical estimates, specific numerical estimates, or estimates of the range or interval of probabilities. Their estimates rely on two main methods (Tversky & Kahneman, 1983). *Extensional* methods depend on deducing the probability of an event from the mutually exclusive ways in which it could occur, which are either equiprobable or of known probabilities. These methods are explicable in terms of mental models of the possibilities (Johnson-Laird, Legrenzi, Girotto, Legrenzi, & Caverni, 1999). *Non-extensional* methods depend on inducing the probability of an event from some relevant evidence, index, or heuristic such as that if an exemplar is typical of a category, it is likely to be a member of the category (Tversky & Kahneman, 1983).

The distinction between the probabilities of events that occur with some frequency and the probabilities of unique events is subtle. The probability that a particular throw of a roulette ball lands on a red number can be inferred from the frequency that it has done so in the past, that is, that particular ball on that particular roulette wheel. The instance is a clear member of a single set of cases in which the ball lands on a red number, whose cardinality can be compared with that of the set in which the ball does not land on a red number. In contrast, the probability that Hillary Clinton is elected U.S. President in 2016 is not a member of a single set of cases that allows a definitive inference of its value. Many different sets could be relevant—women elected President, Democrats elected President, Yale graduates elected President, and so on. They yield different frequencies of the outcome, and none of these frequencies is, or can be, definitive. This article concerns inferences about unique events, such as Clinton's election to the Presidency, and we define a *unique event* as one that is not a member of any single set allowing a definitive inference of the frequency of its occurrence.

Tversky and Kahneman (1983) pioneered the study of the probabilities of unique events, and they demonstrated egregious fallacies in inferring them from hypothetical scenarios. The present research diverges from their studies in three ways. First, we focus on the probabilities of real possibilities in everyday life (cf. Ungar et al., 2012). Second, we do not use scenarios designed to elicit the use of heuristics, but instead simply pose questions to the participants about real possibilities, such as: What is the probability that the music industry will embrace a new and more relaxed version of copyright law in the next 10 years? Third, and most important, the goals of our research were to describe a model-based theory of the mechanisms underlying estimates of the probabilities of unique events and compounds of them, to outline its implementation in a computational model, and to report experimental corroborations of its principal predictions.

A profound mystery about estimates of the probabilities of unique events is where the numbers come from and what determines their magnitudes. It is conceivable, of course, that estimates of such probabilities are unsystematic, unprincipled, and meaningless (cf. Cosmides & Tooby, 1996). And so, one aim of our empirical studies was to check whether or not this possibility is correct. On the presupposition that it would not be, we developed a theory of how people make such estimates using mental models, and we implemented this theory as part of a unified computational model of logical and probabilistic reasoning, *mReasoner*. This model-based theory and its implementation explain how individuals infer the probabilities of unique events, and where the numbers in such estimates come from. The theory also predicts systematic violations of the probability calculus. To explain these violations, however, we must first describe the complete joint probability distribution (the JPD) and the concept of subadditivity.

Given two propositions and their respective negations:

$$\begin{aligned} &A: \text{It will snow tomorrow.} \\ &B: \text{It will freeze tomorrow.} \end{aligned} \tag{1}$$

There are four possible contingencies: $A\&B$, $A\&\neg B$, $\neg A\&B$, and $\neg A\&\neg B$. The respective probabilities of each of these contingencies make up the complete joint probability distribution (JPD) for the propositions A and B. The JPD generalizes to any number of random variables and to variables that are not just binary but that have multiple values. The JPD yields the probability of any compound proposition concerning its individual events. Suppose, for example, the JPD of the events above is as follows:

$$\begin{aligned} &P(\text{snows} \;\;\&\;\; \text{freezes}) = .4 \\ &P(\text{snows} \;\;\&\; \neg\text{freezes}) = .3 \\ &P(\neg\text{snows} \;\;\&\;\; \text{freezes}) = .2 \\ &P(\neg\text{snows} \;\;\&\; \neg\text{freezes}) = .1 \end{aligned} \tag{2}$$

These probabilities yield the probability of any compound of the individual events in it (e.g., the conditional probability that it snows given that it freezes is .4/(.4 + .2) = .66, and the converse conditional probability that it freezes given that it snows is .4/(.4 + .3) = .57). The conditional probability, $P(A|B)$ can also be inferred using Bayes's theorem, which, given $P(B) \neq 0$, can be expressed as follows:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \tag{3}$$

But the three probabilities on the right-hand side of the equation in (3) are just one way to fix the values of the JPD. And the conditional probability, $P(A|B)$, can be inferred from the JPD; however, its values are fixed.

A *subadditive* function is one for which the value for the whole is less than the sum of the values for the parts, as in the square root function where $\sqrt{(4+4)} < \sqrt{(4)} + \sqrt{(4)}$. Probability, however, is not subadditive, that is, $P(A) = P(A\&B) + P(A\&\neg B)$. Hence, $P(A\&B)$ must be less than or equal to $P(A)$, and less than or equal to $P(B)$. Estimates that violate this principle are subadditive and are known as "conjunction fallacies," because the probability of the conjunction is then greater than the probability of one of its conjuncts (see, e.g., Tversky & Kahneman, 1983). Conjunction fallacies yield at least one, and possibly two, negative probabilities in the JPD. A set of estimates of probabilities can also be *superadditive*, yielding a JPD that sums to less than 1. The estimates of $P(A)$, $P(B)$, and $P(A\&B)$ cannot be superaddive, but the addition of a fourth estimate, such as $P(\neg A\&\neg B)$, can be. Subadditivity and superadditivity have normative consequences. Reasoners who produce estimates of probabilities that are subadditive or superadditive violate the probability calculus. Their performance is irrational and not coherent, because in principle a "Dutch book" can be made against them, that is, a set of bets in which they are bound to lose (see, e.g., de Finetti, 1936/1995; Ramsey, 1926/1990).

Suppose someone estimated the following probabilities for snow and freezing:

$$\begin{aligned} &P(A) = .7 \\ &P(B) = .7 \\ &P(A\&B) = .3 \end{aligned} \tag{4}$$

The estimates do not commit the conjunction fallacy, but the probability of the conjunction is too small and so the estimates yield the following JPD:

$$\begin{aligned} &P(A\&B) = .3 \\ &P(A\&\neg B) = .4 \\ &P(\neg A\&B) = .4 \\ &P(\neg A\&\neg B) = -.1 \end{aligned} \tag{5}$$

The three estimates therefore yield a negative probability in the JPD.

Negative probabilities are impossible in the probability calculus, but Dirac (1942) introduced them in his quantum probabilities, and some psychologists have argued for their use to account for human judgments (Pothos & Busemeyer, 2013). A corollary is that the "conjunction fallacy" ceases to be a fallacy, and, as Oaksford (2013) argues, the notion of rationality as the avoidance of inconsistency ceases to apply. In daily life, nothing is less probable than the impossible, and so the number of negative probabilities in the JPD provides a metric for inferences about probabilities. For JPDs based on two possibilities A and B, the metric has three values: 0 negative probabilities (a consistent JPD), 1 negative probability (one violation of the JPD), and 2 negative probabilities (two violations of the JPD). This metric is analogous to a measure of "semantic coherence" (Wolfe & Reyna, 2010), but the latter depends on $P(A)$, $P(B)$, $P(A\&B)$, and $P(A \vee B)$. In contrast, the present metric applies to any set of estimates that fix a JPD.

The model theory predicts the occurrence of systematic violations of the JPD in many different sorts of judgment, and we report studies corroborating the occurrence of such violations. In what follows, the article describes pertinent previous studies, a model-based theory of what people are computing in estimating unique probabilities, and the computer implementation of how they make such computations. It derives the model theory's principal predictions and reports three experiments that corroborate these predictions for conjunctive, disjunctive, and conditional probabilities. Finally, it discusses the implications of the results for theories of reasoning.

2. Previous studies

Extensional and non-extensional estimates of probabilities show that naïve individuals violate the norms of the probability calculus, that is, they commit fallacies. In extensional cases, theorists have proposed various accounts of how to predict estimates of the probability of compound assertions, but they center around the hypothesis that individuals take the mean of the probabilities of the conjuncts or some function thereof (e.g., Fantino, Kulik, Stolarz-Fantino, & Wright, 1997; Wyer, 1976), the mean of the probabilities of disjuncts in estimates of disjunctions (e.g., Bar-Hillel & Neter, 1993; Tversky & Kahneman, 1983; Young, Nussbaum, & Monin, 2007), and a similar procedure in estimates of conditional probabilities (Reyna & Brainerd, 1995, 2008; Wolfe & Reyna, 2010). In non-extensional cases, theorists propose accounts based on the representativeness of the conjunction in comparison with the vignette that is the basis of the estimates (Barbey & Sloman, 2007; Fantino, 1998; but cf. Wallsten, Budescu, Erev, & Diederich, 1997; Wallsten, Budescu, & Tsao, 1997). In their studies of representativeness, Tversky and Kahneman (1983, p. 306) considered the hypothesis that individuals estimate the probability of a conjunction by taking the average of the probabilities of its conjuncts. Tversky and Koehler (1994) proposed "support" theory to account for estimates of the probabilities of disjunctions, which tend to be less than or equal to the sum of the probabilities of their disjuncts (Rottenstreich & Tversky, 1997). Likewise, Zhao, Shah, and Osherson (2009) present convincing evidence that individuals do not compute the ratio of $P(A\&B)$ to $P(B)$ in estimating $P(A|B)$ for real future possibilities. Some authors propose instead that they rely on the Ramsey test (e.g., Evans, 2007; Gilio & Over, 2012)—a possibility to which we return later.

Despite these studies, three main gaps in knowledge exist. First, no account exists of the mental processes underlying the participants' estimates of the probabilities of real but unique possibilities, though Wyer (1976) called for such investigations nearly 40 years ago. Second, as a consequence, psychologists have no account of the mental processes that yield numerical probabilities from non-numerical premises. Third, with few exceptions (Over, Hadjichristidis, Evans, Handley, & Sloman, 2007; Zhao et al., 2009), psychologists have not studied estimates of the probabilities of unique events that are real possibilities. Hence, we cannot be certain that individuals concur in their estimates of such probabilities or whether they are likely to violate the JPD in making them. We now consider a theory designed to fill these gaps in knowledge.

3. A model theory of unique probabilities and their compounds

We describe a theory (the "model" theory) that is founded on the notion that individuals build mental models to understand discourse and make inferences. The model theory of the probability of unique events aims to take into account previous studies, to ground them in terms of the familiar distinction between intuitions and deliberations, and to derive new predictions about the probabilities of unique events. A preliminary study described experiments with conjunctions (Khemlani, Lotstein & Johnson-Laird, 2012), but we now present a comprehensive theory that also deals with the probabilities of disjunctions and conditional probabilities. In this section, we focus on what people are computing. But in the section that follows we outline a computer implementation of the algorithm that the model theory postulates as underlying human estimates of unique probabilities.

The theory postulates that estimates of the probabilities of unique events depend on intuitions (system 1) and on deliberative reasoning (system 2). This distinction is a familiar one in dual-process theories of judgment and reasoning (see, e.g., Evans, 2008; Johnson-Laird, 1983, ch. 6; Kahneman, 2011; Sloman, 1996; Stanovich, 1999; Verschueren, Schaeken, & d'Ydewalle, 2005; Wason & Evans, 1975; Wason & Johnson-Laird, 1970).

These theories have antecedents in Turing (1939) and Pascal (1966/1669). Some accounts (Evans & Stanovich, 2013) have been criticized as not empirically refutable (Keren, 2013), and as not distinguishable from accounts based on a single system (Kruglanski, 2013). One prophylactic to both these problems is to implement a computer model of the two systems and of how they interact. The next section of the article (4) presents such an implementation, which distinguishes between the two systems in terms of their computational power. Here, however, we describe the model theory of intuitions about probabilities (system 1) and of deliberative processes concerning them (system 2).

3.1. Intuitions about probabilities

Intuitions about probabilities are not numerical but reflect the Aristotelian idea that events vary from impossible to certain, with various intermediate cases. They can be expressed informally (Budescu, Karelitz, & Wallsten, 2003; Wallsten & Budescu, 1995) in such Aristotelian terms as:

impossible, almost impossible, highly unlikely, unlikely, as likely as not,
likely, highly likely, almost certain, certain. (6)

The theory postulates that the single most important intuitive guide to improbability is uncertainty. The two seem to be synonymous, but, as we show, they can diverge from one another and lead to errors in estimates of probabilities.

Uncertainty as a state of mind arises in various ways. Suppose, for instance, you assess the probability of the following event:

Hillary Clinton is elected U.S. President in 2016. (7)

You may adduce several pieces of evidence that concur, in which case you can make a corresponding intuitive estimate, such as, “It is likely.” But suppose instead that your evidence conflicts. On the one hand, for example, you think that Clinton is well known as a highly competent politician, and that many people will vote for such a politician. But, on the other hand, you think, she is female and no woman has ever been elected President. Such conflicts are a source of uncertainty: One piece of evidence suggests that her election is likely, but another piece of evidence suggests that it is unlikely. Your problem is to reconcile at least two conflicting judgments. It is analogous in the probability calculus to inferring *P*(*Clinton-is-elected*) from two different conditional probabilities, *P*(*Clinton-is-elected*|*Clinton-is-competent*) and *P*(*Clinton-is-elected*|*Clinton-is-female*). In fact, these two values do not fix *P*(*Clinton-is-elected*), which calls for still more evidence. Normative theories of how to reconcile conflicting probabilities can be founded on Bayesian considerations (e.g., Lindley, Tversky, & Brown, 1979). But, unless you are an expert, your intuitions are unlikely to suggest a need for additional evidence, because you need to reconcile the current conflict, which further evidence might exacerbate. Intuition yields a simple conservative compromise: You split the difference between your two judgments; that is, you take their average (see Wallsten, Budescu, Erev, et al., 1997; Wallsten, Budescu, & Tsao, 1997). Clinton's election is accordingly, "as likely as not." Of course, you may adduce more than two pieces of evidence and have to reconcile a conflict between them. The theory postulates that system 1 deals with them by reconciling its initial estimate of a probability with each successive piece of evidence, and the next section describes a feasible mechanism.

Suppose you have to estimate the probability of a conjunction, such as:

Clinton is elected U.S. President in 2016 and all wars in the world cease in 2017. (8)

If the evidence you adduce suggests that Clinton is likely to be elected, and other evidence—including perhaps the election of Clinton—suggests all wars are not likely to cease, you need to resolve the conflict. And, once again, intuition makes a compromise and takes the average.

Disjunctions are a major source of uncertainty both in reasoning (e.g., Johnson-Laird, Byrne, & Schaeken, 1992; Johnson-Laird, Lotstein, & Byrne, 2012) and in decision making (e.g., Tversky & Shafir, 1992). An assertion about an individual event, such as (7), rules out more possibilities than an inclusive disjunction into which it enters, such as:

Clinton is elected U.S. President in 2016, or all wars in the world cease in 2017, or both. (9)

This disjunction yields three separate mental models to represent each of the three possibilities to which it refers, and they include one that is incompatible with the model of the individual event, namely, the possibility in which Clinton is not elected, but all wars cease in 2017. It follows that the assertion about the individual event conveys more semantic information than the disjunction (Bar-Hillel, 1964). Hence, because uncertainty suggests improbability, intuition assesses the disjunction as less likely than the individual event.

Suppose that you are asked to estimate a conditional probability, such as

Given that all wars in the world end in 2015, what is the probability that Clinton is elected U.S. President in November 2016? (10)

System 1 cannot compute Bayes's theorem, and so the model theory posits that intuition relies on a simple procedure. It treats the event in first clause in (10) as evidence for the event in the second clause. It then uses the same procedure that we described earlier to estimate the probability of an individual event from evidence. According to Bayes's theorem, a conditional probability, $P(B|A)$, depends on the values of three variables: the prior converse conditional probability, $P(A|B)$, and the base rates of the two events: $P(A)$ and $P(B)$. Equivalently, $P(B|A)$ depends on the ratio, $P(A\&B)/P(A)$. Naïve individuals, according to the theory, tend to reduce the problem to a matter of how the dependent variable, $P(B)$, changes given that the independent variable A is true. This procedure is analogous to an anchoring and adjustment heuristic (Tversky & Kahneman, 1983), but the model theory suggests that its application is pervasive, and the result of anchoring the probability on an initial mental model (of B) and adjusting it relative to the evidence for A. Of course, individuals may suppose that the given event in (10) is irrelevant to Clinton's election. In which case, they can adduce their own evidence for her election, using the procedure for individual events. A conditional probability $P(B|A)$ in the standard probability calculus is undefined if the given condition A is false, because the ratio $P(A\&B)$ to $P(A)$ would then call for division by zero. But the model theory allows that individuals can envisage counterfactual models (Byrne, 2005), and so they can make estimates of conditional probabilities in this case, too. Another way to avoid this problem is to formulate the probability calculus so that conditional probability is treated as a primitive rather than introduced by definition (see also Appendix 2 of Adams, 1998).

A consequence of the intuitive methods for estimating the probabilities of compounds is that when system 2 transforms system 1's intuitions into numerical values, the resulting values may violate the complete joint probability distribution (the JPD); that is, it contains at least one negative probability. Even expert probabilists are prone to violate the JPD in judgments about probabilities. Only certain special cases of variables are likely to be free from risks of violation. For example, if one event is inconsistent with another, as in:

Hillary Clinton is elected U.S. President in 2016 and Joe Biden is elected U.S. President in 2016. (11)

individuals know that it is impossible for both events to occur, and so they will estimate the probability of their conjunction as zero regardless of their estimates of the probability of the conjuncts. Likewise, if one event contradicts another, then individuals who evaluate their disjunction, such as:

Clinton is either elected U.S. President in 2016 or not. (12)

know that one or other of these two is bound to occur, and so they will estimate the probability of their disjunction as 1.0 regardless of their estimate of the probability of Clinton's election. But contradictory and tautological combinations of events such as the preceding examples are rare, and so most estimates of the probabilities of unique events depend on evidence.

3.2. Deliberations and numerical probabilities

The deliberative system (system 2) has access to working memory, and so it can carry out recursive or unboundedly iterative processes, such as counting and other arithmetical operations—at least until they overload processing capacity and call for an external notation. And it is this system that maps the intuitive system's judgments into numerical estimates. The deliberative system can also try to keep track of the JPD, though naïve individuals are unlikely to do so in any exact way unless prompted. One corollary of the difference between the two systems concerns the apparent vagueness of verbal descriptions of probabilities in comparison with numerical probabilities. In fact, verbal descriptions make explicit the contrast in polarity between affirmatives such as "probable" and negatives such as "improbable" around a neutral midpoint of "as likely as not," whereas numerical probabilities make no such explicit morphological contrast. Hence, verbal descriptions are more explicit about polarity than degree, whereas numerical probabilities have the converse property (see Teigen & Brun, 2000). Verbal descriptions of the sort that system 1 produces, such as "highly probable," have moderately coherent mappings into numerical estimates (see, e.g., Reagan, Mosteller, & Youtz, 1989; Wallsten & Budescu, 1995), and so too do determiners such as "some" and "not all" (Teigen & Brun, 2003). But if the numerical scale is of a finer grain than intuitive judgments, the mapping calls for an arbitrary choice of a particular number from a range of possibilities. Hence, the mapping from intuitions to a scale should be more likely to yield a consistent JPD with a coarse scale than with a fine scale. Monte Carlo simulations bear out this phenomenon, and a previous study corroborated this prediction (Khemlani, Lotstein, et al., 2012): Coarse scales yielded a greater proportion of consistent JPDs than fine numerical scales, and this result supports system 1's use of a coarse mental scale.

System 2 allows individuals to hold numerical estimates in memory. It is particularly useful for *extensional* judgments for which the input is numerical estimates (see Johnson-Laird et al., 1999). If the arithmetic is easy, then whether the data are, say, natural frequencies or chances out of 100, do not seem to affect estimates. Readers may be tempted to assume that system 2 is fully rational. In fact, it suffers from several impediments. One is the computational intractability of most rational systems, including the probability calculus, logic, and decision theory. Each time a new variable is added to the JPD, the number of contingencies it contains doubles in size. For example, the JPD for three binary variables *A*, *B*, and *C* represents eight contingencies, and, in general, *n* binary variables calls for 2^n contingencies. Bayesian nets (Pearl, 1988), which allow one to infer probabilities without having to construct a complete JPD, are also intractable, even with only approximate calculations (Dagum & Luby, 1993). Some theorists reject intractable accounts of cognition (e.g., Gigerenzer, Hoffrage, & Goldstein, 2008; Oaksford & Chater, 2007, ch. 2); other theorists propose accounts that are intractable (for a review, see van Rooij, 2008). A synthesis of these various standpoints is that human reasoners are bound to make inferences in domains that cannot have tractable algorithms, such as two-dimensional spatial reasoning (Ragni, 2003). In consequence, they can cope only with small-scale problems in intractable domains (Johnson-Laird, 2006, p. 340; van Rooij, 2008, p. 971).

Another impediment to rationality in system 2 is a consequence of a central principle of the model theory: Mental models represent what is true at the expense of what is false.

A corollary is that system 2 deliberations about extensional probabilities can err. Consider, for instance, the following problem:

> There is a box in which there is at least a red marble, or else there is a green marble and there is a blue marble, but not all three marbles. What is the probability of the following situation:
> In the box there is a red marble and a blue marble? (13)

The premise has two mental models: one represents the presence of a red marble, and the other represents the presence of a green and a blue marble. In neither case is there a red marble and a blue marble? As this account predicts, the majority of participants in an experiment inferred that the probability of their conjunction was zero (Johnson-Laird et al., 1999). In fact, when the first disjunct in the premise is true (a red marble is in the box), there are three ways in which the second disjunct can be false, and so the box can contain any of the following four sets of marbles:

red	green	
red		blue
red		
	green	blue

The second of these possibilities shows that a red and blue marble can be in the box, and so their conjunction is not impossible. Despite the occurrence of systematic fallacies of this sort, system 2 should be much better than system 1 in inferring probabilities: Deliberation trumps intuition in this domain. Inferences about the probabilities of unique events cannot be estimated in a thoroughgoing extensional way, and so human reasoners are bound to rely heavily on system 1.

A major impediment to the rationality of system 2 is ignorance. Experts can rely on system 2 to use the numerical probabilities of individual events to compute the correct values for their conjunction:

$$P(A\&B) = P(A) * P(B|A) \tag{14}$$

their disjunction:

$$P(A \vee B) = P(A) + P(B) - P(A\&B) \tag{15}$$

and their conditional probability using Bayes's theorem (3). Naïve individuals, in contrast, do not know these formulas for computing probabilities. When system 2 is engaged, they can at best grasp that the probability of a conjunction of independent events is their product, and that the probability of a disjunction of inconsistent events is the sum of their probabilities. But they do not know Bayes's theorem (Zhao et al., 2009), and even when they are taught it, they find it difficult to use, because it is not intuitive (Shimojo & Ichikawa, 1989). We polled 30 cognitive scientists who had published on probability and asked them the following question: If one knows the probability of *A* and the probability of *B*, what constraints if any do they place on the conditional probability of *B* given *A*? Some replied, "none"; some replied that they did not know; and only a handful knew the correct answer (see Appendix A). The constraints are way beyond the intuitive system; and estimates of conditional probabilities lie on the boundary of naïve competence (Johnson-Laird et al., 1999).

The best definition of a conditional probability, $P(B|A)$, according to Bennett (2003) is provided by Ramsey's (1926/1990) test for fixing your degree of belief in assertions of the form *If A, then B*: You add *A* to your beliefs and then estimate the probability of *B*. Evans and his colleagues have defended the Ramsey test as a psychological account of how individuals estimate conditional probabilities (e.g., Evans, 2007, p. 54). Earlier we mentioned the problems of computing conditional probabilities when the given condition, *A*, is false. The Ramsey test can be extended to deal with these cases. Individuals adjust their beliefs to accommodate its falsity without inconsistency, and then assess the probability of *B* in that context. One drawback with the Ramsey test is that it takes no account of the probability of *A*. Another drawback is that no one has described its underlying mental processes, and any plausible account of them is likely to be computationally intractable (see Oaksford & Chater, 2007, p. 107). In contrast, the model theory, as we described earlier, postulates a simple tractable procedure for estimating conditional probabilities in system 1. We now turn to an account of a computer implementation of the model theory.

4. How people compute unique probabilities: An algorithm

In this section, we describe an algorithmic theory of the mental processes underlying the estimates of the probabilities of unique events. Empirical results place fewer constraints on algorithms than on the functions they compute, because any computable function has a countably infinite number of distinct algorithms (Rogers, 1967, p. 9). Nevertheless, the success of the model theory's predictions, as shown in the experimental results below, lends some support to the present account of how the computations might be carried out. The aim of the implementation, however, was to explain in principle how numerical estimates derive from non-numerical information of an Aristotelian sort. The program, *mReasoner*, is intended to unify deductive and probabilistic reasoning, and it embodies all of the operations of system 1 and system 2 for the probabilities of unique events. It is also capable of many other sorts of inference. Its source code is in the public domain and is available at http://mentalmodels.princeton.edu/models/mreasoner/. The reason for integrating the deductive and probabilistic systems in a single program is to show that inferences about probabilities, whether extensional or not, can be made with mental models. The program copes only with a fragment of natural language, and it does not, at present, deal with relations of the following sort: *A is not less probable than B*. Table 1 summarizes its overall plan (see Hosni, 2013).

4.1. Constraints on the algorithm

The algorithm is based on three main constraints. The first is that the intuitive system has no access to working memory, and so it cannot carry out recursive or iterative processes. Hence, it cannot carry out arithmetical operations, such as counting and addition. It lacks even the computational power of finite-state automata (Hopcroft & Ullman, 1979), because it can carry out a loop of operations for only a small number of times—a restriction that is built into its computer implementation in *mReasoner*.

The second constraint is that the algorithm adduces as little evidence as possible to estimate probabilities. But additional evidence can push the probability represented in the system one way or the other. If a piece of evidence concurs with the current estimate, the system yields the informal estimate that they both support. If a piece of evidence conflicts with the current estimate, then the system resolves the conflict. It can, in principle, add further evidence in a piecemeal way, resolving any conflict between its current estimate and the impact of the new piece of evidence.

The third constraint is that the algorithm relies on primitive analog representations of magnitudes akin to those in animals (Meck & Church, 1983), infants (Barth et al., 2006; Carey, 2009; Dehaene, 1997; McCrink & Wynn, 2007), and adults in numerate (Song & Nakayama, 2008) and non-numerate cultures (Gordon, 2004). These *icons* are not numerical, and they represent magnitudes by a direct analogy: The greater the size of the icon, the greater the magnitude that it represents. For probabilities, icons represent the degree of belief in a proposition as a pointer on a "line" extending between an origin representing impossibility and a maximum representing certainty:

Table 1
A summary of *mReasoner*'s Systems 1 and 2 for inferring the probabilities of unique events and their compounds

System 1 Assessment of the Probability of Unique Events	System 2 Numerical and Arithmetical Computations
1. Represent a piece of evidence as a mental model (Khemlani & Johnson-Laird, 2012a,b)	5. Convert an icon (from 3) into a numerical probability
2. Convert a mental model into an icon representing a probability: magnitudes in icons correspond to probabilities	6. Carry out arithmetical operations in extensional reasoning about the probability of compounds (Johnson-Laird et al., 1999)
3. For cases in which two conflicting estimates occur, use a primitive average to resolve the conflict (see Table 2)	7. Check the consistency of the JPD, where prompted to do so, for simple cases
4. Translate the resulting icon into informal language, for example, "highly probable"	

Notes. Steps 1–4 can be used to compute compound probabilities, for example, $P(A\&B)$. Conjunctive and disjunctive probabilities use primitive averaging to estimate the compound, whereas conditional probabilities, $P(A|B)$, use B as evidence to estimate $P(A)$.

|------ | (16)

The left vertical represents impossibility, the right vertical represents certainty, and the mid-point between them represents the boundary between the probable and the improbable, though for simplicity, we omit midpoints from these diagrams. The rightmost end of the line is a pointer that represents a magnitude corresponding to the strength of a particular belief, that is, a subjective probability. The line is not the real number continuum, which is infinitely and non-denumerably dense, and so cannot have an isomorphic representation in any finite device, such as the human brain. Instead, the line has granularity—it is made up of minimal indivisible elements. The model theory postulates that the granularity of mental icons is quite coarse, and *mReasoner* uses similarly coarse icons with a granularity of eight divisions between impossibility and certainty, corresponding to the informal Aristotelian terms listed in (6) above. Overall, system 1 adduces evidence from knowledge of a relevant domain, simulates it in mental models, and uses these models to form an icon, which it can describe in Aristotelian terms.

The *mReasoner* program can judge probabilities using its small knowledge base, or it can interact with users to acquire such knowledge. When it interacts with users, its first step is to solicit from them any logical relation between the relevant propositions, *A* and *B*. For example, if *A* contradicts *B*, then their conjunction is impossible and has a probability of 0, but their disjunction is certain and has a probability of 1.0. If instead *A* implies *B*, then their conjunction equals the probability of *A*. The program also assesses whether the user believes that the occurrence of *A* has any effect on the probability of *B*, and whether such an influence increases or decreases its value. It solicits evidence for this influence from the user. We now consider how system 1 uses evidence to infer the probabilities of unique events, and how it infers conjunctive, disjunctive, and conditional probabilities.

4.2. The implementation of intuitions about probabilities

System 1 in *mReasoner* can make an intuitive estimate of a unique probability, such as:

What is the probability that Apple is profitable this year? (17)

It uses knowledge, such as:

Apple is a well-managed company. (18)

and:

Most well-managed companies are profitable. (19)

to infer:

Apple is likely to be profitable. (20)

The preceding inference is striking, because the conclusion refers to a probability, but neither premise does, so an obvious question is: Where does the probability come from? System 1 infers it from a mental model of the quantified assertion (19). In principle, it could do so from a pre-existing model in its database, but the present method allows users to input their own premises akin to those above. The program constructs a mental model representing the evidence that most well managed companies are profitable:

well-managed-firm	profitable
well-managed-firm	profitable
well-managed-firm	profitable
well-managed-firm	

(21)

This diagram denotes a mental model of an arbitrary number of entities—four in this case, and so its first row represents a well-managed company that is profitable (for the details of how models are constructed from quantified assertions, see Khemlani & Johnson-Laird, 2012a,b). Unlike the diagram, a real mental model is a simulation of actual entities (Khemlani, Mackiewicz, Bucciarelli, & Johnson-Laird, 2013). Some models, however, also contain abstract symbols (e.g., a symbol for negation; Khemlani, Orenes, & Johnson-Laird, 2012). Given that Apple is a well-managed firm, system 1 uses the model above to construct an icon derived from the *proportion* in the model of well-managed firms that are profitable:

|------ | (22)

System 1 can translate this icon into a verbal description:

It's highly likely. (23)

System 1 copes with determiners ranging from *all* and *any* through *most* and *many* to *few* and *none*. It can represent only a single mental model at any one time, but it can consider more than one piece of relevant evidence. Indeed, users may know that Apple is an innovator and believe that some innovators are profitable. If system 1 uses only this evidence, and ignores the evidence yielding the conclusions in (23), it constructs the following icon:

|---- | (24)

This icon ends at the midpoint, and so it translates into: "as likely as not."

Readers may wonder why the theory postulates the use of an icon given that proportions in models correspond to what the icon represents. The answer is that an icon allows different pieces of evidence for an event to be combined to estimate its probability. And so system 1 can combine the effects of several pieces of evidence about Apple's profitability. It starts with the first piece of evidence, such as that most well-managed companies are profitable, from which given that Apple is well managed, it can make the inference yielding the belief icon in (22). With further evidence, such as that Apple is an innovator, and that some innovators are profitable, it infers that Apple's profitability is "as likely as not." It inserts this probability as a second pointer, "^," into the existing icon in (22) to yield one that represents two different probabilities (the new pointer and the right-hand end of the icon):

`|---^--    |` (25)

System 1 then makes a primitive average of the pointer and the end of the icon to yield:

`|-----     |` (26)

The two pieces of evidence about a single event yield an icon supporting the conclusion that Apple is likely to be profitable (20). Likewise, any subsequent piece of evidence can shift the existing pointer using the same mechanism. The theory, however, postulates that individuals tend to consider only a minimal amount of evidence.

The way in which system 1 makes primitive averages depends on its non-numerical operations, which we now review before considering the computation of the probabilities of compound events. System 1 can carry out six non-numerical, or primitive, operations: p-averages, p-adds, p-subtracts, p-longer, p-multiplies, and p-divides. Table 2 summarizes these operations. Each of them relies on a finite loop that iterates no more than a small fixed number of times depending on the granularity of icons. Both the number of iterations and the granularity of icons may vary from one person to another. The operation, p-averages, is a primitive form of averaging: It moves two pointers toward one another until they meet, and so it yields a rough average of their values. This operation deals with icons representing conflicts in the evidence for a single event, such as Apple's profitability.

The operations in Table 2 underlie intuitive estimates of conjunctive, disjunctive, and conditional probabilities, which we described earlier. Conjunction, $P(A\&B)$, depends on p-averages to resolve the conflict between the probabilities of the two conjuncts, $P(A)$ and $P(B)$. The system assesses the probability of a disjunction, $P(A \vee B)$, guided by the uncertainty of disjunctions, and so it also uses p-averages. Conditional probability, $P(B|A)$, uses A as evidence for B in the same way that system 1 uses evidence to assess the probability of a single event, such as Apple's profitability. If the evidence, A, neither increases nor decreases the magnitude of $P(B)$, the system returns its initial estimate of $P(B)$. System 1 can compute elementary logical relations between A and B, such as:

Clinton is elected U.S. President in 2016 or Clinton is not elected U.S. President in 2016. (27)

and in this case returns the value of "certain" for the disjunction.

In summary, system 1 in *mReasoner* infers intuitive non-numerical probabilities from mental models of evidence, which it represents iconically as magnitudes to combine them. Its combinations depend on primitive analogs of averaging, adding, and multiplying magnitudes in these icons to infer the probabilities of compound events. System 1, of course, has a role in other sorts of cognition, where it is susceptible to the same constraints on its operations. For example, its limitations predict the intuitive conclusions that individuals tend to draw from syllogistic premises (see Khemlani, Lotstein, & Johnson-Laird, unpublished date). In all three probabilistic cases that we have considered—conjunctive, disjunctive, and conditional probabilities—the output of system 1 is an icon representing an intuitive probability. It can be put into qualitative descriptions using Aristotelian terms, as in (6) above, or it can be passed on to system 2 to be expressed numerically.

Table 2
The five primitive arithmetical operations that System 1 in *mReasoner* carries out

Primitive Operations	Input	Operation	Output
p-averages	One icon with two magnitude pointers	Finite loop moves the two pointers an increment at a time so they converge on an approximate mid-point	A magnitude that is an average of the initial pointers' magnitudes
p-adds	One icon with two magnitude pointers	Finite loop joins each element of second magnitude to the first, checks result against maximum magnitude using *p-subtracts*	A magnitude ≤ the one equivalent to "certainty"
p-subtracts	One icon with two magnitude pointers	Finite loop removes each element of second magnitude from the first. Halts if first magnitude becomes empty	A magnitude ≥ empty one equivalent to "impossibility"
p-longer	One icon with two magnitude pointers	Finite loop *p-subtracts* second magnitude from first	If first magnitude is not empty, returns "true"; otherwise "nil"
p-multiplies	One icon with two non-empty magnitude pointers	Finite loop *p-adds* second magnitude to first, and *p-subtracts* an element from second on each iteration. Halts when second becomes empty. Checks result against maximum magnitude using *p-subtracts*	A magnitude ≤ standard one equivalent to "certainty"
p-divides	One icon with two non-empty magnitude pointers	Finite loop *p-subtracts* denominator magnitude from numerator magnitude, *p-adds* an element to output on each iteration until numerator is *p-longer* than denominator	A magnitude > empty one equivalent to "impossibility"

4.3. The implementation of deliberations yielding numerical probabilities

System 2 has access to working memory and so it can carry out more complex operations, including counting and arithmetic. Its role in probabilistic inferences is likely to be confined to transforming non-numerical icons into numerical probabilities, and carrying out slightly more rational operations, such as the exact multiplication of probabilities for conjunctions, and their addition for disjunctions. Multiplication, however, is normatively correct only for the conjunction of independent events, and addition is likewise normatively correct only for disjunctions of mutually inconsistent events. The version of system 1 implemented in *mReasoner* is able to carry out primitive analogues of these operations too. Hence, a remaining open question is whether in fact these operations are carried out by system 2.

4.4. The general predictions of the model theory

The model theory makes four general predictions about estimates of the probabilities of unique events:

1. Probability estimates of unique events should be systematic: Individuals from the same population have available roughly the same sorts of evidence about real possibilities, and so their estimates should concur reliably in rank order.
2. The primitive operations of system 1 should lead to systematic violations of the JPD. In the case of a conjunction, intuition makes a simple compromise between conflicting values of $P(A)$ and $P(B)$, yielding their primitive average. In the case of a disjunction, intuition takes the uncertainty between $P(A)$ and $P(B)$ to elicit their primitive average. In the case of a conditional probability, $P(B|A)$, intuition treats A as evidence for estimates of $P(B)$.
3. When individuals have already made estimates of $P(A)$ and $P(B)$, they should be more likely to use system 2's methods to estimate conjunctive and disjunctive probabilities. Hence, they should tend to estimate $P(A\&B)$ as the product of $P(A)$ and $P(B)$, which is appropriate only for independent events; and they should tend to estimate $P(A \vee B)$ as the sum of $P(A)$ and $P(B)$, which is also appropriate only for inconsistent events. They have no deliberative method for estimating conditional probabilities about unique events.
4. When individuals have already made estimates of $P(A)$ and $P(B)$, they should be faster to make estimates of any sort of compound.

Although some studies have examined estimates of the probabilities of real unique possibilities (e.g., Zhao et al., 2009), none as far as we know provides results that allow us to test these predictions. We therefore carried out three experiments to do so. Three different ways to fix the JPD for two binary variables A and B are as follows (see Appendix B):

1 $P(A)$, $P(B)$, and $P(A\&B)$

2 $P(A)$, $P(B)$, and $P(A \vee B)$
3 $P(A)$, $P(B)$, and $P(B|A)$

A challenge to readers is to discover a set of three judgments of probabilities that fix the JPD for two binary variables *A* and *B*, but that, no matter what their values are, these values cannot violate the JPD. The triple can include probabilities of individual events, such as $P(A)$ or $P(B)$, and any sort of compound probabilities, such as $P(A\&B)$, $P(A \vee B)$, or $P(A|B)$. Solutions to this puzzle are in Appendix C. However, estimates for the three triples above can violate the JPD, and we used them in our experiments to test the model theory's predictions. In each study, participants estimated the probability of pairs of related unique events, $P(A)$ and $P(B)$. In Experiment 1, participants also made an estimate of the probability of their conjunction, $P(A \textit{ and } B)$; in Experiment 2, they made an estimate of the probability of their disjunction, $P(A \textit{ or } B \textit{ or both})$; and in Experiment 3 they made estimates of the two conditional probabilities $P(B|A)$ and $P(A|B)$.

5. Experiment 1: Conjunctions of unique events

When naive individuals estimate the probability of a conjunction of two unique events, the model theory predicts that their intuitions should be based on the primitive average of the probabilities of the two conjuncts, or, in deliberations, they should be based on multiplying them. The former yields violations of the JPD; the latter does not, but it is appropriate only when the two events are independent of one another.

5.1. Method

Forty-two participants completed the study for monetary compensation (a \$10 lottery) on an online platform, Amazon Mechanical Turk (for an evaluation of the validity of this method, see Paolacci, Chandler, & Ipeirotis, 2010). All the participants stated that they were native English speakers. The participants each had to estimate sets of three probabilities of unique events in four different orders:

1 $P(A\&B)$, $P(A)$, $P(B)$
2 $P(A)$, $P(A\&B)$, $P(B)$
3 $P(B)$, $P(A\&B)$, $P(A)$
4 $P(A)$, $P(B)$, $P(A\&B)$

There were 16 sets of materials, which concerned unique events in domains such as politics, sports, and science (Appendix D presents the full set of materials). For half the materials of the problems, *A* was likely to increase the value of $P(B)$, and for the other half, *A* was likely to decrease the value of $P(B)$. In a prior norming study, judges decided independently whether *A* increased $P(B)$, decreased $P(B)$, or had no effect on $P(B)$ for a larger battery of materials. For the two sorts of materials in the present experiment, we used only items for which the majority of judges had categorized *A* as increasing $P(B)$

and as decreasing $P(B)$, respectively. Hence, A and B in the experiment were not independent events. The allocation of the materials to the four orders was random with the constraint that all the participants had four trials in each of the four orders, which were presented to each of them in a different random order. The problems were presented one question at time in the appropriate order on the computer screen. The participants made a response by moving a slider on a scale from 0% to 100%, and the relevant percentage probability of the position of the slider appeared above it. The instructions told the participants to make their best estimate of the probabilities of the events but did not mention that their responses would be timed.

5.2. Results and discussion

The raw data for this experiment and the two subsequent ones are available at http://mentalmodels.princeton.edu/portfolio/unique-events/. The participants concurred in the rank order of their estimates over the different materials for $P(A)$, for $P(B)$, and for $P(A\&B)$ (Kendall's W's = .32, .16 and .21, respectively, all ps < .0001). These significant correlations corroborated the model theory's first prediction that individuals rely at least in part on knowledge and processes in common to estimate unique probabilities.

Fig. 1 shows the percentages of trials for each of the 16 materials in which participants estimated the probability of the conjunctions by taking p-averages of the probabilities of their conjuncts, and p-multiplies of the probabilities of the conjuncts. We scored a response as a p-average or as a p-multiplies provided (a) that it was within 12.5% of the correct arithmetic value, which assumes a granularity of eight divisions in the belief icon; and (b) in the case of p-average, that the conjunctive probability estimate was in between the estimated probabilities of the individual conjuncts. The balances of the percentages are cases in which the participants used other miscellaneous strategies of an unknown nature. A multiplicative response would have been normative had the two events been mutually exclusive, but they were not. Goodness-of-fit metrics showed that a p-average of each participants' estimates of $P(A)$ and $P(B)$, on a trial by trial basis, provided a

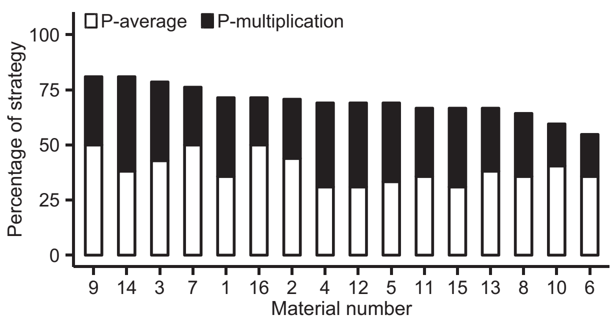

Fig. 1. The percentages of trials on which the participants in Experiment 1 estimated $P(A\&B)$ by taking a p-average or a p-multiplication of $P(A)$ and $P(B)$ for each of the 16 materials in the experiment (see Appendix D). The balances of percentages in each case were miscellaneous responses.

better fit of the data (Kolmogorov–Smirnov test, $D = .43$, $p < .001$; $R^2 = .82$; RMSE = .11) than a p-multiplication of their estimates of the two probabilities (Kolmogorov–Smirnov test, $D = .45$, $p < .0005$; $R^2 = .78$; RMSE = .14). But the best fit with the data was a p-average for those estimates on which JPDs were violated and otherwise their p-multiplication (Kolmogorov–Smirnov test, $D = .14$, n.s.; $p = .79$; $R^2 = .91$; RMSE = .05). Overall, 28 of the 42 participants made conjunctive estimates that fell between their estimates of $P(A)$ and $P(B)$ on at least 50% of trials (Binomial test, $p < .025$, assuming a conservative prior probability of .5). These results corroborate the model theory's second prediction that individuals should make systematic violations of the norms of the probability calculus, and that in the case of conjunctions their estimates were either primitive averages or primitive products of the two probabilities (see Table 2).

Table 3 presents the percentages of trials on which there were violations of the JPD, and the mean percentages that the participants estimated for the three sorts of events, depending on the order of the judgments. As it shows, violations were quite marked when the conjunction came first in the series of judgments. Indeed, order had a highly reliable effect on the frequency of violations: the earlier the position of the conjunction, the greater the likelihood of a violation (Page's $L = 531$, $z = 2.94$, $p < .003$).

The percentages of estimates in which the participants p-multiplied $P(A)$ and $P(B)$ to estimate $P(A\&B)$ were as follows:

$$\begin{array}{ll} P(A\&B)\ P(A)\ P(B) & 25\% \\ P(A)\ P(A\&B)\ P(B) & 26\% \\ P(B)\ P(A\&B)\ P(A) & 30\% \\ P(A)\ P(B)\ P(A\&B) & 42\% \end{array} \tag{28}$$

The trend for more multiplicative responses the later in the series the estimate of $P(A\&B)$ occurred was reliable (Page's $L = 534$, $z = 3.27$, $p = .001$, collapsing the two middle orders). This result corroborates the model theory's third prediction.

The variability in the estimates of probability was similar across the different materials, with standard deviations ranging, for example, from 17% to 31% for estimates of $P(A)$, and comparably for $P(B)$. A reviewer suggested that some participants might have

Table 3
The percentages of trials yielding violations of the JPD in Experiment 1 and the mean percentage judgments for three sorts of events depending on their order

The Estimation Order	The Percentages of Violations of JPD	$P(A)$ %	$P(B)$ %	$P(A\&B)$ %
1. $P(A\&B)$ $P(A)$ $P(B)$	61	41	32	30
2. $P(A)$ $P(A\&B)$ $P(B)$	49	38	34	27
3. $P(B)$ $P(A\&B)$ $P(A)$	51	35	38	28
4. $P(A)$ $P(B)$ $P(A\&B)$	46	38	41	29
Overall results	50	38	37	29

opted for an easy solution in which they estimated 50% for all three probabilities, but in fact such a pattern of responses occurred only on <1% of trials. In general, responses of 50% were relatively rare in this experiment (and the subsequent ones), occurring on <5% of estimates. To assess the variability in the participants' responses, we computed Shannon's measure of information for each participant. Its overall mean was 0.90 bits, and it varied from 0.58 bits in the participant with the least variety in numerical responses to 1.10 bits in the participant the greatest variety in numerical responses. The participants differed reliably in this measure (Friedman non-parametric analysis of variance, $\chi^2 = 60.33$, $p < .001$). This result suggests that participants did not use a single uniform strategy in estimating the probabilities of conjunctions, and they did not make trivial errors, such as estimates of 50% for all three probabilities.

Fig. 2 presents the latencies of the probability judgments. Not surprisingly, estimates of the conjunction take longer than estimates of its conjuncts (Wilcoxon test, $z = 5.07$, $p < .0001$). The latencies show that the earlier in the sequence an estimate of a conjunction is made, the longer it takes (Page's $L = 562$, $z = 6.33$, $p < .0001$), and likewise for the estimates of the conjuncts. The two preceding statistical tests depend only on the rank order of latencies, and so a transformation of the raw data, say, to log latencies, would have no effect on their results. They corroborate the model theory's fourth prediction. When the conjunction occurs first in the set of estimates, the participants think about both *A* and *B* in an intuitive way and map their estimates onto the numerical scale. But, when the judgments of *P*(*A*) or *P*(*B*) precede the judgment of their conjunction, its estimate can take into account the numerical estimates for the conjuncts. Given these numerical estimates, individuals are also more likely to multiply probabilities and to avoid violations of the JPD.

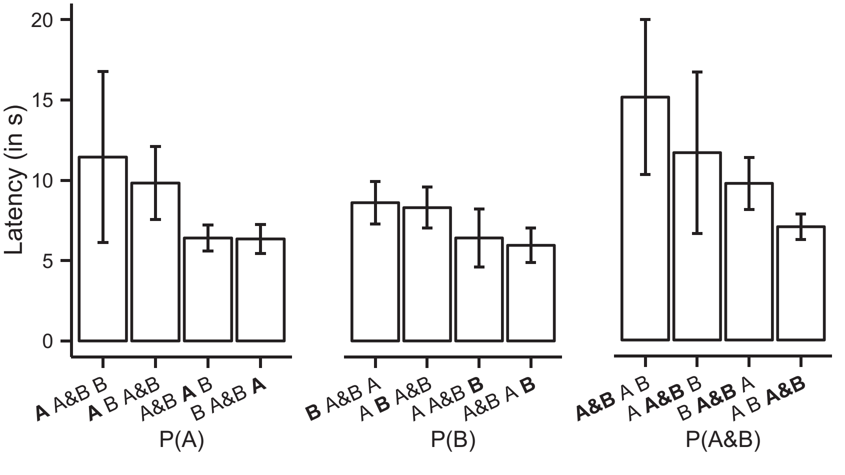

Fig. 2. The latencies (s) of *P*(*A*), *P*(*B*), and *P*(*A*&*B*) in Experiment 1 depending on the order in which they were estimated. Text in bold shows the position in the sequence of estimates at which the relevant estimate was made.

6. Experiment 2: Disjunctions of unique events

Intuitive estimates of the probabilities of inclusive disjunctions should tend to be based on the primitive average of the estimates of the two disjuncts, and, as a result, to violate the JPD. In cases in which participants deliberate, they should tend to add the estimates of the two disjuncts even though this procedure is appropriate only for mutually exclusive events. Experiment 2 tested the predictions using the same materials as those in the previous experiment, but the compound was now a disjunction:

> What is the chance that a nuclear weapon will be used in a terrorist attack in the next decade or there will be a substantial decrease in terrorist activity in the next 10 years, or both? (29)

6.1. Method

The participants each had to estimate sets of three probabilities of unique events in two different orders: $P(A)$, $P(B)$, $P(A \vee B)$, and $P(A \vee B)$, $P(A)$, $P(B)$. They carried out 16 problems in total. The order of the problems and the assignment of materials were randomized, and participants encountered a particular material only once in the experiment. The program that administered the experiment presented each question separately, and it recorded the participant's numerical estimate and its latency. The 41 participants who carried out the experiment were a new sample from the same population as before. All of them stated that they were native English speakers. They participated in a \$10 lottery as in Experiment 1.

6.2. Results and discussion

The participants concurred in the rank order of their estimates over the different materials for $P(A)$, for $P(B)$, and for $P(A \vee B)$ (Kendall's W's = .29, .17, .18, respectively, $p < .0001$ in all three cases). The result corroborates the model theory's first prediction that participants rely at least in part on knowledge and processes in common to make estimates of unique events. Fig. 3 shows the percentages of trials for each of the 16 materials in which participants estimated the probability of the disjunctions by taking p-averages of the probabilities of their disjuncts, and p-additions of the probabilities of the their disjuncts. An additive response would have been normative had the two events been mutually inconsistent, but they were not. Goodness-of-fit metrics showed that the p-average of each participants' estimates of $P(A)$ and $P(B)$, on a trial-by-trial basis, yielded a better fit of the data (Kolmogorov–Smirnov test, $D = .29$, $p = .06$; $R^2 = .60$; RMSE = .09) than the p-addition of their estimates (Kolmogorov–Smirnov test, $D = .71$, $p < .0001$; $R^2 = .52$; RMSE = .25). The best fit was based on a p-average of $P(A)$ and $P(B)$ for violations of the JPD and otherwise their p-addition (Kolmogorov–Smirnov test,

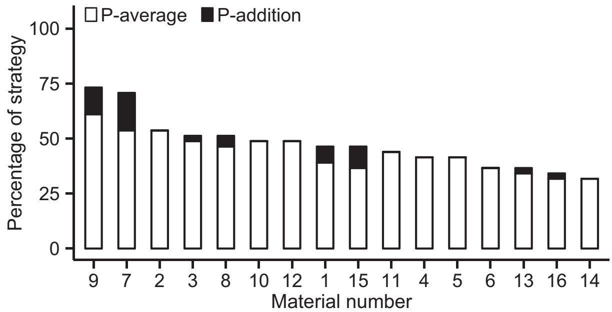

Fig. 3. The percentages of trials on which the participants in Experiment 2 estimated $P(A \vee B)$ by taking a p-average or a p-addition of $P(A)$ and $P(B)$ for each of the 16 materials in the experiment (see Appendix D). The balances of percentages in each case were miscellaneous responses.

$D = .24$, n.s.; $R^2 = .77$; RMSE $= .11$). Overall, 32 of the 41 participants made disjunctive estimates that fell between their estimates of $P(A)$ and $P(B)$ on at least 50% of trials (Binomial test, $p < .0005$). They also bear out the restricted nature of system 1, which cannot compute the correct value of $P(A) + P(B) - P(A\&B)$. These results corroborate the model theory's second prediction that individuals should make systematic violations of the norms of the probability calculus, and that in the case of disjunctions their estimates were either primitive averages or primitive additions of the two probabilities.

Table 4 presents the percentages of trials that violated the JPD and the mean percentages for the three sorts of estimate. Order had no reliable effect on the frequency of JPD violations (Wilcoxon test, $z = 1.58$, $p = .12$). The percentages of estimates in which the participants added $P(A)$ and $P(B)$ to estimate $P(A \vee B)$ were as follows:

$$\begin{array}{ll} P(A \vee B)\ P(A)\ P(B) & 3\% \\ P(A)\ P(B)\ P(A \vee B) & 5\% \end{array} \tag{30}$$

There was no reliable difference between the two orders (Wilcoxon test, $z = 1.29$, $p = .20$), which is contrary to the third prediction of the model theory. But, as Fig. 4 shows, the time to make an estimate speeded up over the triples of estimates (Page's $L = 562$, $z = 6.33$, $p < .0001$). The estimates of the disjunction took longer than estimates of its disjuncts (Wilcoxon test, $z = 4.54$, $p < .0001$), and longer when the disjunction

Table 4
The percentages of violations of the JPD in Experiment 2 and the mean percentage judgments for three sorts of events depending on their order

The Order of the Judgments	The Percentage of Violations of the JPD	$P(A)$ %	$P(B)$ %	$P(A \vee B)$ %
1. $P(A \vee B)$ $P(A)$ $P(B)$	62	42	37	44
2. $P(A)$ $P(B)$ $P(A \vee B)$	69	41	35	42
Overall results	65	41	36	43

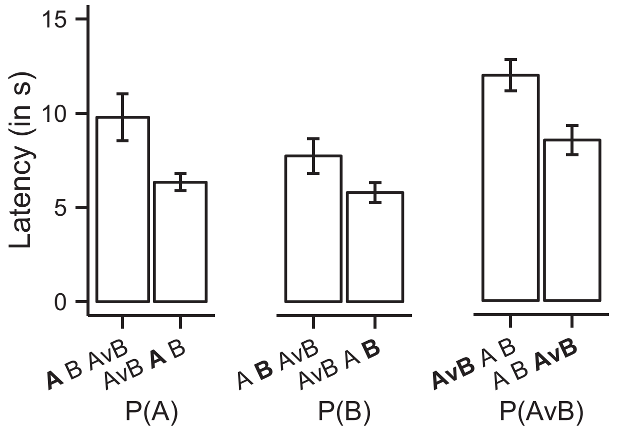

Fig. 4. The latencies (s) of the probability estimates in Experiment 2 depending on their order. Text in bold shows the position in the sequence of estimates at which the relevant estimate was made.

occurred first than when it occurred last (Wilcoxon test, $z = 4.73$, $p < .0001$). This result corroborates the model theory's fourth prediction.

As in the previous experiment, we computed Shannon's measure of information for the estimates of each participant. Its overall mean was 0.96 bits, and it varied from 0.76 bits in the participant with the least variety in numerical responses to 1.07 bits in the participant the greatest variety in numerical responses. The participants differed reliably in this measure (Friedman non-parametric analysis of variance, $\chi^2 = 41.10$, $p = .0002$). This result suggests that participants used a variety of strategies in estimating disjunctive probabilities. In line with this result, only 3 of the 41 participants responded 50% to all three judgments on at least one trial in the experiment.

When individuals estimate the probability of the disjunction last, they have access to their estimates of its disjuncts. However, there was no evidence for an increase in additive responses when the disjunction occurred last, and the low level of these responses is surprising, though probably in part a consequence of the fact that on nearly a third of trials, a sum of the probabilities of the disjuncts would have exceeded a probability of 100%.

7. Experiment 3: Conditional probability estimates

The model theory postulates that individuals estimate the probability of an individual event, $P(B)$, by adducing evidence. It treats estimates of a conditional probability, $P(B|A)$, in an entirely analogous way in which A is evidence pertinent to $P(B)$. In case individuals have a prior estimate of this probability, this new evidence can shift this estimate one way or another (see Sections 3.1 and 4.1). The process is a mechanistic implementation of the anchoring and adjustment heuristic proposed by Tversky and Kahneman (1983), and it can lead to violations of the JPD, because it does not take into account either $P(A)$

or the converse conditional probability. Experiment 3 tested the predicted corollaries of this account: the participants' estimates should not accord with Bayes's theorem, and often violate the JPD.

7.1. Method

The participants made four probability estimates concerning two unique events in a fixed order: $P(A)$, $P(B|A)$, $P(B)$, and $P(A|B)$. Examples of $P(B|A)$ and $P(A|B)$ are as follows:

What is the chance that there will be a substantial decrease in terrorist activity in the next 10 years, assuming that a nuclear weapon will be used in a terrorist attack in the next decade? (31)

What is the chance that a nuclear weapon will be used in a terrorist attack in the next decade, assuming that there will be a substantial decrease in terrorist activity in the next 10 years? (32)

The sample of 43 participants was drawn from the same population as before, and all of them stated that they were native English speakers. They participated in a \$10 lottery as in Experiment 1. They carried out 12 problems in total: Six had materials in which A increased the value of $P(B)$, and six had materials in which A decreased the value of $P(B)$ according to the previous norming study. The experiment accordingly used 12 of the 16 materials in Appendix D (see Fig. 5 for a list of the 12 materials). The order of the problems was randomized for each participant. The program that administered the experiment presented each question separately, and it recorded the participant's numerical estimate and latency.

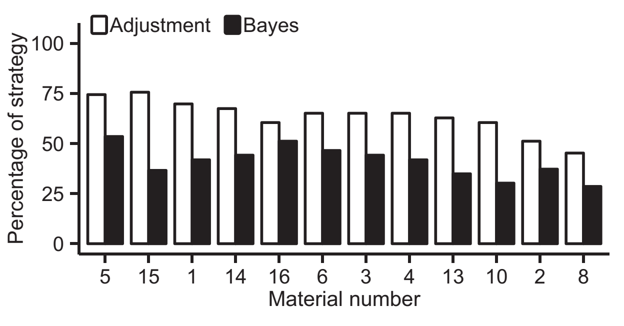

Fig. 5. The percentages of trials on which the participants in Experiment 3 estimated $P(B|A)$ by a minimal adjustment to $P(B)$ and the percentages in accordance with Bayes's theorem for each of the 12 contents in the experiment (see Appendix D).

7.2. Results and discussion

The participants concurred in the rank order of the probabilities that they estimated over the different contents for $P(A)$, for $P(B)$, and for their conditional probabilities, $P(B|A)$ and $P(A|B)$ (Kendall's W's = .28, .14, .09, .17, respectively, $p < .0001$ in all four cases). The result corroborates the model theory's first prediction that participants adduce common evidence to make estimates of unique events, and it suggests that participants are prepared to estimate conditional probabilities in the absence of base rate information, because they estimated $P(B|A)$ before they estimated $P(A)$, which was a subsequent judgment on each trial.

Goodness-of-fit metrics showed that the model theory provided a better fit of the data (Kolmogorov–Smirnov test, $D = .26$, $p = .12$; $R^2 = .57$; RMSE = .10) than Bayes's theorem (Kolmogorov–Smirnov test, $D = .47$, $p = .0002$; $R^2 = .11$; RMSE = .64). But the model theory's fit is noticeably poorer than in the case of conjunctions (Experiment 1) and disjunctions (Experiment 2). One possibility is that the prior norming study of whether A increases $P(B)$ or decreases $P(B)$ failed to do justice to the present participants' beliefs. We therefore carried out an analysis in which we examined the fit taking either one-step up or one-step down from each participant's estimate of $P(B)$ depending on the direction of the participant's estimate of $P(B|A)$, where a step was according to the granularity of *mReasoner's* icon, that is, a step of approximately 12.5%. The resulting fit with the theory was comparable to the results of the previous experiments (Kolmogorov–Smirnov test, $D = .16$, $p = .62$; $R^2 = .74$; RMSE = .07). In contrast, an analogous procedure applied to the Bayesian computations did not yield a significant improvement to its fit to the data (Kolmogorov–Smirnov test, $D = .44$, $p = .0005$; $R^2 = .12$; RMSE = .64). Fig. 5 shows the percentages of trials for each of the 16 contents in which participants estimated the conditional probability, $P(B|A)$, by adjusting their estimate of $P(B)$ in this minimal way and the percentages of trials on which their judgments were in accord with Bayes's theorem (categorized as such if a participants' response fell within 12.5% of the adjustment or the Bayesian calculation). In contrast with the previous studies, the two strategies under investigation in Experiment 3 were independent from one another, and indeed, in some cases they appeared to coincide, that is, an estimate could be interpreted as reflecting both an adjustment and a Bayesian strategy. Accordingly, a post hoc analysis of strategies revealed that participants adopted an adjustment strategy reliably more often than a Bayesian one (63% vs. 41%, Wilcoxon test, $z = 5.41$, $p < .0001$).

Overall, the participants' first three estimates violated the JPD on 24% of the trials, and 42 of the 43 participants made one or more such estimates (Binomial test, $p < .0001$). On each trial, the participants' first three estimates—of $P(A)$, $P(B|A)$, $P(B)$—provide the input for Bayes's theorem to predict their fourth estimate of $P(A|B)$, and so we examined to what extent they might rely on the theorem to make this fourth estimate. The correlation between the prediction based on Bayes's theorem and the participants' estimates was not reliable ($R^2 = .02$; RMSE = 1.25), and the two sets of data were reliably different (Kolmogorov–Smirnov test, $D = .37$, $p < .01$). In many cases, the partici-

Table 5
The percentages of trials violating the JPD in Experiment 3, and the mean percentage judgments for the four sorts of probability depending on the effect of *A* on *P*(*B*) established in a prior norming study

Effect of *A* on *P*(*B*)	The Percentage of Violations of the JPD	*P*(*A*) %	*P*(*B*|*A*) %	*P*(*B*) %	*P*(*A*|*B*) %
A increases *P*(*B*)	19	42	39	43	36
A decreases *P*(*B*)	28	45	30	35	42
Overall results	24	43	35	39	39

Table 6
The latencies (s) of the probability estimates in Experiment 3 depending on the effect of *A* on *P*(*B*).

Effect of *A* on *P*(*B*)	*P*(*A*)	*P*(*B*|*A*)	*P*(*B*)	*P*(*A*|*B*)
A increases *P*(*B*)	10.76	11.54	7.07	10.11
A decreases *P*(*B*)	9.26	12.07	9.16	9.87
Overall means	10.01	11.81	8.12	9.99

pants' first three estimates violated the JPD, but even when we capped the estimates to prevent such violations the correlation with Bayes's prediction was weak (R^2 = .29). These results corroborate the model theory's second prediction that individuals should make systematic violations of the norms of the probability calculus.

Table 5 presents the percentages of trials that violated the JPD, and the mean percentages of the four sorts of estimate depending on whether A increased or decreased the probability of event *B*. This difference, estimated in the prior norming study, had an overall reliable effect on estimates of $P(B|A)$ and $P(A|B)$ (Wilcoxon test, $z = 2.15$, $p = .03$). The fixed order of the four estimates—$P(A)$, $P(B|A)$, $P(B)$, $P(A|B)$—did not allow us to evaluate the model theory's third prediction. Informal studies have sometimes found that individuals give the same values for the two conditional probabilities (J. Baron, personal communication). Our participants did so on 15% of trials.

Table 6 presents the latencies of the probability judgments. The table shows that estimates of $P(B|A)$ take longer than any of the other three estimates (Wilcoxon tests, $z > 3.75$, $p < .0005$ in all cases), and that participants are faster to estimate $P(B)$ than $P(A)$ (Wilcoxon test, $z = 4.83$, $p < .0001$). The data are consistent with the model theory's fourth prediction that to estimate $P(B|A)$, participants tacitly estimate $P(B)$. The result is to speed up their subsequent estimate of $P(B)$.

In general, the results of Experiment 3, like those of the previous experiments, corroborated the model theory.

8. General discussion

Why, depending on the odds, do people accept or reject bets about unique events? After all, an event such as the election of Hillary Clinton to the Presidency will either occur or not—a fact that has led frequentists to declare that a statement of its probability is meaningless (Cosmides & Tooby, 1996, p. 6) or that the probability calculus is irrelevant to these judgments (Gigerenzer, 1994). A Bayesian riposte is that the probability of a unique event reflects the degree to which an individual believes in its occurrence, that is, the event's subjective probability. Hence, granted a good pay-off, when the odds of a bet offered by a bookmaker translate into a probability better than this subjective probability, individuals are apt to accept the bet; but when the odds translate into a probability worse than this subjective probability, they are apt to spurn it (de Finetti, 1936/1995; Ramsey, 1926/1990). The riposte reveals three mysteries that this study was designed to solve: the source of the numbers in subjective probabilities, the mechanism that determines their magnitudes, and the process yielding the probabilities of compound assertions, such as disjunctions. To tackle them, our research has introduced several innovations. Previous studies of the probabilities of compounds reported errors in their estimates (e.g., Abelson, Leddo, & Gross, 1987; Zhao et al., 2009), but did not provide a definitive explanation of them. We developed a dual-process theory postulating intuitions based on uncertainty, simulations of evidence in mental models, and deliberations that translated non-numerical icons into numbers. We implemented the model theory's account of system 1 and system 2 in a computer program, *mReasoner*, which integrates deductive and probabilistic reasoning. The model theory applies to estimates of the probabilities of unique events, and to conjunctive, disjunctive, and conditional probabilities of them.

The model theory distinguishes between system 1 and system 2 in computational power. System 1 has a severely curtailed power, carrying out loops of operations only a small number of times. For a unique event, such as:

What is the probability that Hillary Clinton is elected U.S. President in 2016? (33)

individuals may have access to various sorts of knowledge pertinent to an estimate of probability. They may adduce knowledge of possibilities, of causal relations, and of other sorts of factors, such as:

her age
her gender
her political career
her spouse
possible alternative Democratic candidates
possible Republican candidates

The limitations of system 1 imply that individuals tend to adduce a minimum of evidence, but it is a task for future studies to investigate the number and the factors that determine it. The theory implies that each piece of evidence is considered in relation to general knowledge. Clinton, for example, will be older than most U.S. Presidents when they were elected (a median age of 54 years), and so this renders her election slightly improbable. The model theory proposes that the evidence is used to set an initial magnitude of belief, and to modify it if necessary in the light of subsequent evidence. In *mReasoner*, quantified assertions play an exemplary role in yielding intuitive probabilities. For example, individuals may believe:

Few candidates over the age of 60 years are elected to the U.S. Presidency. (34)

But the theory does not imply that all evidence must correspond to a quantified assertion. Other evidence can be conditional: If a candidate's health is poor, then he or she may not be elected. A reviewer suggested that evidence itself is weighted so that some facts play a bigger role than others in setting a probability. Convincing evidence for the spontaneous use of such weightings would be contrary to the model theory, because a weighting mechanism seems beyond the capability of system 1.

Given the preceding evidence (34), the program simulates it in a mental model, such as:

Over 60 in age	elected U.S. President
Over 60 in age	
Over 60 in age	
Over 60 in age	

(35)

The proportion of cases yields an analog magnitude in an icon corresponding to a degree of belief that Clinton is elected:

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|--          |
```

(36)

The deliberative system 2 can map this icon into a numerical magnitude, such as a probability of 30%.

The model theory postulates that intuitions take uncertainty as a prime sign of improbability. One way in which uncertainty arises is when a conflict occurs. Suppose, for example, that one piece of evidence suggests that the event in (33) is unlikely, whereas another piece of evidence suggests that it is likely. System 1 resolves the conflict with a compromise. It takes a primitive average, and infers that the event is as likely as not.

Conflicts can also occur in compounds of events. Given that no logical relations exist between two unique events, *A* and *B*, a conflict arises in the probability of their conjunction, when system 1 yields different intuitive probabilities for them. System 1 again resolves the conflict by taking their primitive average. A slightly more rational response in the light of the probability calculus is to take their primitive product, though this response is correct only if one event has no effect on the probability of the other. Suppose, now, that such a conflict arises in the probability of an inclusive disjunction of two unique events, *A or B or both*—again, on the assumption that there are no logical implications between them. Disjunctions yield uncertainty (e.g., Bauer & Johnson-Laird, 1993; Tversky & Shafir, 1992), which cues improbability. And so, once again, system 1 uses the compromise of a primitive average. A slightly more rational response in this case is to sum $P(A)$ and $P(B)$, though this response is correct only if A and B cannot both occur.

Some philosophers and psychologists have argued that individuals estimate a conditional probability, $P(B|A)$, using a modified version of Ramsey's test (e.g., Bennett, 2003; Evans, 2007, p. 54). You add A to your beliefs. If you know that A is false, you adjust your beliefs where necessary to ensure their consistency, and then you estimate the probability of B. The problem is that individuals have too many beliefs, and so a test of their consistency is intractable (Johnson-Laird, Girotto, & Legrenzi, 2004; Oaksford & Chater, 2007, p. 107). Given that system 1 lacks computational power, intuitions about a conditional probability, $P(B|A)$, are likely to rely on a much simpler procedure. System 1 treats A as evidence that may increase $P(B)$ or decrease it (see Section 3.1). In the absence of a logical relation between them, the system then makes a minimal adjustment—in terms of the granularity of icons—to revise $P(B)$. If A has no effect on the probability of B, then the two events are independent, and no adjustment to $P(B)$ is necessary. This procedure for conditional probabilities has a family resemblance to Ramsey's test, but, as its implementation in *mReasoner* shows (see Section 4.1), it is tractable and rapid enough to square with the latencies of the participants' estimates in Experiment 3.

The use of intuitions, as the model theory and *mReasoner* imply, leads to numerical estimates of the probabilities of compounds that should tend to violate the probability calculus. To measure the degree of such violations, we introduced a measure based on negative probabilities in the complete joint probability distribution (JPD), where the JPD for two events, A and B, is the following set of probabilities: $P(A\&B)$, $P(A\&\neg B)$, $P(\neg A\&B)$, and $P(\neg A\&\neg B)$, where "¬" denotes negation (see the Introduction). The predictions with one exception were corroborated in all three of our experiments:

First, the participants concurred, weakly but reliably, in the rank order of their estimates of the probabilities of unique events, and their compounds.

Second, they tended to violate the JPD in the ways that the model theory predicts. According to the probability calculus, the probability of a conjunction of events, $P(A\&B)$, is the product of the probability of one event, such as $P(A)$, and the conditional probability of the other event, $P(B|A)$. This computation is beyond the power of system 1, and so, as the model theory predicts, individuals tend to take a primitive average of the two probabilities, or otherwise to multiply them (Experiment 1). The average yields violations of the JPD, but it provided a better fit to the data than a simple multiplicative account. The best fit used both procedures. The use of averaging was even apparent in the overall means in four of the problems. For instance, the mean estimate of the conjunction of millions of people living past 100 and an end to the replacement organ shortage was 22%, a value falling between the mean estimates of the two conjuncts (19% and 52%, respectively), and these means violate the JPD. The probability of an inclusive disjunction of events, $P(A \vee B)$, equals $P(A) + P(B) - P(A\&B)$. This computation is also beyond the power of system 1, and so the model theory predicts that the uncertainty evoked by disjunctions leads individuals to average the probabilities of the disjuncts. The primitive averaging operation yields violations of the JPD, but it provided a better fit to the data than an additive model (Experiment 2). The best fit used both procedures, though the addition of probabilities is appropriate only if the disjunction was exclusive, not inclusive. The conditional probability, $P(B|A)$, can be computed according to Bayes's theorem (3). This computation is again beyond the power of system 1, and so too is the equivalent computation of the ratio of $P(A\&B)$ to $P(A)$. In fact, as the model theory predicts, individuals tend to adjust $P(B)$ a minimal amount according to whether A increases or decreases the probability of B. This simple procedure provided a better fit to the data than Bayes's theorem (Experiment 3). The best fit of all ignored the predicted effect of A on $P(B)$ as determined in a prior norming study, and simply adjusted $P(B)$ up or down by a minimal amount. In other words, participants seem to have differed for certain contents from those tested in the norming study about the effects of A on $P(B)$.

Third, estimates of the probability of a conjunction were more likely to be multiplicative after the participants had estimated the numerical probabilities of their conjuncts (Experiment 1). The one exception to a corroboration of the model theory was there was no reliable sign of a greater tendency for additive estimates of disjunctions when these estimates occurred last in the series. Very few participants in the study made additive estimates, perhaps because many such additions would have exceeded a probability of 100%.

Fourth, estimates of the probabilities of conjunctions and disjunctions were faster after estimates of their constituents (Experiments 1 and 2). The estimates of conditional probabilities, $P(B|A)$ also took longer than the estimates of $P(A)$ or $P(B)$, and the estimate of the conditional probability, as the model theory predicts, speeded up estimates of $P(B)$ (Experiment 3).

The role of deliberations about probabilities (in system 2) is curtailed by ignorance. Most people know little about the probability calculus. Readers of this article know that the probability of a conjunction cannot be larger than that of either of its conjuncts, and that the probability of an inclusive disjunction cannot be smaller than that of either of its disjuncts. They are less aware, we suspect, of the third constraint on each of these estimates (see Appendix B): Given $P(A)$ and $P(B)$, an estimate of $P(A\&B)$ can be too low, and an estimate of $P(A \vee B)$ can be too high. The constraints on conditional probabilities are still more recondite. Before we started this research, our guess would have been that $P(A)$ and $P(B)$ put no constraints on $P(A|B)$ to yield a consistent JPD. Even experts do not know at once what the constraints are but have to calculate them (see Appendix A). We can illustrate the impotence of system 2 with two problems. The first problem is as follows:

$$\text{If } P(A) = .7 \text{ and } P(B) = .3, \text{what is the range of values for the conditional probability } P(A|B) \text{that yield a consistent JPD?} \tag{37}$$

The second problem has the opposite values:

$$\text{If } P(A) = .3 \text{ and } P(B) = .7, \text{what is the range of values for } P(A|B) \text{ that yield a consistent JPD?} \tag{38}$$

The exact numerical values call for calculation, but which of the two problems allows a greater range of values for $P(A|B)$? The answer is the first problem, which for a consistent JPD allows that the conditional probability can be anywhere from 0 to 1, whereas for the second problem it must be in the narrower range from 0 to .42 to two decimal places. Even the use of the Ramsey test to estimate conditional probabilities overlooks the constraint that $P(A)$ can put on $P(B|A)$.

Skeptics might argue that our participants are not making real estimates of probabilities. They are instead indulging in unprincipled guesses. Evidence counts against this conjecture. The participants performed better than chance in all three experiments (see Appendix B for the chance values). They concurred about the relative probabilities of the different contents, and these reliable concordances in all three experiments imply that they relied to some degree on common knowledge. The latencies of responses showed systematic patterns, reflecting greater thought about compounds than about their constituent events, and faster responses to compounds when the participants had already assessed the numerical probabilities of their constituents. Indeed, the robust fit of the processes implemented in *mReasoner* to the patterns of data could not have occurred without the participants' reliance on systematic mental processes.

Other skeptics might argue that alternative theories could account for our results, and that no need exists to postulate a dual-process theory or processes based on mental models. For example, an alternative theory might posit that degrees of belief depend on activations of particular chunks in memory (Anderson, 1990), or else reflect the causal power of instances of one concept on those of another (e.g., Cheng, 1997). Of course, no results can ever show that a theory is the only possible one, and so clearly other theories might be framed to explain the phenomena. It is critical, however, that any alternative theory accounts for the origins of numerical estimates from non-numerical premises, for how this information is mentally represented, and for why estimates violate the JPD. Accounts of reasoning based on formal rules of inference (e.g., Rips, 1994) offer no account of quantifiers, such as "most" and "few," and so they cannot explain their role in probabilistic thinking. Perhaps surprisingly, theories of reasoning based on probability logic (e.g., Oaksford & Chater, 2007) offer no account of how numerical probabilities derive from non-numerical considerations. Hence, at present, the model theory offers the only explanation of the phenomena, it is implemented in a computer program, and its predictions emerge from its unification of deductive and probabilistic reasoning. We have previously compared the model theory's predictions for conjunctions with other extant theories of probabilistic estimates (Khemlani, Lotstein, et al., 2012), though these alternative theories were not framed for the probabilities of unique possibilities (Fantino et al., 1997; Wyer, 1976). The comparison showed that the computational implementation of *mReasoner* yields a better fit to data about the probabilities of conjunctions than these alternatives.

The present results have implications for the role that probabilities play in higher cognition. Marcus and Davis (2013) have shown how the apparent successes of probabilistic approaches to a variety of tasks—from categorization to inferences in naïve physics—may depend on investigators' bias selecting certain tasks and neglecting others, and also on the choice of different statistical models from one domain to another. In the case of reasoning, recent developments—often motivated by the goal of sustaining the rationality of naïve individuals—have led to a cumulative introduction of probabilities into accounts of simple classical deductions. A modest step is to argue that the meaning of a conditional assertion, such as:

If Clinton is elected U.S. President in 2016, then all wars in the world will cease thereafter. (39)

is a conditional probability (e.g., Adams, 1998; Douven & Verbrugge, 2013; Pfeifer, 2013; Pfeifer & Kleiter, 2005). The next step replaces deductive validity with probabilistic validity, where an inference is probabilistically valid provided that the informativeness of its conclusion (1 minus its probability) is not greater than sum of the premises' informativeness (Adams, 1998). One proponent of a probabilistic approach to deduction, however, proposes to abandon its normative evaluation in favor of description alone (Evans, 2012). Others argue: "people are far better probabilistic reasoners than Tversky and Kahneman supposed" (Oaksford & Chater, 1998, p. 262). The burden of the present results on violations of the JPD is that not only were Tversky and Kahneman correct, but that human competence is still worse than one might have supposed: Errors occur even when inferences about probabilities are not elicited in scenarios likely to trigger heuristics such as representativeness.

The most radical probabilistic proposal replaces the classical probability calculus with quantum probabilities, which treat JPDs with negative values as acceptable, and so every response in our experiments is rational. But if psychologists abandon all norms of rationality, or equivalently allow that any set of estimates of probabilities is rational, no way exists to assess competence. Yet people differ in the ability to reason. The Wright brothers were better thinkers about aeronautical matters than their rivals, and their ability to think of counterexamples to their own conjectures helped them to develop the first controllable heavier-than-air craft (Johnson-Laird, 2006, ch. 25). Likewise, John Snow was a better thinker about the mode of communication of cholera than the proponents of the rival "miasma" theory. He understood that the gas laws refuted communication by way of a miasma (ibid., ch. 27). Yet a single counterexample as a decisive refutation plays no role in the approaches to reasoning summarized above, and so the yardstick distinguishing between good and bad thinking is in danger of disappearing too.

Let us put aside issues of rationality and ask what is certain in the light of our research. It establishes five points about naive probabilities:

1. Individuals readily make numerical estimates of the probabilities of unique events.
2. Their estimates yield systematic violations of the probability calculus.
3. These violations are predictable in terms of the intuition that uncertainty cues improbability.
4. The implementation of this principle in a particular dual-process algorithm predicts the violations. Of course, there are infinitely many other possible algorithms that can also do so in principle—though as far as we know no alternatives yet exist.

5. If intuitions about probabilities underlie deductive reasoning, then people will err in making deductions, too, even granted the yardstick of probabilistic validity.

How might individuals avoid violations of the probability calculus in estimating the probabilities of unique possibilities? Conjunctions seem transparent, and perhaps the best way to reveal their true nature is to consider the conjunction of very rare events, such as the probability of winning the New York lottery and being struck by lightning on the same day. Individuals may realize that such a conjunction is less probable than either of its conjuncts. Disjunction creates a veil of uncertainty, and one way to lift the veil is to use disjunctions of mutually exclusive events, which reduce the number of mental models. Deductions from exclusive disjunctions are easier than those from inclusive disjunctions (e.g., Bauer & Johnson-Laird, 1993; Johnson-Laird et al., 1992), and so too should be inferences of their probabilities. Individuals should be more likely to make a primitive addition of probabilities. Still easier should be exclusive disjunctions that are clearly exhaustive, such as the probability that a card is either an ace or not.

In conclusion, the intuitions of naïve individuals accord with the Bayesian interpretation of probabilities as degrees of belief. This interpretation extends to unique events, which are important in daily life and in science. The relative rarity of compound bets in prediction markets may reflect folk wisdom about the difficulty of conforming to the probability calculus, and hence a potential vulnerability to "Dutch books." The model theory, which bases rationality on computations that simulate the world (Craik, 1943), explains this difficulty in terms of the restricted operations of the intuitive system of reasoning. They curtail the ways in which the system can combine evidence and estimate conjunctive, disjunctive, and conditional probabilities.

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References

Abelson, R. P., Leddo, J., & Gross, P. H. (1987). The strength of conjunctive explanations. *Personality & Social Psychology Bulletin*, *13*, 141–155.

Adams, E. W. (1998). *A primer of probability logic*. Stanford, CA: CLSI Publications.

Anderson, J. R. (1990). *The adaptive character of thought*. Hillsdale, NJ: Erlbaum.

Barbey, A. K., & Sloman, S. A. (2007). Base-rate respect: From ecological rationality to dual processes. *Behavioral and Brain Sciences*, *30*, 241–297.

Bar-Hillel, Y. (1964). *Language and information processing*. Reading, MA: Addison-Wesley.

Bar-Hillel, M., & Neter, E. (1993). How alike is it versus how likely is it: A disjunction fallacy in probability judgments. *Journal of Personality and Social Psychology*, *65*, 1119–1131.

Barnes, J. (Ed.) (1984). *The complete works of Aristotle*. Princeton, NJ: Princeton University Press.

Baron, J. (2008). *Thinking and deciding* (4th ed.). New York: Cambridge University Press.

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. S. (2006). Nonsymbolic arithmetic in adults and young children. *Cognition*, *98*, 199–222.

Bauer, M. I., & Johnson-Laird, P. N. (1993). How diagrams can improve reasoning. *Psychological Science*, *4*, 372–378.

Bennett, J. (2003). *A philosophical guide to conditionals*. Oxford, UK: Oxford University Press.

Boole, G. (1854). *An investigation of the laws of thought*. London: Macmillan.

Budescu, D. V., Karelitz, T. M., & Wallsten, T. S. (2003). Predicting the directionality of probability words from their membership functions. *Journal of Behavioral Decision Making*, *16*, 159–180.

Byrne, R. M. J. (2005). *The rational imagination: How people create alternatives to reality*. Cambridge, MA: MIT.

Carey, S. (2009). *The origin of concepts*. New York: Oxford University Press.

Chater, N., Tenenbaum, J. B., & Yuille, A. (2006). Probabilistic models of cognition: Conceptual foundations. *Trends in Cognitive Sciences*, *10*, 287–291.

Cheng, P. W. (1997). From covariation to causation: A causal power theory. *Psychological Review*, *104*, 367–405.

Cosmides, L., & Tooby, J. (1996). Are humans good intuitive statisticians after all?" Rethinking some conclusions of the literature on judgment under uncertainty. *Cognition*, *58*, 1–73.

Craik, K. (1943). *The nature of explanation*. Cambridge, UK: Cambridge University Press.

Dagum, P., & Luby, M. (1993). Approximating probabilistic inference in Bayesian belief networks is NP-hard. *Artificial Intelligence*, *60*, 141–153.

Dehaene, S. (1997). *The number sense*. Oxford, UK: Oxford University Press.

Dirac, P. (1942). The physical interpretation of quantum mechanics. *Proceedings of the Royal Society, London, Series A*, *180*, 1–39.

Douven, I., & Verbrugge, S. (2013). The probabilities of conditionals revisited. *Cognitive Science*, *37*, 711–730.

Evans, J. St. B. T. (2007). *Hypothetical thinking*. Hove, UK: Psychology Press.

Evans, J. St. B. T. (2008). Dual-processing accounts of reasoning, judgment and social cognition. *Annual Review of Psychology*, *59*, 255–278.

Evans, J. St. B. T. (2012). Questions and challenges for the new psychology of reasoning. *Thinking & Reasoning*, *18*, 5–31.

Evans, J. St. B. T., &Stanovich, K. (2013). Dual-process theories of higher cognition: Advancing the debate. *Perspectives on Psychological Science*, *8*, 223–241.

Fantino, E. (1998). Judgment and decision making: Behavioral approaches. *The Behavior Analyst*, *21*, 203–218.

Fantino, E., Kulik, J., Stolarz-Fantino, S., & Wright, W. (1997). The conjunction fallacy: A test of averaging hypotheses. *Psychonomic Bulletin and Review*, *4*, 96–101.

de Finetti, B. (1995). The logic of probability [trans. Angell, R.B. of 1936 original]. The logic of probability. *Philosophical Studies*, *77*, 181–190.

Franklin, J. (2001). *The science of conjecture: Evidence and probability before Pascal*. Baltimore, MD: Johns Hopkins University Press.

Gigerenzer, G. (1994). Why the distinction between single-event probabilities and frequencies is relevant for psychology and vice versa. In G. Wright & P. Ayton (Eds.), *Subjective probability* (pp. 129–162). New York: Wiley.

Gigerenzer, G., Hoffrage, U., & Goldstein, D. G. (2008). Fast and frugal heuristics are plausible models of cognition: Reply to Dougherty, Franco-Watkins, and Thomas (2008). *Psychological Review*, *115*, 1230–1239.

Gilio, A., & Over, D. (2012). The psychology of inferring conditionals from disjunctions: A probabilistic study. *Journal of Mathematical Psychology*, *56*, 118–131.

Gnedenko, B. V., & Khinchin, A. Ya. (1961). *An elementary introduction to the theory of probability*. San Francisco, CA: Freeman.

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*, 496–499.

Hájek, A. (2003). Interpretations of probability. In E. N. Zaltna (Ed.), *The Stanford encyclopedia of philosophy*. Available at http://plato.stanford.edu/archives/sum2003/entries/probability-interpret/. Accessed October 1, 2013.

Hopcroft, J. E., & Ullman, J. D. (1979). *Formal languages and their relation to automata*. Reading, MA: Addison-Wesley.

Hosni, H. (2013). Uncertain reasoning. *The Reasoner*, *7*, 60–61.

Jeffrey, R. (1983). *The logic of decision* (2nd ed.). Chicago: University of Chicago Press.

Jeffrey, R. C. (2004). *Subjective probability: The real thing*. New York: Cambridge University Press.

Johnson-Laird, P. N. (1983). *Mental models*. Cambridge, UK: Cambridge University Press.

Johnson-Laird, P. N. (2006). *How we reason*. New York: Oxford University Press.

Johnson-Laird, P. N., Byrne, R. M. J., & Schaeken, W. S. (1992). Propositional reasoning by model. *Psychological Review*, *99*(418–439), 1992.

Johnson-Laird, P. N., Girotto, V., & Legrenzi, P. (2004). Reasoning from inconsistency to consistency. *Psychological Review*, *111*, 640–661.

Johnson-Laird, P. N., Legrenzi, P., Girotto, V., Legrenzi, M., & Caverni, J.-P. (1999). Naive probability: A mental model theory of extensional reasoning. *Psychological Review*, *106*, 62–88.

Johnson-Laird, P. N., Lotstein, M., & Byrne, R. M. J. (2012). The consistency of disjunctive assertions. *Memory & Cognition*, *40*, 769–778.

Kahneman, D. (2011). *Thinking, fast and slow*. New York: Farrar, Strauss, Giroux.

Keren, G. (2013). A tale of two systems: A scientific advance or a theoretical stone soup? Commentary on Evans & Stanovich (2013). *Perspectives on Psychological Science*, *8*, 257–262.

Khemlani, S., & Johnson-Laird, P. N. (2012a). The processes of inference. *Argument and Computation*, *4*, 4–20.

Khemlani, S., & Johnson-Laird, P. N. (2012b). Theories of the syllogism: A meta-analysis. *Psychological Bulletin*, *138*, 427–457.

Khemlani, S., Lotstein, M., & Johnson-Laird, P. N. (2012). The probabilities of unique events. *PLoS ONE*, *7*, 1–9. Online version.

Khemlani, S. S., Mackiewicz, R., Bucciarelli, M., & Johnson-Laird, P. N. (2013). Kinematic mental simulations in abduction and deduction. *Proceedings of the National Academy of Sciences*, *110*, 16766–16771.

Khemlani, S., Orenes, I., & Johnson-Laird, P. N. (2012). Negation: A theory of its meaning, representation, and use. *Journal of Cognitive Psychology*, *24*, 541–559.

Kruglanski, A. W. (2013). Only one? The default interventionist perspective as a unimodel – Commentary on Evans & Stanovich (2013). *Perspectives on Psychological Science*, *8*, 242–247.

Lindley, D. V., Tversky, A., & Brown, R. V. (1979). On the reconciliation of probability assessments. *Journal of the Royal Statistical Society A*, *142*, 146–180.

Marcus, G. F., & Davis, E. (2013). How robust are probabilistic models of higher-level cognition? *Psychological Science*, *24*, 2351–2360.

McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science*, *18*, 740–745.

Meck, W. H., & Church, R. M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes*, *9*, 320–334.

Nickerson, R. (2004). *Cognition and chance: The psychology of probabilistic reasoning*. Mahwah, NJ: Erlbaum.

Oaksford, M. (2013). Quantum probability, intuition, and human rationality. *Behavioral and Brain Sciences*, *36*, 303.

Oaksford, M., & Chater, N. (1998). *Rationality in an uncertain world*. Hove, UK: Psychology Press.

Oaksford, M., & Chater, N. (2007). *Bayesian rationality: The probabilistic approach to human reasoning*. New York: Oxford University Press.

Over, D. E., Hadjichristidis, C., Evans, J. St. B. T., Handley, S. J., & Sloman, S. A. (2007). The probability of causal conditionals. *Cognitive Psychology*, *54*, 62–97.

Paolacci, G., Chandler, J., & Ipeirotis, P. G. (2010). Running experiments on Amazon Mechanical Turk. *Journal of Decision Making*, *5*, 411–419.

Pascal, B. (1966). *Pensées*. London: Penguin. (Originally published 1669.)

Pearl, J. (1988). *Probabilistic reasoning in intelligent systems: Networks of plausible inference*. San Francisco, CA: Morgan-Kaufman.

Pfeifer, N. (2013). The new psychology of reasoning: A mental probability logical perspective. *Thinking & Reasoning*, *19*, 329–345.

Pfeifer, N., & Kleiter, G. D. (2005). Towards a mental probability logic. *Psychologica Belgica*, *45*, 71–99.

Pothos, E. M., & Busemeyer, J. R. (2013). Can quantum probability provide a new direction for cognitive modeling? *Behavioral and Brain Sciences*, *36*, 255–327.

Ragni, M. (2003). An arrangement calculus, its complexity and algorithmic properties. *Advances in Artificial Intelligence: Lecture Notes in Computer Science*, *2821*, 580–590.

Ramsey, F. P. (1990). Truth and probability. In D. H. Mellor (Ed.), *F.P. Ramsey: Philosophical Papers* (pp. 52–94). Cambridge, UK: Cambridge University Press. (Essay originally published in 1926.)

Reagan, R. T., Mosteller, F., & Youtz, C. (1989). Quantitative meanings of verbal probability expressions. *Journal of Applied Psychology*, *74*, 433–442. doi:10.1037/0021- 9010.74.3.433.

Reyna, V. F., & Brainerd, C. J. (1995). Fuzzy-trace theory: An interim synthesis. *Learning and Individual Differences*, *7*, 1–75. doi:10.1016/1041-6080(95)90031-4.

Reyna, V. F., & Brainerd, C. J. (2008). Numeracy, ratio bias, and denominator neglect in judgments of risk and probability. *Learning and Individual Differences*, *18*, 89–107. doi:10.1016/j.lindif.2007.03.011.

Rips, L. (1994). *The psychology of proof*. Cambridge, MA: MIT Press.

Rogers, H. (1967). *Theory of recursive functions and effective computability*. New York: McGraw-Hill.

van Rooij, I. (2008). The tractable cognition thesis. *Cognitive Science*, *32*, 939–984. doi:10.1080/03640210801897856.

Rottenstreich, Y., & Tversky, A. (1997). Unpacking, repacking, and anchoring: Advances in support theory. *Psychological Review*, *104*, 406–415.

Scott, D., & Krauss, P. (1966). Assigning probabilities to logical formulas. In H. Hintikka & P. Suppes (Eds.), *Aspects of inductive logic* (pp. 219–264). Amsterdam: North-Holland.

Shimojo, S., & Ichikawa, S. (1989). Intuitive reasoning about probability: Theoretical and experimental analyses of the "Problem of three prisoners." *Cognition*, *32*, 1–24.

Sloman, S. A. (1996). The empirical case for two systems of reasoning. *Psychological Bulletin*, *119*, 3–22.

Song, J.-H., & Nakayama, K. (2008). Numeric comparison in a visually-guided manual reaching task. *Cognition*, *106*, 994–1003.

Stanovich, K. E. (1999). *Who is rational? Studies of individual differences in reasoning*. Mahwah, NJ: Erlbaum.

Teigen, K. H., & Brun, W. (2000). Ambiguous probabilities: When does p = 0.3 reflect a possibility, and when does it express a doubt? *Journal of Behavioral Decision Making*, *13*, 345–362.

Teigen, K. H., & Brun, W. (2003). Verbal probabilities: A question of frame? *Journal of Behavioral Decision Making*, *16*, 53–72. doi:10.1002/bdm.432.

Turing, A. (1939). Systems of logic based on ordinals. *Proceedings of the London Mathematical Society*, *s2–45*, 161–228.

Tversky, A., & Kahneman, D. (1983). Extension versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review*, *90*, 292–315.

Tversky, A., & Koehler, D. J. (1994). Support theory: A nonextentional representation of subjective probability. *Psychological Review*, *101*, 547–567.

Tversky, A., & Shafir, E. (1992). The disjunction effect in choice under uncertainty. *Psychological Science*, *3*, 305–309.

Ungar, L., Mellers, B., Satopää, V., Baron, J., Tetlock, P., Ramos, J., & Swift, S. (2012). The good judgment project: A large scale test of different methods of combining expert predictions. In *AAAI 2012 Fall symposium on machine aggregation of human judgment*, AAAI Technical Report FS-12-06 (pp. 37–42). Palo Alto, CA: AAAI Press.

Verschueren, N., Schaeken, W., & d'Ydewalle, G. (2005). A dual-process specification of causal conditional reasoning. *Thinking & Reasoning*, *11*, 278–293.

Wallmann, C., & Kleiter, G. D. (2013). Probability propagation in generalized inference forms. *Studia Logica*, *102*, 913–929.

Wallsten, T. S., & Budescu, D. V. (1995). A review of human linguistic probability processing: General principles and empirical evidence. *The Knowledge Engineering Review*, *10*, 43–62.

Wallsten, T. S., Budescu, D. V., Erev, I., & Diederich, A. (1997). Evaluating and combining subjective probability estimates. *Journal of Behavioral Decision Making*, *10*, 243–268.

Wallsten, T. S., Budescu, D. V., & Tsao, C. J. (1997). Combining linguistic probabilities. *Psychlogische Beiträge*, *39*, 27–55.

Wason, P. C., & Evans, J. St. B. T. (1975). Dual processes in reasoning? *Cognition*, *3*, 141–154.

Wason, P. C., & Johnson-Laird, P. N. (1970). A conflict between selecting and evaluating information in an inferential task. *British Journal of Psychology*, *61*, 509–515.

White, D., Engelberg, R., Wenrich, M., Lo, B., & Curtis, J. R. (2010). The language of prognostication in intensive care units. *Medical Decision Making*, *30*, 76–83.

Williamson, J. (2010). *In defence of objective Bayesianism*. Oxford, UK: Oxford University Press.

Wolfe, C. R., & Reyna, V. F. (2010). Semantic coherence and fallacies in estimating joint probabilities. *Journal of Behavioral Decision Making*, *23*, 203–223.

Wyer, R. S. (1976). An investigation of the relations among probability estimates. *Organizational Behavior & Human Decision Processes*, *15*, 1–18.

Young, S. A., Nussbaum, A. D., & Monin, B. (2007). Potential moral stigma and reactions to sexually transmitted diseases: Evidence for a disjunction fallacy. *Personality & Social Psychology Bulletin*, *33*, 789–799.

Zhao, J., Shah, A., & Osherson, D. (2009). On the provenance of judgments of conditional probability. *Cognition*, *113*, 26–36.

Appendix A: The constraints on conditional probabilities

Depending on the values of $P(A)$ and $P(B)$, sometimes there are no constraints on $P(A|B)$ other than it is in the unit interval [0, 1], sometimes $P(A|B)$ must be larger than a certain value greater than 0, sometimes it must be smaller than a certain value less than 1, and sometimes it must lie between two such values, to yield a coherent JPD. With higher values of $P(B)$, the narrower the constraints. One way to derive the bounds is from the formulas for each component of the JPD given $P(A)$, $P(B)$, and $P(A|B)$:

$$\begin{aligned}
&P(A\&B) = P(B) * P(A|B) \\
&P(A\&\neg B) = P(A) - [P(B) * P(A|B)] \\
&P(\neg A\&B) = P(B) - [P(B) * P(A|B)] \\
&P(\neg A\&\neg B) = [1 + [P(B) * [P(A|B) - 1]]] - P(A)
\end{aligned}$$

Each of these values must be in the unit interval [0, 1] and the resulting inequalities from these four formulas simplify to two:

$$P(A) \geq [P(B) * P(A|B)] \geq [P(A) + P(B) - 1]$$

A more general and rigorous derivation, which we owe to personal communications from Osherson, Kleiter, and Gilio, 5/26/13 (see also Wallmann & Kleiter, 2013), is:

$$\text{If } P(A) = x, P(B) = y, \text{ then } \max\{0, x + y - 1/x\} \leq P(B|A) \leq \min\{1, y/x\}.$$

In standard treatments of conditional probabilities, $P(B|A)$ is undefined in case $P(A) = 0$, but in treatments in which conditional probabilities are treated as undefined primitives, as in the analysis due to de Finetti (1995), the value of $P(B|A)$ can be anywhere in the unit interval.

Appendix B: Formulas and constraints of the complete joint probability distributions

Table B1
The formulas for the complete joint probability distributions (JPD), the constraints that they yield on the values of triples for conjunctive, disjunctive, and conditional probabilities for consistent JPDs, and the percentages of 0, 1, and 2 violations of the JPDs from exhaustive searches of possible values for the different triples of probability judgments

The Formulas for the JPD	The Constraints to be Satisfied for a Consistent JPD	Chance Percentages of n Violations of the JPD: 0	1	2
Conjunctions: $P(A)$, $P(B)$, $P(A\&B)$				
$P(A\&B)$ = given				
$P(A\&\neg B) = P(A) - P(A\&B)$	$P(A) \geq P(A\&B)$	17	50	33
$P(\neg A\&B) = P(B) - P(A\&B)$	$P(B) \geq P(A\&B)$			
$P(\neg A\&\neg B) = 1 - [P(A)+P(B) - P(A\&B)]$	$1.0 \geq P(A) + P(B) - P(A\&B)$			
Disjunctions: $P(A)$, $P(B)$, $P(A \vee B)$				
$P(\neg A\&\neg B) = 1 - P(A \vee B)$	$P(A \vee B) \geq P(A)$	17	50	33
$P(\neg A\&B) = [1 - P(A)] - P(\neg A\&\neg B)$	$P(A \vee B) \geq P(B)$			
$P(A\&B) = P(B) - P(\neg A\&B)$	$P(B) \geq P(A) - P(A \vee B)$			
$P(A\&\neg B) = P(A) - P(A\&B)$				
Conditional probabilities: $P(A)$, $P(B)$, $P(B\|A)$				
$P(A\&B) = P(A) * P(B\|A)$	$P(B) \geq P(A) * P(B\|A)$	50	50	0
$P(A\&\neg B) = P(A) - P(A) * P(B\|A)$	$P(A) * P(B\|A) \geq P(A) + P(B) - 1$			
$P(\neg A\&B) = P(B) - P(A) * P(B\|A)$				
$P(\neg A\&\neg B) = 1 - [P(A\&B) + P(A\&\neg B) + P(\neg A\&B)]$				

Notes. "¬" denotes negation, "&" denotes conjunction, "∨" denotes inclusive disjunction, and "|" denotes a conditional probability. 1% of instances of cases of conditional probabilities are undefined because of potential divisions by 0, and those instances were dropped from the table.

Appendix C: Solutions to the challenge posed in Section 4.3

At the end of Section 4.3, we raised the problem of a set of three probabilities that fix the JPD for two binary variables A and B, but whose values can never violate it. A simple case is an instance of the law of total probability (e.g., Gnedenko & Khinchin, 1961, p. 31):

$$P(A) = P(A|B) * P(B) + P(A|\neg B) * P(\neg B) \tag{40}$$

If we substitute b for $P(B)$, x for $P(A|B)$, and y for $P(A|\neg B)$, then:

$$P(A) = xb + y(1 - b) \tag{41}$$

The values of b, x, and y, are free to vary in the unit interval [0, 1], and they fix the value of $P(A)$. Hence, no matter what their values are the following three probabilities fix the JPD and cannot violate it:

$$P(B), P(A|B), \text{ and } P(A|\neg B). \tag{42}$$

We owe this argument to an anonymous reviewer. There are other answers too, for example, the three conditional probabilities also fix the JPD and cannot violate it:

$$P(B|A), P(A|B), \text{ and } P(A|\neg B). \tag{43}$$

Appendix D: The 16 sets of materials used in the three experiments were based on the following conjunctions, where the first conjunct is *A* and the second conjunct is *B* in the estimates of probabilities

Material Number	Assertion 1 What is the probability that...	Assertion 2 What is the probability that...
1	space tourism will achieve widespread popularity in the next 50 years?	advances in material science will lead to the development of anti-gravity materials in the next 50 years?
2	in <5 years, millions of people will live past 100?	advances in genetics will end the shortage of replacement organs in the next 15 years?
3	the Supreme Court rules on the constitutionality of gay marriage in the next 5 years?	that a gay person will be elected as president in the next 50 years?
4	Greece will make a full economic recovery in the next 10 years?	Greece will be forced to leave the EU?
5	a nuclear weapon will be used in a terrorist attack in the next decade?	there will be a substantial decrease in terrorist activity in the next 10 years?

(continued)

Appendix (continued)

Material Number	Assertion 1 What is the probability that...	Assertion 2 What is the probability that...
6	the United States will sign the Kyoto Protocol and commit to reducing CO_2 emissions in next 15 years?	global temperatures reach a theoretical point of no return in the next 100 years?
7	intelligent alien life is found outside the solar system in the next 10 years?	world governments dedicate more resources to contacting extra-terrestrials?
8	the United States adopts an open border policy of universal acceptance?	English is legally declared the official language of the United States?
9	Honda will go bankrupt in 2012?	Ford will go bankrupt before the end of 2013?
10	a significant upturn in the economy occurs next year?	Obama will be reelected President in 2012?
11	a new illegal but synthetic drug becomes popular in the USA over the next two years?	the movement to decriminalize drugs doubles its numbers by 2015?
12	U.S. companies focus their advertising on the Web next year?	the *New York Times* becomes more profitable?
13	three-dimensional graphics will be required to contain explicit markers to indicate their unreal nature by 2020	competitive video game playing will achieve mainstream acceptance by 2020?
14	intellectual property law in the United States will be updated to a reflect advances in technology by the year 2040?	Russia will become the world center for software development by 2040?
15	at least one head of state will be assassinated by 2012?	NATO will grant military support to Arab Spring movements in several countries?
16	scientists will discover a cure for Parkinson's disease in 10 years?	the number of patients who suffer from Parkinson's disease will triple by 2050?

Notes. Data from Experiments 1, 2, and 3 were collected in 2010 and 2011. Several of the materials reference years that have since passed and therefore have definitive answers, but they were controversial for the participants in our experiments.