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Matrix Transformations on Some Difference Sequence Spaces

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Abstract: *The sequence spaces $l_\infty(u,v,\Delta)$, $c_0(u,v,\Delta)$ and $c(u,v,\Delta)$ were recently introduced. The matrix classes $(c(u,v,\Delta):c)$ and $(c(u,v,\Delta):l_\infty)$ were characterized. The object of this paper is to further determine the necessary and sufficient conditions on an infinite matrix to characterize the matrix classes $(c(u,v,\Delta):bs)$ and $(c(u,v,\Delta):l_p)$. It is observed that the later characterizations are additions to the existing ones.*

Keywords- *Difference operators, Duals, Generalized weighted mean, Matrix transformations*

I. Introduction

The sequence spaces $l_\infty(\Delta)$, $c_0(\Delta)$ and $c(\Delta)$ were first introduced by Kizmaz [6] in 1981. Similar to the sequence spaces $l_\infty(p)$, $c_0(p)$ and $c(p)$ for $p_k > 1$ of Maddox [7] and Simons [10], the Δ- sequence spaces above were extended to $\Delta l_\infty(p)$, $\Delta c_0(p)$ and $\Delta c(p)$ by Ahmad and Mursaleen [1] in … The concept of difference operators has been discussed and used by Polat and Başar [8] and by Altay and Başar [2], both in 2007.

The idea of generalized weighted mean was applied by Altay and Başar [3], in 2006. This concept depends on the idea of $G(u,v)$- transforms which has been used by Polat, *et al* [10] and by Basarir and Kara [4]. We shall need the following sequence spaces:

$\omega = \{x = (x_k) : x \text{ is any sequence}\}$
$c = \{x = (x_k) \in \omega : x_k \text{ converges, i.e. } \lim_{k\to\infty} x_k \text{ exists}\}$
$c_0 = \{x = (x_k) \in \omega : \lim_{k\to\infty} x_k = 0\}$, the set of all null sequences
$l_\infty = m = \{x = (x_k) \in \omega : \|x\|_\infty = sup_n |x_k| < \infty\}$
$l_1 = l = \{x = (x_k) \in \omega : \|x\|_1 = \sum_{k=0}^{\infty} |x_k| < \infty\}$
$l_p = \{x = (x_k) \in \omega : \|x\|_p = \sum |x_k|^p < \infty;\ 1 \le p < \infty\}$
$\phi = \{x = (x_k) \in \omega : \exists\, N \in \mathbb{N} \text{ such that } \forall\, k \ge N, x_k = 0\}$, the set of finitely non- zero sequences
$bs = \{x = (x_k) \in \omega : \|x\|_{bs} = sup_n |\sum_{k=0}^{n} x_k| < \infty\}$, the set of all sequences with bounded partial sums

$X^\beta = \{a = (a_k) \in \omega : \sum_{k=0}^{\infty} a_k x_k \in c, \forall\, x \in X\}$

Note that $x = (x_k)$ is used throughout for the convention $(x_k) = (x_k)_{k=0}^{\infty}$. We take $e = (1,1,1,\dots)$ and e^k for the sequence whose only nonzero term is 1 in the kth place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,3,\dots\}$. Any vector subspace of ω is called a sequence subspace. A sequence space X is *FK* if it is a complete linear metric space with continuous coordinates $P_n : X \to \mathbb{C}$, defined by $P_n(x) = x_n\ \forall\, x = (x_k) \in X$ with $n \in \mathbb{N}$. A normed *FK* space is *BK*-space or Banach space with continuous coordinates. An *FK* space has *AK*- property if $x^{[m]} \to x$ in X, where $x^{[m]} = \sum_{k=0}^{n} x_k e^k$ is the m$^{\text{th}}$ section of x. If φ is dense in X then it has an *AD*- property (see Boos [5]). A matrix domain of a sequence space X, is defined as $X_A = \{x = (x_k) \in \omega : Ax \in X\}$.

Let $\mathcal{U}$ be the set of all sequences $u = (u_k)$ with $u_k \ne 0\ \forall\, k \in \mathbb{N}$, and for $u \in \mathcal{U}$ let $\frac{1}{u} = \left(\frac{1}{u_k}\right)$. Then for $u, v \in \mathcal{U}$ define the matrix $G(u,v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k, & \text{for } 0 \le k \le n, \\ 0, & \text{for } k > n \end{cases} \quad \forall\, k, n \in \mathbb{N}$$

This matrix is called the generalized weighted mean. The sequence $y = (y_k)$ in the sequence spaces

$$\lambda(u,v,\Delta) = \{x = (x_k) \in \omega : y = \sum_{i=0}^{k} u_k v_i \Delta x_i \in X\},\quad \lambda \in \{l_\infty,\ c,\ c_0\} \tag{1}$$

is the $G(u,v,\Delta)$ –transform of a given sequence $x = (x_k)$. It is defined by

$$y = \sum_{i=0}^{k} u_k v_i \Delta x_i = \sum_{i=0}^{k} u_k \nabla v_i x_i$$

where,

$$\nabla v_i = v_i - v_{i+1} \text{ and } \Delta x = (\Delta x_i) = x_i - x_{i-1},$$

and taking all negative subscripts to be naught. The spaces (1) were defined in [9]. If X is any normed sequence space the matrix domain $X_{G(u,v,\Delta)}$ is the generalized weighted mean difference sequence space [9]. Our object is to characterize the matrix classes $(c(u,v,\Delta) : l_p)$ and $(c(u,v,\Delta) : bs)$. However, matrix class characterizations are done with help of β –duals, and so we need the following

Lemma 1.1 [9]**:** Let $u, v, \in \mathcal{U}$, $a = (a_k) \in \omega$ and the matrix $D = (d_{nk})$ by

$$d_{nk} = \begin{cases} \left(\frac{1}{u_n v_k} - \frac{1}{u_n v_{k+1}}\right) a_k; & (0 \le k < n), \\ \frac{1}{u_n v_n} a_n; & (k = n) \\ 0; & (k > n) \end{cases}$$

and let d_1, d_2, d_3, d_4 and d_5 be the sets

$$d_1 = \{a = (a_k) \in \omega : sup_n \sum_n |\sum_{k \in \mathcal{K}} d_{nk}| < \infty\};$$
$$d_2 = \{a = (a_k) \in \omega : sup_n \sum_n |d_{nk}| < \infty\};$$
$$d_3 = \{a = (a_k) \in \omega : lim_{n \to \infty} d_{nk} \text{ exists for each } n \in \mathbb{N}\}$$

Then, $[c_0(u, v, \Delta)]^\beta = d_1 \cap d_2 \cap d_3$.

II. Methodology

If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then $A = (a_{nk})$ is used for $A = (a_{nk})_{n,k=0}^{\infty}$ and A_n is the sequence in the n^th^ row of A, or $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. The A- transform of a sequence x is defined as

$$Ax = (A_n(x))_{n=0}^{\infty}$$
$$= \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N})$$

provided the series on the right converges for each n and for all $x \in X$. The pair (X, Y) is referred to as a matrix class, so that

$$A \in (X, Y) \Leftrightarrow \begin{cases} A_n \in X^\beta \ \ \forall\, n \in \mathbb{N} \\ \text{and} \\ Ax \in Y \ \ \forall\, x \in X, \ \text{ in the norm of } Y \end{cases} \tag{2}$$

In this paper we shall take $X = c(u, v, \Delta)$ and $Y \in \{l_p, bs\}$. We shall need the following lemma for the proof of Theorems 3.1 and 3.2 as our main results in section 3:

Lemma 2.1 [9]**:** The sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{l_\infty, \ c, \ c_0\}$ are complete normed linear spaces with the norm $\|x\|_{\lambda(u,v,\Delta)} = \sup_k \left|\sum_{i=0}^{k} u_k \Delta x_i\right| = \|y\|_\lambda$. They are also *BK* spaces with both *AK*- and *AD*- properties. Further, let $y \in c_0$ and define $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) y_i + \frac{1}{u_k v_k} y_k; \quad k \in \mathbb{N}$$

then $x \in c_0(u, v, \Delta)$.

An infinite matrix A maps a *BK* space X continuously into the space bs if and only if the sequence the sequence of functional $\{f_n\}$ defined by

$$f_n(x) = \sum_{n=1}^{m} \sum_{k=1}^{\infty} a_{nk} x_k, \ \ n = 1, 2, 3, \ldots$$

is bounded in the dual space of X.

III. Main Results

Theorem 3.1. $A \in \left(c(u, v, \Delta) : l_p\right)$ *for* $p > 1$, *if and only if*

(i) $$sup_n \left|\sum_{k \in \mathcal{K}} \left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk} + \frac{1}{u_k v_k} a_{nk}\right]\right|^p < \infty,$$

(ii) $$lim_{n \to \infty} \left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk} + \frac{1}{u_k v_k} a_{nk}\right] = a_k, \text{ exists}$$

(iii) $$lim_{n \to \infty} \sum_{k=0}^{n} \left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk} + \frac{1}{u_k v_k} a_{nk}\right] = a, \text{ exists}$$

Proof: Since $c(u, v, \Delta)$ and l_p are *BK* spaces, we suppose that (i), (ii) and (iii) hold and take $x = (x_k) \in c(u, v, \Delta)$. Then by (2) and Lemma 1.1, $A_n \in [c(u, v, \Delta)]^\beta$ for all $n \in \mathbb{N}$, which implies the existence of the A-transform of x, or Ax exists for each n. It is also clear that the associated sequence $y = (y_k)$ is in c and hence $y \in c_0$. Again, since $c(u, v, \Delta)$ has *AK* (Lemma 2.1) and contains ϕ, by the m^th^ partial sum of the series $\sum_{k=0}^{\infty} a_{nk} x_k$ we have

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) + \frac{1}{u_k v_k}\right] a_{nk} y_k,$$

which becomes

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk} + \frac{1}{u_k v_k} a_{nk}\right] y_k, \text{ for } p > 1,$$

$$\Rightarrow \; \|Ax\|_{l_p} \le \sup_n \sum_k \left[\sum_{k=0}^{k-1} \left|\frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk}\, y_k + \frac{1}{u_k v_k} a_{nk}\, y_k\right|^p\right]^{1/p}$$

$$\le \|y_k\|_{l_p} sup_n \left(\sum_k \left[\sum_{k=0}^{k-1} \left|\frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk}\right|^p\right]^{1/p} + \left[\sum_{k=0}^{k-1} \left|\frac{a_{nk}}{u_k v_k}\right|^p\right]^{1/p}\right) < \infty$$

$$\Rightarrow \; Ax \in l_p \text{ and hence } A \in \big(c(u,v,\Delta) : l_p\big).$$

Conversely, let $A \in \big(c(u,v,\Delta) : l_p\big)$, $1 < p < \infty$. Then again by (2) and Lemma 1.1, $A_n \in [c(u,v,\Delta)]^{\beta}$ for all $n \in \mathbb{N}$ implying (ii) and (iii) for all $x \in c(u,v,\Delta)$ and $y \in l_p$. To prove (i), let the continuous linear functional f_n $(n \in \mathbb{N})$ be defined on $(c(u,v,\Delta))^*$, the continuous dual of $c(u,v,\Delta)$. Since the series $\sum_{k=0}^{\infty} a_{nk}\, x_k$ converges for each x and for each n, then $f_{A_n} \in (c(u,v,\Delta))^*$; where

$$f_{A_n}(x) = \sum_{k=0}^{\infty} a_{nk}\, x_k \;\; \forall\, x \in c(u,v,\Delta).$$

$$\Rightarrow \;\; \|f_{A_n}\| = \|A_n\|_{l_p} = \left(\sum_{k=0}^{\infty} |a_{nk}|^p\right)^{\frac{1}{p}} < \infty, \text{ for all } n \in \mathbb{N},$$

with $A_n \in [c(u,v,\Delta)]^{\beta}$. This means that the functional defined by the rows of A on $c(u,v,\Delta)$ are pointwise bounded, and by the Banach-Steinhaus theorem these functional are uniformly bounded. Hence there exists a constant $M > 0$, such that $\|f_{A_n}\| \le M, \;\forall\, n \in \mathbb{N}$, yielding (i).

Theorem 3.2: $A \in (c(u,v,\Delta) : bs)$ *if and only if conditions (ii) and (iii) of Theorem 3.1 hold, and*

(iv) $\quad sup_m \sum_k \sum_{n=1}^{m} \left|\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_{nk} + \frac{1}{u_k v_k} a_{nk}\right| < \infty.$

Proof. Suppose$A \in (c(u,v,\Delta) : bs)$. Then $A_n \in [c(u,v,\Delta)]^{\beta}$ for all $n \in \mathbb{N}$. Since $e_k = (\delta_{nk})$, where $\delta_{nk} = 1$ $(n = k)$ and $= 0$ $(n \ne k)$, belongs to $c(u,v,\Delta)$, the necessity of (ii) holds. Similarly by taking $x = e = (1,1,1,\dots) \in c(u,v,\Delta)$ we get (iii). We prove the necessity of (i) as follows:

Suppose $A \in (c(u,v,\Delta) : bs)$. Then it implies

$$\sum_{n=1}^{m} |A_r(x)| < \infty,\; m = 1,2,3,\dots,$$

where,

$$A_r(x) = \sum_k a_{rk} \left(\sum_{i=0}^{k-1} \left(\frac{y_k}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) + \frac{y_k}{u_k v_k}\right)\right)$$

converges for each r whenever $x \in c(u,v,\Delta)$, which follows by the Banach-Steinhaus theorem that $sup_k |a_{nk}| < \infty$, each r. Hence A_r defines an element of $[c(u,v,\Delta)]^*$ for each r.

Now define

$$q_m(x) = \sum_{n=1}^{m} |A_r(x)|, \quad r = 1,2,3,\dots$$

q_m is subadditive. Moreover, A_r is a bounded linear functional on $c(u,v,\Delta)$ implies each q_m is a sequence of continuous seminorms on $c(u,v,\Delta)$ such that

$$sup_m\, q_m(x) = \sum_{r=1}^{\infty} |A_r(x)| < \infty \text{ for each } x \in c(u,v,\Delta).$$

Thus there exists a constant $M > 0$ such that

$$\sum_{r=1}^{\infty} |A_r(x)| \le M\|x\|_{c(u,v,\Delta)}$$

which implies (i).

Sufficiency: Suppose (i) – (iii) of the theorem hold. Then $A_n \in [c(u,v,\Delta)]^{\beta}$. If $x \in c(u,v,\Delta)$, it suffices to show that $A_n(x) \in bs$ in the norm of the sequence space bs.

Now,

$$\sum_{k=0}^{n} a_{nk}\, x_k = \sum_{k=0}^{n} \left[\left[\sum_{i=1}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) + \frac{1}{u_k v_k}\right] a_{nk}\right] y_k$$

$$\leq sup_n \sum_{k=0}^{n} \left[\sum_{i=0}^{k-1} \left(\frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{a_{nk}}{u_k v_k} \right) y_k \right] \quad \text{by (i)}$$

$$\leq \|y_k\| sup_n \sum_{k=0}^{\infty} \left[\sum_{i=1}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) + \frac{1}{u_k v_k} \right] a_{nk} < \infty, \text{ as } n \to \infty.$$

This implies $A_n(x) \in bs$ or $A \in (c(u, v, \Delta) : bs)$. □

Concluding Remarks

The generalization obtained here still admit improvement in the sense that the conditions obtained here may further be simplified resulting in less restrictions on the involved matrices.

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