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Control of an Underactuated Three-Link Passive–Active–Active Manipulator Based on Three Stages and Stability Analysis

This paper presents a novel three-stage control strategy for the motion control of an underactuated three-link passive–active–active (PAA) manipulator. First, a nonlinear control law is designed to make the angle and angular velocity of the third link convergent to zero. Then, a swing-up control law is designed to increase the system energy and control the posture of the second link. Finally, an integrated method with linear control and nonlinear control is introduced to stabilize the manipulator at the straight-up position. The stability of the control system is guaranteed by Lyapunov theory and LaSalle's invariance principle. Compared to other approaches, the proposed strategy innovatively introduces a preparatory stage to drive the third link to stretch-out toward the second link in a natural way, which makes the swing-up control easy and quick. Besides, the intergraded method ensures the manipulator moving into the balancing stage smoothly and easily. The effectiveness and efficiency of the control strategy are demonstrated by numerical simulations. [DOI: 10.1115/1.4028051]

1 Introduction

This paper concerns the motion control of a class of underactuated mechanical systems, called a PAA manipulator, which have fewer actuators than degrees of freedom (DOF) [1–3]. The PAA manipulator is a rigid three-link gymnastic robot operating in a vertical plane with its first joint being passive and the rest being actuated. A common control objective for the PAA manipulator is to drive it away from any arbitrary initial position and balance it at the straight-up unstable equilibrium position. The merits of this system are downsizing, lightening, energy-saving, and cost-reduction. Moreover, it is subject to a second-order nonholonomic constraint and cannot be completely linearized in the whole motion space. Therefore, the investigation of the PAA manipulator is of great importance in both control theory and applications.

Over the past few years, a remarkable research activity has been devoted to the control of underactuated mechanical systems, such as acrobot/pendulum-like robot, overhead crane and underwater vehicle [4–7]. Among them, the acrobot is considered as a highly simplified model of a human gymnast on a high bar, where the underactuated first joint models the gymnast's hands on the bar, and the actuated second joint models the gymnast's hips. A number of methodologies have been proposed, such as partial-feedback linearization [8], fuzzy control [9], impulse–momentum approach [10], rewinding approach [11], interconnection and damping assignment passivity-based approach [12–14] and energy-based method [15–17] etc. More recently, a three-link underactuated manipulator that can describe the mechanical systems in real world more realistically and precisely has drawn increasing attention. For example, the three-link planar manipulator [18,19], the three-link human-like walking robot [20], and the three-link PAA manipulator [21–24]. The three-link PAA manipulator, which consists of arm, trunk, and leg, can mimic a gymnastic routine more realistically than the acrobot. But the motion control problem is more complicated and challenging due to the strong coupling characteristic of its control inputs.

Although Takashima [21] and Jian and Zushu [22] studied the dynamic model of the PAA manipulator and gave some fundamental motion analysis, they did not present effective control methods. Spong [23] studied the motion control based on collocated partial feedback linearization, but few analysis of the swing-up control for the PAA manipulator was found. Xin and Kaneda [24] studied the swing-up control problem based on the concept of virtual composite link by devising a virtual composite-link formulation. However, the coordinate transformation is complicated. Recently, researchers are paying more and more attention to find a single controller to realize the motion control of a 2-DOF underactuated system. For example, an equivalent-input-disturbance approach was proposed in Ref. [25]. However, it is very hard to extend this single controller to the 3-DOF case because of its complex structure. In Ref. [26], we discussed a motion control design method for the PAA manipulator based on the combination of energy and the posture of the third link, but the posture of the entire manipulator was unconcerned, which makes the stability analysis of the overall control system hard.

Motivated by the above considerations, this paper aims to propose an efficient control strategy for the motion control of the three-link PAA manipulator. The control strategy contains three stages. The first stage, called *preparatory stage*, is to force both the angle and angular velocity of third link to converge to zero, which makes it stretch-out toward the second link in a natural way. The control law of the third link is first designed based on a Lyapunov function, and is maintained throughout the whole control process. Whereas the control law of the second link is first set to constant to simplify the control law design. Due to the control

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Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received March 18, 2014; final manuscript received July 16, 2014; published online September 10, 2014. Assoc. Editor: Heikki Handroos.

of this stage, the influence of the control input and angular velocity of the third link to the control of the next stages is explicitly eliminated. The second stage, named *swing-up stage*, is to increase the system energy and control the posture of the second link to fulfill the swing-up operation. During this stage, the control law of the second link is no longer a constant, but designed based on another Lyapunov function constructed by the system energy and the posture of the second link. The third stage, called *balancing stage*, is to capture the manipulator and stabilize it at the straight-up unstable equilibrium position. An integrated method with linear control and nonlinear control is introduced to ensure that the manipulator can move into the third stage smoothly and easily. Finally, the stability of the proposed control strategy is analyzed and guaranteed by LaSalle's invariance principle.

The paper is organized as follows: The PAA dynamics are presented in Sec. 2. The three-stage control strategy and the design of control laws are described in Sec. 3. Stability issues of the first stage and second stage are proved in Sec. 4. Section 5 presents the simulation and comparison results of the control strategy. Conclusions are finally drawn in Sec. 6.

2 Dynamics of PAA Manipulator

The model of the PAA manipulator is shown in Fig. 1. For $j=1, 2, 3$, q_j is the angle of jth link measured relative to the vertical for the link attached to the base or relative to the front link; m_j and L_j are the mass and length of the jth link, respectively; L_{cj} is the distance from the jth joint to the center of mass of the jth link; J_j is the moment of inertia of the jth link about its centroid; τ_j is the torque applied to the jth joint. g is the gravitational acceleration (9.80665 m/s^2).

Let $q(t)=[q_1(t)\, q_2(t)\, q_3(t)]^\top$ and assume that the viscous friction effect, matched and mismatched uncertainties, and external disturbances can be ignored, the dynamics of the PAA manipulator [23,24] are given by

$$M(q)\ddot{q}+H(q,\dot{q})+G(q)=\tau \tag{1}$$

where q is the abbreviation of $q(t)$; $M(q)\in\mathbb{R}^{3\times 3}$ is the symmetric positive definite inertia matrix; $H(q,\dot{q})\in\mathbb{R}^3$ is the combination of the Coriolis and centrifugal forces; $G(q)\in\mathbb{R}^3$ is the force of gravity; and $\tau=[0,\tau_2,\tau_3]^\top\in\mathbb{R}^3$ is the vector of the driving torque on the joints. Detailed structures of $M(q)$, $H(q,\dot{q})$ and $G(q)$ are given in the Appendix.

Assuming $x=[x_1\, x_2\, x_3\, x_4\, x_5\, x_6]^\top=[q_1\, q_2\, q_3\, \dot{q}_1\, \dot{q}_2\, \dot{q}_3]^\top$, and rewriting Eq. (1) in the state space yields the nonlinear model

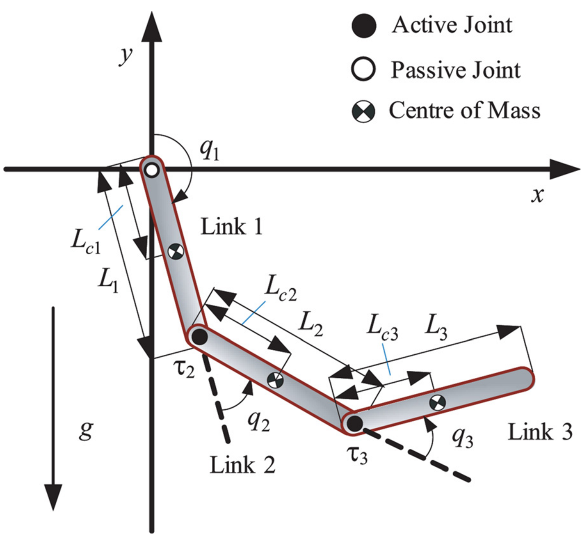

Fig. 1 Model of a PAA manipulator

$$\begin{cases}\dot{x}_1=x_4, \quad \dot{x}_2=x_5, \quad \dot{x}_3=x_6\\ \dot{x}_4=f_1(x)+b_1(x)\tau_2+c_1(x)\tau_3\\ \dot{x}_5=f_2(x)+b_2(x)\tau_2+c_2(x)\tau_3\\ \dot{x}_6=f_3(x)+b_3(x)\tau_2+c_3(x)\tau_3\end{cases} \tag{2}$$

where

$$\begin{bmatrix}f_1(x)\\ f_2(x)\\ f_3(x)\end{bmatrix}=M^{-1}(x)[-H(x)-G(x)] \tag{3}$$

$$\begin{bmatrix}b_1(x) & c_1(x)\\ b_2(x) & c_2(x)\\ b_3(x) & c_3(x)\end{bmatrix}=M^{-1}(x)\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix} \tag{4}$$

The total mechanical energy $E(x)$ of the PAA manipulator is

$$E(x)=T(x)+P(x) \tag{5}$$

$$T(x)=\frac{1}{2}[x_4\, x_5\, x_6]M(x)[x_4\, x_5\, x_6]^\top \tag{6}$$

$$P(x)=\beta_1\cos x_1+\beta_2\cos(x_1+x_2)+\beta_3\cos(x_1+x_2+x_3) \tag{7}$$

where β_i ($i=1, 2, 3$) are the structure parameters (see Appendix); $T(x)$ and $P(x)$ are the kinetic and potential energy of the PAA manipulator, respectively.

For simplifying the design of the control system, a three-stage control strategy is studied based on the following division of the motion state.

Definition 1. *Let all the motion state of the PAA manipulator be in a compact set $\mathcal{S}$, namely, $x\in\mathcal{S}$, and define $\mathcal{C}_i(i=1,\cdots,5)$ as*

$$\mathcal{C}_1=\min\left\{\mathrm{mod}\left(\frac{x_3}{2\pi}\right),\left|\mathrm{mod}\left(\frac{x_3}{-2\pi}\right)\right|\right\} \tag{8}$$

$$\mathcal{C}_2=|x_6| \tag{9}$$

$$\mathcal{C}_3=\min\left\{\mathrm{mod}\left(\frac{x_1}{2\pi}\right),\left|\mathrm{mod}\left(\frac{x_1}{-2\pi}\right)\right|\right\} \tag{10}$$

$$\mathcal{C}_4=\min\left\{\mathrm{mod}\left(\frac{x_{12}}{2\pi}\right),\left|\mathrm{mod}\left(\frac{x_{12}}{-2\pi}\right)\right|\right\} \tag{11}$$

$$\mathcal{C}_5=|E(x)-E_0| \tag{12}$$

where $x_{12}=x_1+x_2$; mod(x/y) is the remainder of x divided by y, and its sign is with y; E_0 is the potential energy at the straight-up unstable equilibrium position, i.e., $x=[0\ 0\ 0\ 0\ 0\ 0]^\top$. Then, the preparatory stage (Σ_1), swing-up stage (Σ_2) and balancing stage (Σ_3) are defined to be

$$\Sigma_1:(\mathcal{C}_1>\varepsilon_1)\cup(\mathcal{C}_2>\varepsilon_2) \tag{13}$$

$$\Sigma_2:\{(\mathcal{C}_1\le\varepsilon_1)\cap(\mathcal{C}_2\le\varepsilon_2)\}\cup\{(\mathcal{C}_3>\varepsilon_3)\cup(\mathcal{C}_4>\varepsilon_4)\cup(\mathcal{C}_5>\varepsilon_5)\} \tag{14}$$

$$\Sigma_3:\{(\mathcal{C}_1\le\varepsilon_1)\cap(\mathcal{C}_2\le\varepsilon_2)\}\cap\{(\mathcal{C}_3\le\varepsilon_3)\cap(\mathcal{C}_4\le\varepsilon_4)\cap(\mathcal{C}_5\le\varepsilon_5)\} \tag{15}$$

respectively, where ε_1, ε_2, ε_3, ε_4 and ε_5 are small positive real numbers.

Remark 1. Note that, according to those operations (8)–(15), if both the angle and angular velocity of the third link are guaranteed in small scales, then the manipulator is said to be in swing-up stage or balancing stage. Otherwise, it is located in preparatory stage. Moreover, operation (15) indicates that if the angles of the three links and angular velocity of the third link are all small enough, meanwhile the system energy is close to E_0, then the manipulator reaches balancing stage. Otherwise, it is located in swing-up stage.

3 Design of Control Strategy

A good combination of energy and posture makes it easy to swing-up a two-link underactuated manipulator [15–17]. But this kind of method cannot be directly employed to control the three-link manipulator. In this section, a three-stage control strategy is proposed and the control laws for each are designed.

3.1 Control Laws for the First Stage. The first stage aims to drive the third link to stretch out toward the second one in a natural way, that is, to force both the angle and angular velocity of the third link to converge to zero. To this end, the first control Lyapunov function candidate $V_1(x)$ is defined as

$$V_1(x) = \frac{1}{2}x_3^2 + \frac{1}{2}x_6^2 \tag{16}$$

Note that $V_1(0) = 0$. Taking the time derivative of $V_1(x)$ yields

$$\dot{V}_1(x) = x_6[x_3 + f_3(x) + b_3(x)\tau_2 + c_3(x)\tau_3] \tag{17}$$

If we choose

$$\tau_3 = \frac{-x_3 - f_3(x) - b_3(x)\tau_2 - \gamma_1 x_6}{c_3(x)} \tag{18}$$

where $\gamma_1 > 0$ is a design parameter, then Eq. (17) becomes

$$\dot{V}_1(x) = -\gamma_1 x_6^2 < 0, \quad \forall x_6 \neq 0 \tag{19}$$

Note that the denominator $c_3(x)$ in Eq. (18) is a smooth positive function. From Eq. (4), it is known that

$$c_3(x) = \frac{m_{11}(x)m_{22}(x) - m_{12}(x)m_{21}(x)}{\det[M(x)]} \tag{20}$$

where $\det[M(x)]$ is the determinant of the square matrix $M(x)$. Since $M(x)$ is a symmetric, and positive matrix containing only q_2 and q_3, which means $\det[M(x)] > 0$ for all q_2 and q_3. Furthermore, notice from the structure of $M(x)$ in the Appendix that the numerator of Eq. (20) is exactly a second-order principal minor determinant of $M(x)$. Since $M(x)$ is positive definite, according to the property of positive definite matrix, it follows that $(m_{11}(x)m_{22}(x) - m_{12}(x)m_{21}(x) > 0)$. Therefore, $c_3(x) > 0$, which guarantees that Eq. (18) has no singularities.

Generally speaking, the control law (18) of τ_3 is employed to make the states of the third link converge to zero. However, from Eq. (18), it is known that there exists a certain relationship between τ_3 and τ_2, which means τ_3 and τ_2 are coupled. In this situation, one feasible treatment for Eq. (18) in the first stage is to assume τ_2 be constant. For simplification, set

$$\tau_2 = 0 \tag{21}$$

Therefore, the control laws of the first stage consist of Eqs. (18) and (21). Under these two control laws, the third link of the PAA manipulator stretches out toward the second one in a natural way (see Fig. 2).

Remark 2. Note that, as long as the control law Eq. (18) is employed for τ_3 in the whole control progress, the third link of the PAA manipulator will stretch out all the time, that is, the angle and the angular velocity of the third link are maintained to be zero. In addition, τ_2 is temporarily set to zero in this stage, which brings an easy way to handle the relationship between τ_2 and τ_3. In the following stages, it will show that control laws are designed for τ_2 respectively to achieve a swing-up control and a balancing control.

3.2 Control Laws for the Second Stage. The second stage aims to increase the system energy and drive the second link to stretch out toward the first one in a natural way, that is, to increase $E(x)$ to approach E_0, and force both the angle and angular velocity of the second link to converge to zero. Before proceeding to the control laws design, the following assumptions are made.

Assumption 1. *The control law Eq. (18) is still employed to guarantee the convergence of x_3 and x_6.*

Assumption 2. *When the PAA manipulator enters into the second stage, the states x_3 and x_6 are assumed to be zero, i.e.,*

$$x_3 = 0, \quad x_6 = 0 \tag{22}$$

Notice from Eq. (14) that the PAA manipulator moves into the second stage on the condition that x_3 and x_6 are less than ε_1 and ε_2, respectively. Since ε_1 and ε_2 are defined to be very small (generally the order of 10^{-5}), and with Assumption 1, Assumption 2 becomes practically reasonable for the second stage.

Now, consider the second control Lyapunov function candidate $V_2(x)$ as

$$V_2(x) = \frac{k_1}{2}E_x^2 + \frac{k_2}{2}x_2^2 + \frac{1}{2}\alpha(x)x_5^2 \tag{23}$$

where $k_1 > 0$ and $k_2 > 0$ are constants; $\alpha(x)$ is a positive time-varying design parameter; and

$$E_x = E(x) - E_0 \tag{24}$$

Note that $E(0) = E_0$, it is easy to know that $V_2(0) = 0$. From Eq. (5), the time derivative of E_x in Eq. (24) is

$$\dot{E}_x = \dot{E}(x) = x_5\tau_2 + x_6\tau_3 \tag{25}$$

Hence, the time derivative of $V_2(x)$ is

$$\begin{aligned}\dot{V}_2(x) = {} & k_1E_x[x_5\tau_2 + x_6\tau_3] + k_2x_2x_5 + \frac{1}{2}\dot{\alpha}(x)x_5^2 \\ & + \alpha(x)x_5[f_2(x) + b_2(x)\tau_2 + c_2(x)\tau_3]\end{aligned} \tag{26}$$

Substituting Eq. (22) into Eq. (26) obtains

$$\dot{V}_2(x) = [\phi(x)\tau_2 + \varphi(x)]x_5 \tag{27}$$

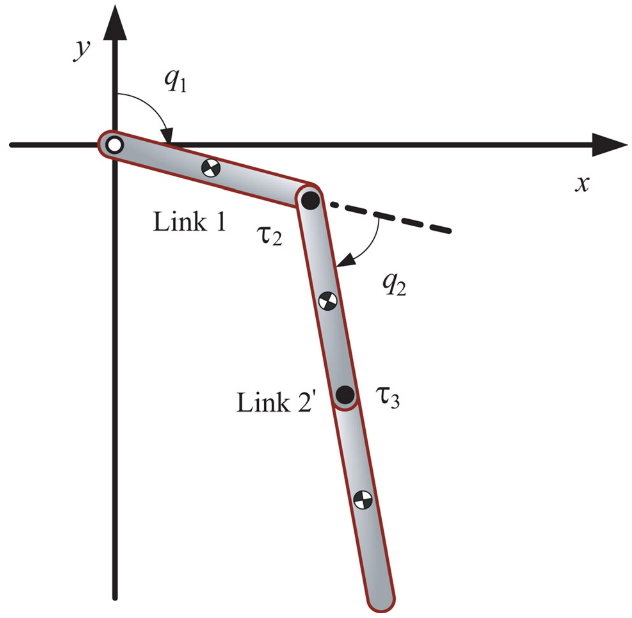

Fig. 2 Control result of the first stage

where

$$\phi(x) = k_1 E_x + \alpha(x)\frac{\mu(x)}{c_3(x)} \tag{28}$$

$$\varphi(x) = k_2 x_2 + \alpha(x)\frac{\nu(x)}{c_3(x)} + \frac{1}{2}\dot{\alpha}(x)x_5 \tag{29}$$

$$\mu(x) = b_2(x)c_3(x) - c_2(x)b_3(x) \tag{30}$$

$$\nu(x) = f_2(x)c_3(x) - c_2(x)f_3(x) \tag{31}$$

When $\phi(x) \neq 0$, the control law of τ_2 is chosen as

$$\tau_2 = \frac{-\varphi(x) - \gamma_2 x_5}{\phi(x)} \tag{32}$$

where $\gamma_2 > 0$ is a design parameter. Then, this control law ensures that

$$\dot{V}_2(x) = -\gamma_2 x_5^2 < 0, \quad \forall x_5 \neq 0 \tag{33}$$

Note that, due to the employment of the first stage, τ_3 does not appear explicitly in Eq. (27), which makes it easy to swing-up the PAA manipulator. However, there is no guarantee that $\phi(x) \neq 0$. Since from Eq. (3), we have

$$M^{-1}(x) = \begin{bmatrix} a_1(x) & b_1(x) & c_1(x) \\ b_1(x) & b_2(x) & c_2(x) \\ c_1(x) & b_3(x) & c_3(x) \end{bmatrix} \tag{34}$$

where $a_1(x) = [m_{22}(x)m_{33}(x) - m_{23}^2(x)]/\det[M(x)]$. As $M(x)$ is a positive definite matrix, $M^{-1}(x)$ is also a positive definite matrix. From Eq. (34), it is observed that $\mu(x) = b_2(x)c_3(x) - c_2(x)b_3(x)$ is exactly a second-order principal minor of $M^{-1}(x)$, so $\mu(x) > 0$. Furthermore, $\alpha(x)$ is chosen to be a positive time-varying parameter, and $c_3(x)$ in Eq. (20) is positive, so the second term of $\phi(x)$ in Eq. (28) is positive. On the other hand, the inequalities $-2E_0 \leq E_x < 0$ hold before $E(x)$ reaches E_0. Thus, the denominator $\phi(x)$ of Eq. (32) may equal to zero, that is, singularity problem may occur. To avoid any singularities, the parameter $\alpha(x)$ in Eq. (23) is chosen as

$$\alpha(x) = \frac{\eta}{\mu(x)} \tag{35}$$

where $\eta > 2k_1\rho E_0$ with $\rho = \max\{c_3(x)\} > 0$. Then $\phi(x) > 0$ always holds.

Therefore, the control laws of the second stage consist of Eqs. (18) and (32), and the choice of Eq. (35) avoids any singularities in the control process. At the end of this stage, the PAA manipulator is controlled as Fig. 3.

Remark 3. It is well-known that the three-link PAA manipulator is a strong-coupling nonlinear system, for which it is hard to design effective controllers by using general control methods. In the proposed control strategy, a preparatory stage is first introduced, and a control law is designed to stretch out the third link. From the above design procedures, it is known that, due to the control of the first stage, the influence of τ_3 and the angular velocity of the third link to the swing-up control of the second stage is explicitly eliminated. In consequence, the swing-up control guarantees that the second link stretches out, meanwhile the system energy increases continuously until reaching E_0. This makes it easy and quick to swing-up the PAA manipulator.

3.3 Control Laws for the Third Stage.

The third stage performs a balancing control to reach the final objective, which is to stabilize the manipulator at the straight-up unstable equilibrium position, i.e., $x = [0\ 0\ 0\ 0\ 0\ 0]^\top$. For this stage, the following assumptions are made.

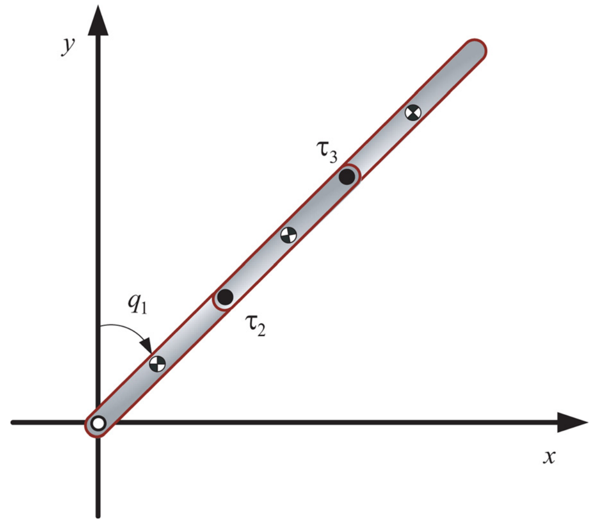

Fig. 3 Control result of the second stage

Assumption 3. *The posture of the third link is maintained under the control law (18), i.e., $x_3 = 0$ and $x_6 = 0$,*

Assumption 4. *The following approximations are taken to obtain a linear model of the PAA manipulator at the straight-up position.*

$$\begin{cases} \cos x_2 \approx 1, \quad \dot{x}_i \approx 0, \quad i = 1, 2 \\ \sin x_1 \approx x_1, \quad \sin(x_1 + x_2) \approx x_1 + x_2 \end{cases} \tag{36}$$

Notice from Eq. (15) that ε_3 and ε_4 are two important parameters for defining the balancing stage, and they are usually chosen to be less than $\pi/4$. Since $\sin(\pi/4) = 0.7071$ and $\pi/4 = 0.7854$, roughly speaking, $\sin\varepsilon \approx \varepsilon$ is true for $0 \leq |\varepsilon| \leq \pi/4$. This leads us to make Assumption 4.

Based on the above assumptions, a reduced model of the PAA manipulator is obtained by substituting Eq. (18) into Eq. (2)

$$\begin{cases} \dot{x}_1 = x_4 \\ \dot{x}_2 = x_5 \\ \dot{x}_4 = f_{a1}(\tilde{x}) + b_{a1}(\tilde{x})\tau_2 \\ \dot{x}_5 = f_{a2}(\tilde{x}) + b_{a2}(\tilde{x})\tau_2 \end{cases} \tag{37}$$

where $\tilde{x} = x|_{(x_3=0, x_6=0)}$ and

$$f_{a1}(\tilde{x}) = f_1(\tilde{x}) - \frac{c_1(\tilde{x})}{c_3(\tilde{x})}f_3(\tilde{x}), \quad b_{a1}(\tilde{x}) = b_1(\tilde{x}) - \frac{c_1(\tilde{x})}{c_3(\tilde{x})}b_3(\tilde{x}),$$
$$f_{a2}(\tilde{x}) = f_2(\tilde{x}) - \frac{c_2(\tilde{x})}{c_3(\tilde{x})}f_3(\tilde{x}), \quad b_{a2}(\tilde{x}) = b_2(\tilde{x}) - \frac{c_2(\tilde{x})}{c_3(\tilde{x})}b_3(\tilde{x})$$

By assuming $z = [x_1\ x_2\ x_4\ x_5]^\top$ and taking Assumption 4, we linearize Eq. (37) at $z = [0\ 0\ 0\ 0]^\top$ as

$$\dot{z} = Az + B\tau_2 \tag{38}$$

An optimization objective is defined as

$$J = \int_0^\infty \left(z^\top Qz + R\tau_2^2\right)\mathrm{d}t \tag{39}$$

where $Q > 0$ and $R > 0$ are weighting matrices. The third control Lyapunov function candidate $V_3(z)$ is defined as

$$V_3(z) = z^\top P z \tag{40}$$

where P is a positive symmetric matrix. Then, the optimal control law is

$$\tau_2 = -Kz, \tag{41}$$

with $K = R^{-1}B^\top P$; and $P = P^\top > 0$ is the solution of the Riccati equation

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0 \tag{42}$$

Note that Eq. (41) ensures $\dot{V}_3(z) < 0, \forall z \neq 0$, so the state z of the reduced system (37) converges to zero. Meanwhile, $\dot{V}_1(x) < 0$ is always guaranteed by the control law (18). Therefore, the control laws (18) and (41) for the third stage force the PAA manipulator to converge to the straight-up unstable equilibrium position.

Remark 4. In conventional design methods, the balancing control laws are usually designed by linearizing system (2) at the straight-up unstable equilibrium position based on the approximations

$$\begin{cases} \cos x_3 \approx 1, \quad \cos(x_2 + x_3) \approx 1, \quad \dot{x}_i \approx 0, \quad i = 1,2,3 \\ \sin x_1 \approx x_1, \sin(x_1 + x_2) \approx x_1 + x_2 \\ \sin(x_1 + x_2 + x_3) \approx x_1 + x_2 + x_3 \end{cases} \tag{43}$$

Notice that Eq. (43) is very rigorous for the three-link PAA manipulator to enter into balancing stage, since it requires all the states of the three links should be small enough. Especially, for $\sin(x_1 + x_2 + x_3) \approx x_1 + x_2 + x_3$, it requires $|x_1 + x_2 + x_3| \leq \varepsilon$, where $\varepsilon \in (0\ \pi/4]$. That means all the angles of the three links should be very small when they have the identical sign. However, in the proposed control strategy, the nonlinear control law (18) designed in preparatory stage guarantees that the third link stretches out all the time, even when τ_2 is switched from Eq. (32) to Eq. (41). So, the approximations (43) is replaced by Eq. (36) in balancing stage, and in particular, $|x_1 + x_2 + x_3| \leq \varepsilon$ is replaced by $|x_1 + x_2| \leq \varepsilon$. As a result, it is much easier for the PAA manipulator to enter into balancing stage, and the switch from the second stage to the third stage becomes smooth.

4 Stability Analysis

In the first and second stages, it is known that, although $V_1(x)$ and $V_2(x)$ in Eqs. (16) and (23) decrease monotonically, there is no guarantee that the system states (x_3, x_6) and (x_2, x_5) converge to zero. Therefore, the stabilities of the PAA manipulator in the first and second stage need to be analyzed. Similar to Refs. [16,17,27], Lyapunov theory and LaSalle's invariance principle are used in this section to guarantee the system stability.

4.1 Stability of the First Stage. It is known that $V_1(x)$ is continuously differentiable, and under the control laws (18) and (21), it is a weak-control Lyapunov function (WCLF) [16,28]. Substituting Eqs. (18) and (21) into Eq. (2) to obtain the following closed-loop system

$$\dot{x} = F_1(x) \tag{44}$$

Since $\dot{V}_1(x) < 0$, $V_1(X)$ is bounded. Define

$$\Phi_1 = \{x \in \mathbb{R}^6 | V_1(x) \leq c_1\} \tag{45}$$

where c_1 is a positive constant. Then, any solution x of Eq. (44) starting in Φ_1 remains in Φ_1 for all $t \geq 0$. Let Ψ_1 be an invariant set of Eq. (44), which is

$$\Psi_1 = \{x \in \Phi_1 | \dot{V}_1(x) = 0\} \tag{46}$$

Since $\dot{V}_1(x) = 0$, then $x_6 = 0$. From Eq. (2), we have

$$f_3(x) + b_3(x)\tau_2 + c_3(x)\tau_3 = 0 \tag{47}$$

Substituting Eq. (18) into Eq. (47) yields

$$f_3(x) + b_3(x)\tau_2 + [-x_3 - f_3(x) - b_3\tau_2 - \gamma_1 x_6] = 0 \tag{48}$$

Simplifying Eq. (48) to obtain $x_3 = 0$. Therefore, the largest invariant set of the PAA manipulator in the first stage is

$$\mathcal{M}_1 = \{x \in \Psi_1 \,|\, x_3 = 0,\ x_6 = 0\} \tag{49}$$

According to LaSalle's invariance theorem [29], it is known that every solution x of (44) starting in Φ_1 approaches to $\mathcal{M}_1$ as $t \to +\infty$.

Summarizing the above obtained result to obtain the following theorem for the first stage.

THEOREM 1. *Consider the PAA manipulator described by Eq. (2) in the preparatory stage. Let $\mathcal{M}_1$ be the largest invariant set of the system (44). $V_1(x)$ is a WCLF, Φ_1 is a compact closed and bounded set that contains all the initial states of the system (44), and Ψ_1 is a set with states in Φ_1 where $\dot{V}_1(x) = 0$. If the control laws (18) and (21) are employed, then every solution x of the closed-loop system (44) converges to the invariant set $\mathcal{M}_1$ given by Eqn. (49), that is, the third link of the manipulator stretches out toward the second one in a natural way.*

4.2 Stability of the Second Stage. In the second stage, $V_2(x)$ is also continuously differentiable. Substituting Eqs. (18) and (32) into Eq. (2) to obtain the following closed-loop system:

$$\dot{x} = F_2(x) \tag{50}$$

Since $\dot{V}_2(x) < 0$, then $V_2(x)$ is bounded. Meantime, the control law of τ_3 designed in the first stage always is maintained, which guarantees x_3 and x_6 converge to zero. Hence, define

$$\Phi_2 = \{x \in \mathbb{R}^6 | V_1(x) \leq c_1, V_2(x) \leq c_2\} \tag{51}$$

where c_1 and c_2 are positive constants. Then, any solution x of Eq. (50) starting in Φ_2 remains in Φ_2 for all $t \geq 0$. Let Ψ_2 be an invariant set of Eq. (50), which is

$$\Psi_2 = \{x \in \Phi_2 | \dot{V}_1(x) = 0, \dot{V}_2(x) = 0\} \tag{52}$$

When $\dot{V}_1(x) = 0$ and $\dot{V}_2(x) = 0, x_6 = 0$ and $x_5 = 0$. Meanwhile, x_3 converges to zero as analyzed previously. Then, from Eq. (25), we know $\dot{E}_x = \dot{E}(x) = 0$, that is, E_x is constant, which may have two cases: $E_x = 0$ and $E_x = \text{constant} \neq 0$. These two cases are now addressed, separately.

Case 1: $E_x = 0$, $x_3 = 0$ and $x_6 = 0$.

Since $x_5 = 0$, it follows from (2) that

$$f_2(x) + b_2(x)\tau_2 + c_2\tau_3 = 0 \tag{53}$$

Substituting Eqs. (18) and (32) with ($x_3 = 0$, $x_6 = 0$) into Eq. (53) yields

$$-\frac{\mu(x)}{c_3(x)} \frac{k_2 x_2 + \alpha(x)\dfrac{\nu(x)}{c_3(x)}}{\alpha(x)\dfrac{\mu(x)}{c_3(x)}} + \frac{\nu(x)}{c_3(x)} = 0 \tag{54}$$

Simplifying the above equation yields $x_2 = 0$. On the other hand, $E_x = 0$, which means $E(x) = E_0$. So, the trajectory is such that (x_2, x_5) = (0, 0) and $E(x) = E_0$ for any variables. Meanwhile, the passive link travels in a periodic circle orbit, which can be derived from $E(x) = E_0$, that is,

$$\frac{1}{2} m_{11}(0) x_4^2 + (\beta_1 + \beta_2 + \beta_3) \cos x_1 = \beta_1 + \beta_2 + \beta_3 \tag{55}$$

Case 2: $E_x = \text{constant} \neq 0$, $x_3 = 0$ and $x_6 = 0$.

If $E_x = \text{constant} \neq 0$, then $E(x)$ is constant. So, from Eqs. (5)–(7), we have

$$\begin{aligned} E(x) &= \frac{1}{2} m_{11}(x) x_4^2 + \beta_1 \cos x_1 + (\beta_2 + \beta_3) \cos(x_1 + x_2) \\ &= \frac{1}{2} m_{11}(x) x_4^2 + [\beta_1 + (\beta_2 + \beta_3) \cos x_2] \cos x_1 \\ &\quad - [(\beta_2 + \beta_3) \sin x_2] \sin x_1 \end{aligned} \tag{56}$$

Since $x_5 = 0$, x_2 is a constant, then $m_{11}(x)$ is also a constant. Hence, Eq. (56) can be written as

$$\mathcal{T}_1 = \mathcal{U}_1 x_4^2 + \mathcal{V}_1 \cos x_1 + \mathcal{W}_1 \sin x_1 \tag{57}$$

where $\mathcal{T}_1, \mathcal{U}_1, \mathcal{V}_1$, and $\mathcal{W}_1$ are constants. On the other hand, substituting Eq. (18) into Eq. (53) yields

$$\tau_2 = \frac{c_3(x) f_2(x) - c_2(x) f_3(x)}{c_2(x) b_3(x) - c_3(x) b_2(x)} = -\frac{\nu(x)}{\mu(x)} \tag{58}$$

Moreover, from Eqs. (28) to (32), it is known that

$$\tau_2 = -\frac{\varphi(x)}{\phi(x)} = -\frac{k_2 x_2 c_3(x) + \alpha(x) \nu(x)}{k_1 E_x c_3(x) + \alpha(x) \mu(x)} \tag{59}$$

Comparing Eq. (58) with Eq. (59) yields

$$\frac{c_3(x) f_2(x) - c_2(x) f_3(x)}{c_2(x) b_3(x) - c_3(x) b_2(x)} = -\frac{k_2 x_2}{k_1 E_x} = \text{constant} \tag{60}$$

Moreover, from Eqs. (2) and (3), Eq. (60) is simplified to

$$\mathcal{T}_2 = \mathcal{U}_2 x_4^2 + \mathcal{V}_2 \cos x_1 + \mathcal{W}_2 \sin x_1 \tag{61}$$

where $\mathcal{T}_2, \mathcal{U}_2, \mathcal{V}_2$, and $\mathcal{W}_2$ are constants. Taking the difference between Eq. (57) multiplied by $\mathcal{U}_2$ and Eq. (61) multiplied by $\mathcal{U}_1$ yields

$$\mathcal{T}_3 = \mathcal{V}_3 \cos x_1 + \mathcal{W}_3 \sin x_1 \tag{62}$$

where $\mathcal{T}_3 = \mathcal{T}_1 \mathcal{U}_2 - \mathcal{T}_2 \mathcal{U}_1, \mathcal{V}_3 = \mathcal{V}_1 \mathcal{U}_2 - \mathcal{V}_2 \mathcal{U}_1$, and $\mathcal{W}_3 = \mathcal{W}_1 \mathcal{U}_2 - \mathcal{W}_2 \mathcal{U}_1$. Note that the vectors $A_1 = (\mathcal{T}_1, \mathcal{V}_1, \mathcal{W}_1)^\top$ and $A_2 = (\mathcal{T}_2, \mathcal{V}_2, \mathcal{W}_2)^\top$ constructed from Eqs. (57) and (61) are nonzero and linearly independent, which means that none of them can be written as a linear combination of the other one. Thus, the vector $A_3 = (\mathcal{T}_3, \mathcal{V}_3, \mathcal{W}_3)^\top$ from Eq. (62) will also be nonzero, that is, $\mathcal{V}_3, \mathcal{W}_3$ and $\mathcal{T}_3$ will not be zero simultaneously. Therefore, from Eqs. (57), (61), and (62), x_1 must be a constant, i.e., $x_4 = 0$.

When $x_4 = x_5 = x_6 = 0$ holds, the robot is in an equilibrium point. From Eq. (2), it obtains

$$f_1 + b_1 \tau_2 + c_1 \tau_3 = 0 \tag{63}$$

$$f_2 + b_2 \tau_2 + c_2 \tau_3 = 0 \tag{64}$$

$$f_3 + b_3 \tau_2 + c_3 \tau_3 = 0 \tag{65}$$

Furthermore, it follows from Eq. (1) that $g_1 = 0$, i.e.,

$$\beta_1 \sin x_1 + (\beta_2 + \beta_3) \sin(x_1 + x_2) = 0 \tag{66}$$

From Eq. (3), we have

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -b_1 g_2 - c_1 g_3 \\ -b_2 g_2 - c_2 g_3 \\ -b_3 g_2 - c_3 g_3 \end{bmatrix} \tag{67}$$

Substituting Eq. (65) into Eq. (64) and using Eq. (67) yields

$$-(b_2 c_3 - c_2 b_3) g_2 + (b_2 c_3 - c_2 b_3) \tau_2 = 0 \tag{68}$$

Note that $\mu(x) = b_2 c_3 - c_2 b_3$ defined in Eq. (30) is positive. Then Eq. (68) gives

$$\tau_2 = g_2 = -(\beta_2 + \beta_3) \sin(x_1 + x_2) \tag{69}$$

Together with Eq. (60), it is clear that

$$\frac{k_2 x_2}{k_1 E_x} = (\beta_2 + \beta_3) \sin(x_1 + x_2) \tag{70}$$

where $E_x \neq 0$. Following the similar analysis in Ref. [27], it is straightforward to conclude that Eqs. (66) and (70) may have the solution

$$(x_1 = 0, x_2 = 0) \quad \text{or} \quad (x_1 = \pi, x_2 = 0) \ [\text{mod } (2\pi)] \tag{71}$$

provided that the control parameters are selected satisfying

$$k_2 \geq 2\beta_1 (\beta_2 + \beta_3) k_1 \tag{72}$$

Meanwhile, $x_3 = 0$, then the PAA manipulator in *case 2* converges to either the downward point $x_{\text{down}} = [\pi \ 0 \ 0 \ 0 \ 0 \ 0]^\top$ or the straight-up point $x_{\text{up}} = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^\top$. If it converges to x_{up}, then the energy converges to E_0, which is exactly in *case 1*, and the control objective is fulfilled. If it converges to x_{down}, then the energy is ($-E_0$), which is the minimum potential energy during the control. Since the control objective of the second stage is to increase the system energy until it reaches E_0, when the control laws are designed as Eqs. (18) and (32), the manipulator has already left the downward point and gained kinetic energy, that is, $E(x) = T(x) + P(x) > -E_0$. Therefore, the manipulator cannot stay still at x_{down}, and *case 2* cannot be the steady state.

Summarizing the above results to obtain the following theorem for the second stage.

THEOREM 2. *Consider the PAA manipulator described by Eq. (2) in swing-up stage. Let $\mathcal{M}_2$ be the largest invariant set of the system (50), which is*

$$\mathcal{M}_2 = \{x \in \Psi_2 | (55),\ x_2 = x_3 = 0,\ x_5 = x_6 = 0\} \tag{73}$$

$V_1(x)$ and $V_2(x)$ are WCLFs, Φ_2 is a compact closed and bounded set that contains all the initial states of the system (50), and Ψ_2 is a set with states in Φ_2 where $\dot{V}_1(x) = 0$ and $\dot{V}_2(x) = 0$. If the control laws in Eqs. (18) and (32) are employed with $\alpha(x)$, k_1 and k_2 satisfying conditions (35) and (72), then every solution x of the closed-loop system (50) converges to the invariant set $\mathcal{M}_2$. In other words, both the two actuated links stretch out toward the front link in a natural way, whereas the passive link travels in a periodic circle orbit given as Eq. (55).

5 Numerical Simulations and Comparisons

To demonstrate the validity of the proposed control strategy, this section presents simulation results for the PAA manipulator. In particular, a comparison result with literature [24] is presented to show the advantages of the presented strategy.

The physical parameters of the PAA manipulator are $m_1 = 5.4$ kg, $m_2 = 29.5$ kg, $m_3 = 18.5$ kg, $L_1 = 0.58$ m, $L_2 = 0.5$ m, $L_3 = 0.79$ m,

$L_{c1} = 0.31$ m, $L_{c2} = 0.20$ m, $L_{c3} = 0.33$ m, $J_1 = 0.15$ kg m^2, $J_2 = 1.93$ kg m^2, and $J_3 = 1.03$ kg m^2. The parameters in operations (13)–(15) are chosen as $\varepsilon_1 = 10^{-5}$ rad, $\varepsilon_2 = 10^{-5}$ rad/s, $\varepsilon_3 = \varepsilon_4 = (\pi/6)$ rad, $\varepsilon_5 = 0.5$ J. To achieve a fast swing-up control, the control parameters in Eqs. (18), (23), (32), and (35) are selected by trial and error, which are $\gamma_1 = 3.5$, $k_1 = 0.05$, $k_2 = 6090$, $\gamma_2 = 1000$, $\eta = 135.10$, $\rho = 1.36$, and $E_0 = 497.54$ J. The weighting matrices $Q = I_4$, $R = 0.5$ are chosen to obtain the state-feedback control law for Eq. (41)

$$K = -10^3 \times [5.9067\ 2.4764\ 2.0091\ 0.9592]$$

5.1 Simulation Results. Without loss of generality, we first assume that the PAA manipulator starts with a nonzero initial state $x = [\pi\ \pi/8\ \pi\ 0\ 0\ 1.7]^\top$. The simulation results are shown in Fig. 4.

Notice that τ_2 switches twice, at $t = 11.535$ s and $t = 22.586$ s, respectively. First, when $t < 11.535$ s, the PAA manipulator is governed by the first stage control laws (18) and (21), and up to $t = 11.535$ s, the third link has stretched out toward the second one in a natural way. At $t = 11.535$ s, the PAA manipulator enters into the second stage. During this stage, the control law (18) of the third link is still employed to guarantee the third link stretches out all the time, and the control law of τ_2 is switched from Eqs. (21) to (32). Then, at $t = 22.586$ s, the PAA manipulator enters into the third stage, and the control laws (18) and (41) are used to stabilize the PAA manipulator at the straight-up position.

5.2 Comparison Results. Then, another case with initial state $x = [17\pi/18\ 0\ 0\ 0\ 0\ 0]^\top$ in Ref. [24] is considered. Note that the control parameters in Ref. [24] are also selected by trial and error,

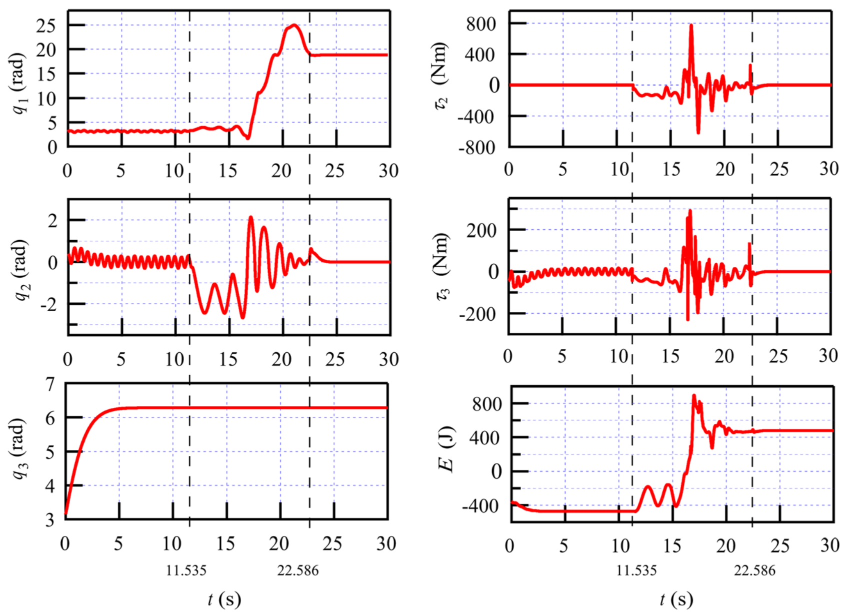

Fig. 4 The angles q_i ($i = 1, 2, 3$), the torques τ_2 and τ_3, and the energy E of the PAA manipulator with initial state $x = [\pi\ \pi/8\ \pi\ 0\ 0\ 1.7]^\top$

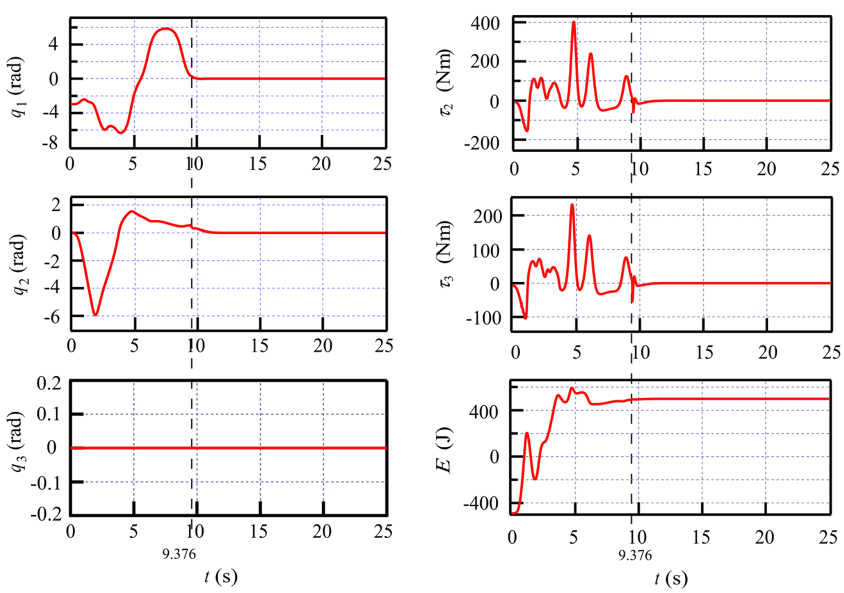

Fig. 5 The angles q_i ($i = 1, 2, 3$), the torques τ_2 and τ_3, and the energy E of the PAA manipulator with initial state $x = [17\pi/18\ 0\ 0\ 0\ 0\ 0]^\top$

and a fast swing-up control is achieved. The simulation results are depicted in Fig. 5.

Notice that the initial state satisfies condition (14), so the PAA manipulator skips the first stage, and directly starts from the second stage. It also observes that there is only one switch in the whole control process, which takes at $t=9.376$ s. Before $t=9.376$ s, the manipulator is in *swing-up stage*, and when $t=9.376$ s, the manipulator enters into *balancing stage*. Finally, the PAA manipulator is stabilized at the straight-up unstable position in no more than 13 s.

Comparing the simulation results between Figs. 4 and 5, we observe that for different initial state of the third link, i.e., q_3 and $\dot{q}_3$, the settling time of the first stage is totally different. The smaller are q_3 and $\dot{q}_3$, the shorter time is taken in the first stage. For example, the q_3 and $\dot{q}_3$ in Fig. 5 are both zero, and the PAA manipulator directly skips the first stage. In comparison with Ref. [24], the settling time of the proposed method is only about 13 s, while that is more than 30 s in Ref. [24]. In addition, from the curves of τ_2 and τ_3, it observes that when τ_3 has the same amplitude, the amplitude of τ_2 is much smaller than that of Ref. [24]. These demonstrate the efficiency of the proposed control strategy.

6 Conclusions

This paper describes a novel control strategy for a three-link underactuated manipulator called PAA manipulator. The control strategy is studied based on three stages. In the first stage, the third link is forced to stretch out toward the second one in a natural way, which provides a prerequisite and basis for the control of the following stages. In the second stage, a swing-up control law is designed based on a Lyapunov function to increase the energy and stretch out the second link. In the third stage, an integrated method with linear control and nonlinear control is introduced to guarantee that the PAA manipulator can move into the third stage easily and smoothly. The stability of the control system is rigorously guaranteed by LaSalle's invariance principle. Simulation and comparison results show that the proposed control strategy swings up the PAA manipulator easily and quickly, and stabilizes the robot at the straight-up position efficiently. Two points of the proposed control strategy are worth noticing:

(1) The employment of the first stage not only realizes the posture control of the third link, but also eliminates the influences of the control input and angular velocity of the third link to the control of the second stage, which makes the design of control laws for the next two stages easy.
(2) The maintained control law of the third link ensures that the third link stretches out all the time in the next two stages, even when the balancing controller is switched to stabilize the manipulator at the straight-up position. This guarantees the manipulator move into the third stage easily and smoothly.

In the future work, it will be interesting and challenging to study the robust issues of the control strategy under uncertainties, such as external disturbances and uncertain model parameters.

Acknowledgment

This work was supported in part by the National Natural Science Foundation of China under Grant Nos. 61074112 and 61374106.

Appendix

Dynamics of the PAA Manipulator

The structures of $M(q)$, $H(q,\dot{q})$, and $G(q)$ in Eq. (1) are defined as

$$M(q)=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},\quad H(q,\dot{q})=\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix},\quad G(q)=\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

where

$$\begin{aligned}
m_{11} &= \alpha_1+\alpha_2+\alpha_4+2\alpha_3\cos q_2+2\alpha_5\cos(q_2+q_3)+2\alpha_6\cos q_3\\
m_{12} &= \alpha_2+\alpha_4+\alpha_3\cos q_2+\alpha_5\cos(q_2+q_3)+2\alpha_6\cos q_3\\
m_{13} &= \alpha_4+\alpha_5\cos(q_2+q_3)+\alpha_6\cos q_3\\
m_{22} &= \alpha_2+\alpha_4+2\alpha_6\cos q_3\\
m_{23} &= \alpha_4+\alpha_6\cos q_3\\
m_{33} &= \alpha_4\\
h_1 &= -\alpha_5(2\dot{q}_1+\dot{q}_2+\dot{q}_3)(\dot{q}_2+\dot{q}_3)\sin(q_2+q_3)\\
&\quad -\alpha_6(2\dot{q}_1+2\dot{q}_2+\dot{q}_3)\dot{q}_3\sin q_3\\
&\quad -\alpha_3(2\dot{q}_1+\dot{q}_2)\dot{q}_2\sin q_2\\
h_2 &= \alpha_3\dot{q}_1^2\sin q_2+\alpha_5\dot{q}_1^2\sin(q_2+q_3)\\
&\quad -\alpha_6(2\dot{q}_1+2\dot{q}_2+\dot{q}_3)\dot{q}_3\sin q_3\\
h_3 &= \alpha_5\dot{q}_1^2\sin(q_2+q_3)+\alpha_6(\dot{q}_1+\dot{q}_2)^2\sin q_3\\
g_1 &= -\beta_1\sin q_1-\beta_2\sin(q_1+q_2)-\beta_3\sin(q_1+q_2+q_3)\\
g_2 &= -\beta_2\sin(q_1+q_2)-\beta_3\sin(q_1+q_2+q_3)\\
g_3 &= -\beta_3\sin(q_1+q_2+q_3)
\end{aligned}$$

In the above equations, $\alpha_i (i=1, \cdots, 6)$ and $\beta_j (j=1, 2, 3)$ are defined as the structure parameters, which are $\alpha_1=J_1+m_1L_{c1}^2+(m_2+m_3)L_1^2$, $\alpha_2=J_2+m_2L_{c2}^2+m_3L_2^2$, $\alpha_3=(m_2L_{c2}+m_3L_2)L_1$, $\alpha_4=J_3+m_3L_{c3}^2$, $\alpha_5=m_3L_1L_{c3}$, $\alpha_6=m_3L_2\ L_{c3}$ and $\beta_1=(m_1L_{c1}+m_2L_1+m_3L_1)g$, $\beta_2=(m_2L_{c2}+m_3L_2)g$, $\beta_3=m_3L_{c3}g$.

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