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Lookback options and diffusion hitting times: A spectral expansion approach

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Abstract. Lookback options have payoffs dependent on the maximum and/or minimum of the underlying price attained during the option's lifetime. Based on the relationship between diffusion maximum and minimum and hitting times and the spectral decomposition of diffusion hitting times, this paper gives an analytical characterization of lookback option prices in terms of spectral expansions. In particular, analytical solutions for lookback options under the constant elasticity of variance (CEV) diffusion are obtained.

Key words: Lookback options, diffusion maximum and minimum, hitting times, spectral expansions, CEV model, Bessel process

JEL Classification: G13

Mathematics Subject Classification (1991): 60J35, 60J60, 60G70

1 Introduction

Lookback options are an important family of path-dependent options. Their payoffs depend on the maximum or minimum underlying asset price attained during the option's life. A *standard lookback call* gives the option holder the right to buy at the lowest price recorded during the option's life. A *standard lookback put* gives the right to sell at the highest price. These options were first studied by Goldman et al. (1979a) and Goldman et al. (1979b) who derived closed-form pricing formulas under the geometric Brownian motion assumption. Lookbacks materialize

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every investor's desire to buy at the ex-post low and sell at the ex-post high. In addition to standard lookback options, Conze and Vishwanathan (1991) introduce and price calls on maximum and puts on minimum. A *call on maximum* pays off the difference between the realized maximum price and some prespecified strike or zero, whichever is greater. A *put on minimum* pays off the difference between the strike and the realized minimum price or zero, whichever is greater. These options are called *fixed-strike lookbacks*. In contrast, the standard lookback options are also called *floating-strike lookbacks*, because the floating terminal underlying asset price serves as the strike in standard lookback options. He et al. (1998) introduce and study *double lookback options*.

The above-referenced papers assume that the underlying asset price follows geometric Brownian motion. However, it is well established that the geometric Brownian motion assumption contradicts the accumulated empirical evidence. A major empirical finding is that equity option prices exhibit pronounced *implied volatility smiles*. Quoting Jackwerth and Rubinstein (1998), "These volatility smiles (implied volatilities which are largely convex and monotonically decreasing functions of strike prices) contradict the assumption of geometric Brownian motion which would imply a flat line. Described with an alternative metric, the implied risk-neutral probability densities are heavily skewed to the left and highly leptokurtic relative to the Black-Scholes lognormal presumption. These differences are large and seemingly well beyond the capacity of market imperfections to provide a solution." The *constant elasticity of variance* (CEV) model of Cox (1975) and Cox and Ross (1976) allows the instantaneous conditional variance of asset return to depend on the asset price level. The CEV model exhibits an implied volatility smile that is a convex and monotonically decreasing function of strike, similar to empirically observed volatility smile curves.

Recently, Boyle and Tian (1999) and Boyle et al. (1999) have initiated a study of barrier and lookback options in the CEV model. Boyle and Tian (1999) approximate the CEV process by a trinomial lattice and use it to value barrier and lookback options numerically. They show that the differences in prices of these extrema-dependent options under the CEV and geometric Brownian motion assumptions can be far more significant than the differences for standard European options. Boyle et. al (1999) value lookback options in the CEV model by Monte Carlo simulation. While the approach of Boyle and co-authors is purely numerical, Davydov and Linetsky (2001) obtain closed-form solutions for Laplace transforms of CEV barrier and lookback options with respect to time remaining to expiration and numerically invert these Laplace transforms via the Abate-Whitt numerical Laplace inversion algorithm. Davydov and Linetsky (2003) analytically invert the Laplace transforms for barrier options in terms of spectral expansions associated with the infinitesimal generator of the CEV diffusion (see also Linetsky 1999 and references therein for various alternatives to standard barrier options). [Remark: Barrier option prices obtained with the trinomial lattice by Boyle and Tian (1999) are in agreement with the prices obtained with the Laplace transform method by Davydov and Linetsky (2001). In contrast, lookback option prices obtained by Boyle and Tian (1999) are in disagreement with the prices obtained by Davydov and Linetsky (2001). The cause is a subtle problem with their trinomial lattice algorithm for lookbacks (see

Boyle et al. 1999, Sect. 1). Subsequently, Boyle et al. (1999) computed lookback prices by Monte Carlo simulation. Their Monte Carlo prices are in agreement with the lookback prices computed in Davydov and Linetsky (2001) via the Laplace inversion and in the present paper via the spectral method.]

In the present paper we apply the spectral expansion approach to lookback options. In Sect. 2 we give some general results for lookback options when the underlying follows a one-dimensional diffusion and express lookback prices in terms of hitting time distributions. In Sect. 3, following McKean (1956) and Kent (1980, 1982) (see also Linetsky 2002b), we give a spectral decomposition of the first hitting time distribution of a one-dimensional diffusion. In Sect. 4 we specialize to the CEV diffusion and obtain analytical solutions for lookback options in terms of spectral expansions. These spectral expansions can serve as benchmarks for numerical methods. Three specific advantages of spectral expansions are: (1) the Greeks can be calculated analytically by taking derivatives without any loss of precision; (2) long-dated contracts are easy to value (the longer the time to expiration, the faster the spectral expansion converges); (3) in the case of the CEV model, it turns out that the steeper is the volatility skew, the faster the spectral expansion convergence. These three properties of the spectral expansion method are in contrast with simulation.

While in this paper we do not consider interest rate models, the same approach can also be used to price lookback options on yield in the CIR term structure model. Leblanc and Scaillet (1998) have recently obtained expressions for lookback options on yield in terms of Laplace transforms of hitting times and suggested the application of the Abate-Whitt Laplace inversion algorithm to invert these transforms. The approach of the present paper can also be applied to their setting to express lookback options on yield in terms of the spectral decomposition of the CIR diffusion hitting time (see also Linetsky 2003 for related results on CIR and OU hitting times).

To conclude this introduction, we note that spectral expansions are a powerful analytical tool in derivatives pricing. Among the papers that employ the spectral method in finance we mention Davydov and Linetsky (2003), Gorovoi and Linetsky (2004), Lewis (1998, 2000), Linetsky (2002a–c, 2003, 2004), Lipton (2001, 2002) and Lipton and McGhee (2002) among others. Further details and references can be found in Linetsky (2002b).

2 Pricing lookback options

In this paper we model asset prices as one-dimensional diffusions. We take an equivalent martingale measure $\mathbb{P}$ as given and assume that, under $\mathbb{P}$, the underlying asset price process $\{S_t, t \geq 0\}$ is a regular diffusion on $(0, \infty)$ starting at some point $S_0 = x \in (0, \infty)$ and with the infinitesimal generator

$$(\mathcal{G}f)(x) = \frac{1}{2}\sigma^2(x)x^2 f''(x) + \mu x f'(x) \tag{1}$$

acting on functions on $(0, \infty)$ subject to appropriate regularity and boundary conditions. Here μ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are the constant risk-free interest rate and the constant dividend yield, respectively), and $\sigma = \sigma(x)$ is a given *local volatility function* assumed continuous and strictly positive for all $x \in (0, \infty)$. We assume $+\infty$ is a natural boundary and 0 is either natural, exit, or regular specified as a killing boundary by sending the process to a *cemetery* (or, in financial terms, *bankruptcy*) state ∂ at the first hitting time of zero (see Borodin and Salminen 1996, Chapt. 2 for Feller's classification of boundaries for one-dimensional diffusions).

Throughout this paper t denotes the running time variable. We assume that all options are written at time $t = 0$ and expire at time $t = T > 0$. To price lookback options we need distributions of the maximum and minimum prices. Let M_t and m_t be the maximum and minimum recorded to date $t \geq 0$, $M_t = \max_{0 \leq u \leq t} S_u$ and $m_t = \min_{0 \leq u \leq t} S_u$. Define the two functions (the probabilities are calculated with respect to $\mathbb{P}$ and the subscript x in $\mathbb{P}_x$ indicates that the process is starting at $S_0 = x$ at $t = 0$):

$$F(y; x, t) := \mathbb{P}_x(m_t \leq y) \quad \text{for} \quad y \leq x, \quad G(y; x, t) := \mathbb{P}_x(M_t \geq y) \quad \text{for} \quad x \leq y. \tag{2}$$

Proposition 1 (Davydov and Linetsky 2001, Proposition 4) *The prices of the standard lookback call, the standard lookback put, call on maximum and put on minimum at some time $t \in [0, T]$ during the option's life are:*

$$e^{-r\tau}\mathbb{E}_t[(S_T - m_T)^+] = e^{-q\tau}S_t - e^{-r\tau}m_t + e^{-r\tau}\int_0^{m_t} F(Y; S_t, \tau)dY, \tag{3}$$

$$e^{-r\tau}\mathbb{E}_t[(M_T - S_T)^+] = e^{-r\tau}M_t - e^{-q\tau}S_t + e^{-r\tau}\int_{M_t}^{\infty} G(Y; S_t, \tau)dY, \tag{4}$$

$$e^{-r\tau}\mathbb{E}_t[(M_T - K)^+] = e^{-r\tau}\begin{cases} \int_K^\infty G(Y; S_t, \tau)dY, & M_t \leq K, \\ M_t - K + \int_{M_t}^\infty G(Y; S_t, \tau)dY, & M_t > K, \end{cases} \tag{5}$$

$$e^{-r\tau}\mathbb{E}_t[(K - m_T)^+] = e^{-r\tau}\begin{cases} \int_0^K F(Y; S_t, \tau)dY, & m_t \geq K, \\ K - m_t + \int_0^{m_t} F(Y; S_t, \tau)dY, & m_t < K, \end{cases} \tag{6}$$

where all contracts are initiated at time zero, m_t and M_t are the minimum and maximum prices recorded to date $t \geq 0$ (known at time t), S_t is the current underlying price at time t, $\tau = T - t$ is the time remaining to expiration, and $\mathbb{E}_t[\bullet] \equiv \mathbb{E}[\bullet|\mathcal{F}_t]$, $\mathcal{F}_t := \sigma\{S_u : u \leq t\}$.

Proposition 1 expresses prices of *seasoned lookback options* at some time $t \in [0, T]$ during the option's lifetime in terms of the spot price S_t at time t, minimum m_t (maximum M_t) to date t, and the probability distribution F (G) of the minimum (maximum) (the term "seasoned" refers to the fact that the option contract has been initiated before the current valuation date t). To value newly-written contracts at time $t = 0$, set $m_0 = M_0 = S_0$ as there is only one price observation. *Partial lookbacks* where the lookback period for the computation of the maximum (minimum) covers only the final period $[t, T]$, $t > 0$, of the contract life can be valued by valuing a newly-written lookback option at time t conditional on the spot price S_t at t via Proposition 1 and then integrating the result with the density of S_t at t conditional on S_0 at time zero and discounting back to time zero to obtain the price of the partial lookback at time zero.

To compute Eqs. (3–6), we need the distributions of the maximum and minimum. First notice that

$$\mathbb{P}_x(M_t \geq y) = \mathbb{P}_x(\mathcal{T}_y \leq t) \text{ for } x \leq y, \;\; \mathbb{P}_x(m_t \leq y) = \mathbb{P}_x(\mathcal{T}_y \leq t) \text{ for } y \leq x, \tag{7}$$

where $\mathcal{T}_y := \inf\{t \geq 0 : S_t = y\}$ is the first hitting time of $y \in (0, \infty)$. One way to characterize the first hitting time distribution is to express its Laplace transform $\mathbb{E}_x[e^{-s\mathcal{T}_y}]$ for $s > 0$ in terms of the increasing and decreasing solutions of the ODE $\mathcal{G}u = s\,u$ (Borodin and Salminen 1996, p. 18). For the CEV diffusion, Davydov and Linetsky (2001) use explicit expressions for the increasing and decreasing solutions in terms of the Whittaker functions and then invert the Laplace transform numerically via the Abate and Whitt numerical Laplace inversion algorithm. In this paper we take a different approach. Instead of using Laplace transforms, we rely on the spectral decomposition, and no numerical Laplace inversion is required.

3 Spectral expansions for hitting times

Consider a one-dimensional, time-homogeneous regular diffusion $\{X_t, t \geq 0\}$ whose state space is some interval $I \subseteq \mathbb{R}$ with end-points e_1 and e_2, $-\infty \leq e_1 < e_2 \leq \infty$, and the infinitesimal generator

$$(\mathcal{G}f)(x) = \frac{1}{2}a^2(x)f''(x) + b(x)f'(x), \;\; x \in (e_1, e_2), \tag{8}$$

acting on functions on I subject to appropriate regularity and boundary conditions. We assume that the diffusion coefficient $a(x)$ is continuous and strictly positive on the open interval (e_1, e_2) and drift $b(x)$ is continuous on (e_1, e_2). The infinitesimal generator can be re-written in the symmetric form

$$(\mathcal{G}f)(x) = \frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{\mathfrak{s}(x)}\right)', \;\; x \in (e_1, e_2), \tag{9}$$

where $\mathfrak{s}(x)$ and $\mathfrak{m}(x)$ are the diffusion scale and speed densities (Borodin and Salminen 1996, p. 17):

$$\mathfrak{s}(x) := \exp\left(-\int^x \frac{2b(y)}{a^2(y)}dy\right), \;\; \mathfrak{m}(x) := \frac{2}{a^2(x)\mathfrak{s}(x)}. \tag{10}$$

The endpoints e_i, $i = 1, 2$, are either natural, entrance, exit, or regular boundaries for the diffusion X. In this section we consider two types of boundary conditions at regular boundaries: killing or instantaneous reflection. For regular instantaneously reflecting boundaries, the boundary point e is included in the state space I and $(\mathcal{G}f)(e) := \lim_{x \to e}(\mathcal{G}f)(x)$. For natural, entrance, exit and regular killing boundaries, the boundary point is not included in I. For exit and regular killing boundaries, the process X is sent to a *cemetery state* ∂ at the first hitting time of the boundary.

Fix some $y \in (e_1, e_2)$ and consider the first hitting time distribution $\mathbb{P}_x(\mathcal{T}_y \leq t)$, $\mathcal{T}_y := \inf\{t \geq 0 : X_t = y\}$, for $x < y$ (*first hitting time up*). Observe that

$$\mathbb{P}_x(\mathcal{T}_y \leq t) = \mathbb{P}_x(\mathcal{T}_y < \infty) - \mathbb{P}_x(t < \mathcal{T}_y < \infty). \tag{11}$$

For $e_1 < x < y$ we have:

$$\Phi_{e_1}(x, y) := \mathbb{P}_x(\mathcal{T}_y < \infty)$$

$$= \begin{cases} 1 & \text{if } e_1 \text{ is entrance, reflecting, or non-attracting natural} \\ \frac{\mathcal{S}(e_1,x]}{\mathcal{S}(e_1,y]} & \text{if } e_1 \text{ is exit, killing, or attracting natural} \end{cases}, \tag{12}$$

where $\mathcal{S}(e_1, x] := \int_{(e_1,x]} \mathfrak{s}(z)dz$ (Karlin and Taylor 1981, p. 227). Then, observing that

$$\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\mathbb{E}[\mathbf{1}_{\{\mathcal{T}_y<\infty\}}|\mathcal{F}_t]] = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)], \tag{13}$$

Eq. (11) can be re-written in the form:

$$\mathbb{P}_x(\mathcal{T}_y \leq t) = \Phi_{e_1}(x, y) - \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)]. \tag{14}$$

We thus need to calculate $\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)]$.

Let $I^y := [e_1, y]$ if e_1 is regular instantaneously reflecting or $I^y := (e_1, y]$ otherwise. The operators $(\mathcal{P}^y_t f)(x) := \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}} f(X_t)]$ form a Feller semigroup $\{\mathcal{P}^y_t, t \geq 0\}$ on the Banach space $C_b(I^y)$ of real-valued, bounded continuous functions on I^y and $\Phi_{e_1}(\cdot, y) \in C_b(I^y)$.

Let $L^2(I^y, \mathfrak{m})$ be the Hilbert space of real-valued functions on I^y square-integrable with the speed density $\mathfrak{m}$ and with the inner product

$$(f, g)_y = \int_{I^y} f(x)g(x)\mathfrak{m}(x)dx. \tag{15}$$

The Feller semigroup $\{\mathcal{P}^y_t, t \geq 0\}$ restricted to $C_b(I^y) \cap L^2(I^y, \mathfrak{m})$ extends uniquely to a strongly-continuous semigroup of self-adjoint contractions in $L^2(I^y, \mathfrak{m})$ with the infinitesimal generator $\mathcal{G}^y$, an unbounded self-adjoint, non-positive operator in $L^2(I^y, \mathfrak{m})$ (see McKean 1956; Langer and Schenk 1990; Linetsky 2002b). The domain of $\mathcal{G}^y$ is: $D(\mathcal{G}^y) := \{f \in L^2(I^y, \mathfrak{m}) : f, f' \in AC(I^y), \mathcal{G}^y f \in L^2(I^y, \mathfrak{m}), \text{boundary conditions at } y \text{ and } e_1\}$, where $AC(I^y)$ is the space of absolutely continuous functions, the boundary condition at y is

$$f(y) = 0, \tag{16}$$

and the boundary condition at e_1 is:

– If e_1 is exit or regular killing, then

$$f(e_1+) = 0. \tag{17}$$

- If e_1 is entrance, regular instantaneously reflecting, or natural with $\int_{e_1}^{\epsilon} \mathfrak{m}(x)dx < +\infty$, $\epsilon \in (e_1, y)$, then

$$\lim_{x \downarrow e_1} \frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{18}$$

- If e_1 is natural with $\int_{e_1}^{\epsilon} \mathfrak{m}(x)dx = +\infty$, $\epsilon \in (e_1, y)$, then $\int_{e_1}^{\epsilon} f^2(x)\mathfrak{m}(x)dx < +\infty$ (this implies $\lim_{x \downarrow e_1} f(x) = 0$).

The operator $\mathcal{G}^y$ acts on its domain by $(\mathcal{G}^y f)(x) := \frac{1}{2}a^2(x)f''(x) + b(x)f'(x)$.

Applying the Spectral Theorem for self-adjoint semigroups (Hille and Phillips 1957, Theorem 22.3.1, for $f \in L^2(I^y, \mathfrak{m})$ we thus have a spectral expansion for $\mathcal{P}_t^y f$. To apply the Spectral Theorem to the calculation of $\mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}} \Phi_{e_1}(X_t, y)]$, we need to verify that $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$.

Lemma 1 *If e_1 is not a natural boundary, then $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$. If e_1 is a non-attracting natural boundary, then $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$ if and only if*

$$\int_{e_1}^{y} \mathfrak{m}(x)dx < \infty. \tag{19}$$

If e_1 is an attracting natural boundary, then $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$ if and only if

$$\int_{e_1}^{y} \mathcal{S}^2(e_1, x]\mathfrak{m}(x)dx < \infty. \tag{20}$$

Proof The proof follows from Eq. (12) and the integrability properties recorded in Table 6.2, Karlin and Taylor (1981, p. 234). □

When e_1 is not a natural boundary, $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$, no further conditions are needed to write down the spectral expansion of $(\mathcal{P}_t^y \Phi_{e_1})(\cdot, y)$, and the spectrum of $\mathcal{G}^y$ is simple, non-positive and purely discrete (McKean 1956, Theorem 3.1). When e_1 is natural, the situation is more complicated. When e_1 is non-attracting (attracting) natural, the condition Eqs. (19–20)) must be satisfied to insure $\Phi_{e_1}(\cdot, y) \in L^2(I^y, \mathfrak{m})$. Furthermore, when e_1 is natural there may be some non-empty continuous spectrum.

Linetsky (2002b) classifies all natural boundaries into two further subcategories based on the oscillation of solutions of the *Sturm-Liouville (SL) equation*

$$\mathcal{A}u = \lambda u, \quad \mathcal{A} := -\mathcal{G}. \tag{21}$$

The *Sturm-Liouville operator* $\mathcal{A}$ is the negative of the diffusion infinitesimal generator (8); while the infinitesimal generator $\mathcal{G}$ is non-positive, the SL operator $\mathcal{A}$ is non-negative (see Dunford and Schwartz 1963; Fulton et al. 1996; Linetsky 2002b for details and bibliography on Sturm-Liouville operators).

For a given real λ, Eq. (21) is said to be *oscillatory* at an endpoint $e \in \{e_1, e_2\}$ if and only if every solution has infinitely many zeros clustering at e. Otherwise it is said to be *non-oscillatory* at e. This classification is mutually exclusive for a fixed real λ, but can vary with λ. For Eq. (21), there are two distinct possibilities at

each endpoint. Let $e \in \{e_1, e_2\}$ be an endpoint of Eq. (21). Then e belongs to one and only one of the following two cases:
(i) (NONOSC) Equation (21) is non-oscillatory at e for *all* real λ. Correspondingly, e is called non-oscillatory.
(ii) (O-NO) There exists a real number $\Lambda \geq 0$ such that Eq. (21) is oscillatory at e for all $\lambda > \Lambda$ and non-oscillatory at e for all $\lambda < \Lambda$. Correspondingly, e is called *O-NO with cutoff* Λ. Equation (21) can be either oscillatory or non-oscillatory at e for $\lambda = \Lambda > 0$. It is always non-oscillatory for $\lambda = 0$. Correspondingly, e is said to be NONOSC (O-NO with cutoff Λ) if Eq. (21) is NONOSC (O-NO with cutoff Λ) at e.

Based on the oscillatory/non-oscillatory classification of boundaries, the spectrum of the non-negative SL operator $\mathcal{A}$ associated with the diffusion process X is classified as follows.
(i) *Spectral Category I.* If both endpoints are NONOSC, then the spectrum is simple, non-negative and purely discrete.
(ii) *Spectral Category II.* If one of the endpoints is NONOSC and the other endpoint is O-NO with cutoff $\Lambda \geq 0$, then the spectrum is simple and non-negative, the essential spectrum is nonempty, $\sigma_e(\mathcal{A}) \subset [\Lambda, \infty)$, and Λ is the lowest point of the essential spectrum. Under some technical conditions given in Linetsky (2002b), $\sigma_e(\mathcal{A}) = [\Lambda, \infty)$ and the spectrum above Λ is purely absolutely continuous. If the SL equation is non-oscillatory at the O-NO endpoint for $\lambda = \Lambda \geq 0$, then there is a *finite* set of simple eigenvalues in $[0, \Lambda]$ (it may be empty). If the SL equation is oscillatory at the O-NO endpoint for $\lambda = \Lambda > 0$, then there is an *infinite* sequence of simple eigenvalues in $[0, \Lambda)$ clustering at Λ.
(iii) *Spectral Category III.* If e_1 is O-NO with cutoff $\Lambda_1 \geq 0$ and e_2 is O-NO with cutoff $\Lambda_2 \geq 0$, then the essential spectrum is nonempty, $\sigma_e(\mathcal{A}) \subset [\underline{\Lambda}, \infty)$, $\underline{\Lambda} := \min\{\Lambda_1, \Lambda_2\}$, and $\underline{\Lambda}$ is the lowest point of the essential spectrum. The spectrum is simple (has multiplicity one) below $\overline{\Lambda} := \max\{\Lambda_1, \Lambda_2\}$ and is not simple (has multiplicity two) above $\overline{\Lambda}$. Under some technical conditions given in Linetsky (2002b), $\sigma_e(\mathcal{A}) = [\underline{\Lambda}, \infty)$ and the spectrum above $\underline{\Lambda}$ is purely absolutely continuous. If the SL equation is non-oscillatory for $\lambda = \underline{\Lambda} \geq 0$, then there is a *finite* set of simple eigenvalues in $[0, \underline{\Lambda}]$ (it may be empty). If the SL equation is oscillatory for $\lambda = \underline{\Lambda} > 0$, then there is an *infinite* sequence of simple eigenvalues in $[0, \underline{\Lambda})$ clustering at $\underline{\Lambda}$.

If there are no natural boundaries, the spectrum of the infinitesimal generator is purely discrete. Hence, regular, exit and entrance boundaries are always NONOSC. Natural boundaries can be either NONOSC or O-NO with cutoff $\Lambda \geq 0$. Sufficient conditions for the oscillatory/non-oscillatory classification of natural boundaries can be formulated directly in terms of behavior of the infinitesimal parameters $a(x)$ and $b(x)$ near the boundary. See Linetsky (2002b) where a more general case including a killing rate $r(x) \geq 0$ is considered. Due to space limitations we do not reproduce these results here.

In our application to the first hitting time up, for the SL operator $\mathcal{A}^y := -\mathcal{G}^y$ the right boundary y is regular killing and, hence, NONOSC. The left boundary e_1 can be either NONOSC or O-NO with some cutoff $\Lambda \geq 0$. In the former (latter) case the operator is in the Spectral Category I (II). First, assume e_1 is NONOSC.

Proposition 2 *Suppose e_1 is either regular, entrance, exit, or NONOSC natural boundary with the condition Eq. (19) (Eq. (20)) satisfied if e_1 is non-attracting (attracting). For $\lambda \in \mathbb{C}$ and $x \in I^y$, let $\psi(x,\lambda)$ be the unique (up to a multiple independent of x) non-trivial solution of the SL equation (21) square-integrable with $\mathfrak{m}$ near e_1, satisfying the appropriate boundary condition at e_1 and such that $\psi(x,\lambda)$ and $\psi'(x,\lambda) \equiv \frac{\partial\psi(x,\lambda)}{\partial x}$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each fixed $x \in I^y$. Let $\{\lambda_{n,y}\}_{n=1}^{\infty}$, $0 < \lambda_{1,y} < \lambda_{2,y} < \ldots$, $\lambda_{n,y} \uparrow \infty$ as $n \uparrow \infty$, be the simple positive zeros of $\psi(y,\lambda)$,*

$$\psi(y, \lambda_{n,y}) = 0. \tag{22}$$

Then the spectral expansion of $\mathbb{P}_x(t < \mathcal{T}_y < \infty)$ with $x < y$ and $t > 0$ takes the form ($\psi_\lambda(x,\lambda) \equiv \frac{\partial\psi(x,\lambda)}{\partial\lambda}$):

$$\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)] = -\sum_{n=1}^{\infty} e^{-\lambda_{n,y}t}\frac{\psi(x,\lambda_{n,y})}{\lambda_{n,y}\psi_\lambda(y,\lambda_{n,y})}. \tag{23}$$

Proof We need to solve the SL problem $\mathcal{A}^y u = \lambda u$ with the boundary condition $u(y) = 0$ at y and the appropriate boundary condition at e_1. Since both boundaries are NONOSC and y is killing, the SL problem has a simple, purely discrete and positive spectrum $\{\lambda_{n,y}\}_{n=1}^{\infty}$, $0 < \lambda_{1,y} < \lambda_{2,y} < \ldots$, $\lim_{n\uparrow\infty}\lambda_{n,y} = \infty$. Let $\lambda_{n,y}$ and $\varphi_{n,y}(x)$ be the eigenvalues and the corresponding normalized eigenfunctions, $\|\varphi_{n,y}\|_y^2 = 1$ ($\|\varphi_{n,y}\|_y^2 \equiv (\varphi_{n,y},\varphi_{n,y})_y$). Then, from the Spectral Theorem for self-adjoint semigroups in Hilbert space (see Linetsky 2002b for details), we have

$$\mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)] = \sum_{n=1}^{\infty} c_{n,y}e^{-\lambda_{n,y}t}\varphi_{n,y}(x), \quad c_{n,y} = (\Phi_{e_1}(\cdot,y),\varphi_{n,y})_y. \tag{24}$$

The expansion coefficients are calculated as follows. Since $\varphi_{n,y}(x)$ is the eigenfunction of $\mathcal{A}^y$ with the eigenvalue $\lambda_{n,y}$, we have:

$$\lambda_{n,y}c_{n,y} = \int_{e_1}^{y}\Phi_{e_1}(x,y)(\mathcal{A}^y\varphi_{n,y})(x)\mathfrak{m}(x)dx = -\int_{e_1}^{y}\Phi_{e_1}(x,y)\left(\frac{\varphi'_{n,y}(x)}{\mathfrak{s}(x)}\right)' dx$$

$$= -\frac{\varphi'_{n,y}(y)}{\mathfrak{s}(y)}, \quad \text{where} \quad \varphi'_{n,y}(y) = \left.\frac{d\varphi_{n,y}(x)}{dx}\right|_{x\uparrow y}, \tag{25}$$

where we integrated by parts, used Eq. (12), the boundary conditions at y and e_1 and, when e_1 is a natural boundary, the conditions in Lemma 1. Thus,

$$c_{n,y} = -\frac{\varphi'_{n,y}(y)}{\lambda_{n,y}\mathfrak{s}(y)}. \tag{26}$$

The solution $\psi(x,\lambda)$ with the required properties exists by Lemma 1 in Linetsky (2002b). For $\lambda \in \mathbb{C}$ and $x \in I^y$, let $\phi(x,\lambda)$ be the unique solution of the SL Eq. (21) with the initial conditions at y:

$$\phi(y,\lambda) = 0, \quad \phi'(y,\lambda) = -1. \tag{27}$$

Both $\phi(x,\lambda)$ and $\phi'(x,\lambda) \equiv \frac{\partial \phi(x,\lambda)}{\partial x}$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each fixed $x \in I^y$ (Lemma 1 in Linetsky 2002b). Since $\psi(x,\lambda)$ and $\phi(x,\lambda)$ are solutions of the SL equation and $\phi(x,\lambda)$ satisfies the initial Conditions (27), the Wronskian of $\psi(x,\lambda)$ and $\phi(x,\lambda)$ is independent of x and:

$$\psi(x,\lambda)\frac{\phi'(x,\lambda)}{\mathfrak{s}(x)} - \phi(x,\lambda)\frac{\psi'(x,\lambda)}{\mathfrak{s}(x)} = -\frac{\psi(y,\lambda)}{\mathfrak{s}(y)} =: w(\lambda). \tag{28}$$

The eigenfunction $\varphi_{n,y}(x)$ is square-integrable with $\mathfrak{m}$ in a neighborhood of e_1 and satisfies the appropriate boundary condition at e_1, hence it must be equal to $\psi(x,\lambda_{n,y})$ up to a non-zero constant multiple. But $\varphi_{n,y}(x)$ also satisfies the boundary condition at y, hence it must also be equal to $\phi(x,\lambda_{n,y})$ up to a non-zero constant multiple. Thus, for $\lambda = \lambda_{n,y}$ $\psi(x,\lambda_{n,y})$ and $\phi(x,\lambda_{n,y})$ are linearly dependent:

$$\phi(x,\lambda_{n,y}) = A_{n,y}\psi(x,\lambda_{n,y}), \quad A_{n,y} = -\frac{1}{\psi'(y,\lambda_{n,y})}, \tag{29}$$

and, hence, their Wronskian must vanish for $\lambda = \lambda_{n,y}$. Thus, from Eq. (28), the eigenvalues are zeros of $\psi(y,\lambda)$. Conversely, let $\lambda_{n,y}$ be a zero of $\psi(y,\lambda)$. Then $\psi(x,\lambda_{n,y})$ and $\phi(x,\lambda_{n,y})$ are linearly dependent and, hence, $\psi(x,\lambda_{n,y})$ is a solution of the SL equation that is square-integrable with $\mathfrak{m}$ on I^y and satisfies the required boundary conditions at e_1 and y, i.e., $\psi(x,\lambda_{n,y})$ is a (non-normalized) eigenfunction corresponding to the eigenvalue $\lambda_{n,y}$. Finally, the normalized eigenfunctions can be taken in the form (Lemma 2 in Linetsky 2002b; $w_\lambda(\lambda_{n,y}) \equiv \left.\frac{\partial w(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_{n,y}}$):

$$\varphi_{n,y}(x) = \frac{\psi(x,\lambda_{n,y})}{\|\psi(\cdot,\lambda_{n,y})\|_y} = \frac{\phi(x,\lambda_{n,y})}{\|\phi(\cdot,\lambda_{n,y})\|_y}, \tag{30}$$

$$\|\psi(\cdot,\lambda_{n,y})\|_y^2 = \frac{w_\lambda(\lambda_{n,y})}{A_{n,y}} = \frac{1}{\mathfrak{s}(y)}\psi'(y,\lambda_{n,y})\psi_\lambda(y,\lambda_{n,y}), \tag{31}$$

$$\|\phi(\cdot,\lambda_{n,y})\|_y^2 = w_\lambda(\lambda_{n,y})A_{n,y} = \frac{1}{\mathfrak{s}(y)}\frac{\psi_\lambda(y,\lambda_{n,y})}{\psi'(y,\lambda_{n,y})}. \tag{32}$$

Substituting this and (26) into (24), we arrive at the eigenfunction expansion (23). $\square$

Now assume e_1 is O-NO natural with some cutoff $\Lambda \geq 0$. We are in the Spectral Category II with some non-empty continuous spectrum.

Proposition 3 *Suppose e_1 is an O-NO with cutoff $\Lambda \geq 0$ natural boundary with the condition Eq. (19) (Eq. (20)) satisfied if e_1 is non-attracting (attracting). For $\lambda \in \mathbb{C}$ and $x \in (e_1, y]$, let $\phi(x,\lambda)$ be the unique solution of the SL equation (21) with the* *initial conditions (27) at y. Then the spectral expansion of* $\mathbb{P}_x(t < \mathcal{T}_y < \infty)$ *with* $x < y$ *and* $t > 0$ *takes the form:*

$$\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi_{e_1}(X_t, y)] = \int_0^\infty e^{-\lambda t}\frac{\phi(x,\lambda)}{\lambda\mathfrak{s}(y)}d\rho^y(\lambda), \quad (33)$$

where $\rho^y(\lambda)$ *is the non-decreasing, right-continuous* spectral function *of the singular SL problem on* $(e_1, y]$ *with the Dirichlet boundary condition at* y *and normalized relative to the solution* $\phi(x, \lambda)$.

Proof We follow McKean (1956) (see also Sect. 2 in Linetsky 2002b). Pick some $l \in (e_1, y)$ and kill the process at $\mathcal{T}_l$, the first hitting time of l. Let $\mathcal{T}_y^{(l)} := \inf\{t \geq 0 : X_t^{(l)} = y\}$ be the first hitting time of y for the process $X^{(l)}$ killed at l. The left endpoint l is regular killing and Proposition 2 can be applied to write (use (23), (29) and (32); $(f,g)_{l,y} := \int_l^y f(x)g(x)\mathfrak{m}(x)dx$, $\|f\|_{l,y}^2 \equiv (f,f)_{l,y}$, $\Phi_l(x,y) = \frac{\mathcal{S}(l,x]}{\mathcal{S}(l,y]}$)

$$\mathbb{P}_x(t<\mathcal{T}_y^{(l)}<\infty)=\mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y^{(l)}\}}\Phi_l(X_t^{(l)},y)]=-\sum_{n=1}^\infty e^{-\lambda_{n,y}^{(l)}t}\frac{\psi^{(l)}(x,\lambda_{n,y}^{(l)})}{\lambda_{n,y}^{(l)}\psi_\lambda^{(l)}(y,\lambda_{n,y}^{(l)})}$$

$$=\sum_{n=1}^\infty e^{-\lambda_{n,y}^{(l)}t}\frac{\phi(x,\lambda_{n,y}^{(l)})}{\lambda_{n,y}^{(l)}\mathfrak{s}(y)}\frac{1}{\|\phi(\cdot,\lambda_{n,y}^{(l)})\|_{l,y}^2}, \quad (34)$$

where $\psi^{(l)}(x,\lambda)$ is the solution entire in $\lambda \in \mathbb{C}$ with the boundary condition $\psi^{(l)}(l,\lambda) = 0$, and $\lambda_{n,y}^{(l)}$ are zeros of $\psi^{(l)}(y,\lambda)$ ($\phi(x,\lambda)$ is the same as for the problem on the interval $(e_1, y]$ and is independent of l since the initial condition (27) is applied at y).

Introduce a nondecreasing right-continuous jump function (the spectral function of the problem on $[l, y]$):

$$\rho_l^y(\lambda) := \sum_{n=1}^\infty \frac{1}{\|\phi(\cdot,\lambda_{n,y}^{(l)})\|_{l,y}^2}\mathbf{1}_{\{\lambda_{n,y}^{(l)}\leq\lambda\}}, \quad (35)$$

where $\mathbf{1}_{\{\lambda_{n,y}^{(l)}\leq\lambda\}} = 1\,(0)$ if $\lambda_{n,y}^{(l)} \leq \lambda$ ($\lambda_{n,y}^{(l)} > \lambda$). It jumps by $\frac{1}{\|\phi(\cdot,\lambda_{n,y}^{(l)})\|_{l,y}^2}$ at an eigenvalue $\lambda = \lambda_{n,y}^{(l)}$. The spectral expansion (34) can now be re-written in the integral form:

$$\mathbb{P}_x^{(l)}(t < \mathcal{T}_y^{(l)} < \infty) = \int_{[0,\infty)} e^{-\lambda t}\frac{\phi(x,\lambda)}{\lambda\mathfrak{s}(y)}d\rho_l^y(\lambda). \quad (36)$$

The limit $\lim_{l\downarrow e_1}\rho_l^y(\lambda) = \rho^y(\lambda)$ produces a nondecreasing right-continuous function, the spectral function of the original problem on $(e_1, y]$. In the limit $l \downarrow e_1$ (36) converges to (33). □

Remark 1 When e_1 is O-NO natural, the spectral expansion involves an integral. Generally this integral has to be computed numerically. The approach in the proof of Proposition 3 provides a practical alternative to numerical integration. Select l close enough to e_1 and compute the series (34) as an approximation to the integral (33). For large n the eigenvalues $\lambda_{n,y}^{(l)}$ grow as n^2:

$$\lambda_{n,y}^{(l)} \sim \frac{n^2\pi^2}{2B^2} + O(1), \quad \text{where } B = \int_l^y \frac{dx}{a(x)}. \tag{37}$$

Remark 2 When a regular diffusion is described by three Borel measures, the speed measure, natural scale, and killing measure that do not have to be absolutely continuous (Ito and McKean 1974) and e_1 is *not* a natural boundary, Proposition 2 follows from Theorem 6.1 and Remark 1 in Kent (1980). The case when e_1 is a natural boundary is considered in Kent (1982) without requiring absolute continuity of the speed, scale and killing measures. Theorem 5.1 and Remark 1 in Kent (1982) give a spectral expansion in terms of the spectral measure associated with the diffusion infinitesimal generator, a *generalized* second-order differential operator. However, no detailed information about the qualitative nature of the spectrum is available in this general context. By restricting to the absolutely continuous case important in financial applications and appealing to the Sturm-Liouville theory of *classical* differential operators to sub-classify natural boundaries into non-oscillatory and oscillatory (NONOSC and O-NO with cutoff $\Lambda \geq 0$), we are able to give a complete characterization of the spectral expansion when e_1 is a natural boundary. In particular, the essential spectrum is fully characterized (see Linetsky 2002b for further details).

Remark 3 First hitting time down $\mathcal{T}_y$ with $y < x < e_2$ is treated similarly. The function $\Phi^{e_2}(x,y)$ is defined by:

$$\Phi^{e_2}(x,y) := \begin{cases} 1, & \text{if } e_2 \text{ is entrance, reflecting or non-attracting natural} \\ \frac{\mathcal{S}[x,e_2)}{\mathcal{S}[y,e_2)}, & \text{if } e_2 \text{ is exit, killing or attracting natural} \end{cases} \tag{38}$$

The solution $\phi(x,\lambda)$ satisfies the appropriate boundary condition at e_2. The solution $\psi(x,\lambda)$ satisfies the initial conditions $\psi(y,\lambda)=0,\ \ \psi'(y,\lambda)=1$ at y (consistent with the notation in Linetsky (2002b), $\psi(x,\lambda)$ ($\phi(x,\lambda)$) is the solution satisfying the boundary condition at the left (right) boundary and entire in $\lambda \in \mathbb{C}$ for each fixed x). The results for the first hitting time down read:

$$\mathbb{P}_x(\mathcal{T}_y \leq t) = \Phi^{e_2}(x,y) - \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi^{e_2}(X_t,y)]. \tag{39}$$

If e_2 is NONOSC ($\lambda_{n,y}$ are zeros of $\phi(y,\lambda)$):

$$\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi^{e_2}(X_t,y)] = -\sum_{n=1}^{\infty} e^{-\lambda_{n,y}t}\frac{\phi(x,\lambda_{n,y})}{\lambda_{n,y}\phi_\lambda(y,\lambda_{n,y})}. \tag{40}$$

If e_2 is O-NO with cutoff $\Lambda \geq 0$ ($\rho_y(\lambda)$ is the spectral function for the problem on $[y,e_2)$):

$$\mathbb{P}_x(t < \mathcal{T}_y < \infty) = \mathbb{E}_x[\mathbf{1}_{\{t<\mathcal{T}_y\}}\Phi^{e_2}(X_t,y)] = \int_0^\infty e^{-\lambda t}\frac{\psi(x,\lambda)}{\lambda\mathfrak{s}(y)}d\rho_y(\lambda). \tag{41}$$

4 Pricing lookbacks in the CEV model

4.1 The CEV process

We now return to the set-up of Sect. 2 and specialize our discussion to the *constant elasticity of variance* (CEV) process of Cox (1975) (see also Beckers 1980; Schroder 1989; Delbaen and Shirakawa 1996; Andersen and Andreasen 2000; Davydov and Linetsky 2001, 2003). We assume that the asset price process $\{S_t, t \geq 0\}$ is a CEV diffusion on $(0, \infty)$ with the infinitesimal generator (1) with the local volatility function

$$\sigma(x) = \delta x^{\beta}, \;\; \delta > 0, \;\; \beta \leq 0. \tag{42}$$

The CEV specification nests the Black-Scholes-Merton geometric Brownian motion model ($\beta = 0$) and the absolute diffusion ($\beta = -1$) and square-root ($\beta = -1/2$) models of Cox and Ross (1976) as particular cases. For $\beta < 0$, the local volatility $\sigma(x) = \delta x^{\beta}$ is a decreasing function of the asset price. We have two model parameters β and δ; β is the elasticity of the local volatility function, $\beta = x\sigma'/\sigma$, and δ is the scale parameter. Cox originally restricted β to the range $-1 \leq \beta \leq 0$. However, Jackwerth and Rubinstein (1998) empirically find that typical values of β implicit in the post-crash of 1987 S&P 500 stock index option prices are as low as $\beta = -4$. They call the model with $\beta < -1$ *unrestricted CEV*. For $\beta < 0$, $+\infty$ is a natural boundary; attracting for $\mu > 0$ and non-attracting for $\mu \leq 0$. For $-1/2 \leq \beta < 0$, the origin is an exit boundary. For $-\infty < \beta < -1/2$, the origin is a regular boundary and is specified as killing. In this paper we focus on the CEV process with $\beta < 0$ and $\mu \geq 0$. This process is used to model the *volatility smile/skew* phenomenon observed in equity options.

Let $\{S_t, t \geq 0\}$ be the CEV diffusion with $\beta < 0$ and $\mu \geq 0$. Define a new process $\{Y_t, t \geq 0\}$ by: $Y_t := \frac{1}{\delta^2\beta^2} S_t^{-2\beta}$ for $0 \leq t < \mathcal{T}_0$ and $Y_t := \partial$ for $t \geq \mathcal{T}_0$ ($\mathcal{T}_0 := \inf\{t \geq 0 : S_t = 0\}$). Then $\{Y_t, t \geq 0\}$ is a *square-root diffusion* on $(0, \infty)$ with the infinitesimal generator

$$(\mathcal{G}f)(x) = 2xf''(x) + (ax + b)f'(x), \;\; a = 2\mu|\beta|, \;\; b = 2 + \frac{1}{\beta}. \tag{43}$$

The origin is exit for $-\infty < b \leq 0$ and regular killing for $0 < b < 2$. Further, take the square root: $R_t := \sqrt{Y_t} = \frac{1}{\delta|\beta|} S_t^{-\beta}$ for $0 \leq t < \mathcal{T}_0$ and $R_t := \partial$ for $t \geq \mathcal{T}_0$. Then $\{R_t^{(\nu,c)}, t \geq 0\}$ is a diffusion on $(0, \infty)$ with the infinitesimal generator

$$(\mathcal{G}f)(x) = \frac{1}{2}f''(x) + \left(\frac{\nu + 1/2}{x} + cx\right) f'(x), \;\; \nu = \frac{1}{2\beta}, \;\; c = \mu|\beta|. \tag{44}$$

The origin is exit for $-\infty < \nu \leq -1$ and regular killing for $-1 < \nu < 0$. For $c = 0$, $R^{(\nu,0)}$ is a standard Bessel process of order $\nu < 0$ killed at the origin (see Kent 1978; Borodin and Salminen 1996; Revuz and Yor 1999 for details on Bessel processes, and Delbaen and Shirakawa 1995 for a discussion of CEV processes with $-1 < \beta < 0$ in terms of Bessel processes). Processes with the generator (44) and $\nu > -1$ (for $-1 < \nu < 0$ the origin is regular instantaneously reflecting and for $\nu \geq 0$ the origin is entrance) are known as *radial Ornstein-Uhlenbeck processes* (Shiga and Watanabe 1973; Pitman and Yor 1982; Göing-Jaeschke and Yor 2003b). Here we consider the process with $\nu < 0$ and $c > 0$ killed at the origin. We call the processes with generator (44) a *generalized Bessel processes* (it has an additional linear term cx in the drift in addition to the standard Bessel drift $(\nu + 1/2)/x$; for Bessel processes with drift $(\nu + 1/2)/x + c$ with additional constant term see Linetsky 2004). [Remark: For $\beta < 0$ the process is killed at zero. If this feature is undesirable, an approach in Andersen and Andreasen (2000) can be used to make the origin unattainable by modifying the volatility as follows. Pick a small $\epsilon > 0$ and modify the volatility function (42) as follows: $\sigma_\epsilon(x) = \delta \min\{x^\beta, \epsilon^\beta\}$. The regularized volatility function is bounded, and the origin is a natural boundary for the regularized process. Andersen and Andreasen (2000) call the regularized process *limited CEV* (*LCEV*). The original CEV process is recovered in the limit $\epsilon \to 0$.]

4.2 Positive drift

In this section we assume $\mu > 0$. The problem of pricing lookback options under the CEV process with $\beta < 0$ and $\mu > 0$ reduces to the problem of calculating the first hitting time distribution for $R^{(\nu,c)}$ with $\nu < 0$ and $c > 0$. The scale and speed densities are:

$$\mathfrak{s}(x) = x^{-2\nu-1}e^{-cx^2}, \quad \mathfrak{m}(x) = 2x^{2\nu+1}e^{cx^2}. \tag{45}$$

First hitting time up. First consider the case $0 < x < y < \infty$. The function $\Phi_0(x, y)$ (12) takes the form:

$$\Phi_0(x, y) = \frac{\gamma(-\nu, cx^2)}{\gamma(-\nu, cy^2)}, \tag{46}$$

where $\gamma(a, x) = \int_0^x z^{a-1}e^{-z}dz$ is the incomplete gamma function.

Notice that the substitution

$$u(x) = x^{-\nu-1}e^{-\frac{cx^2}{2}}w(cx^2) \tag{47}$$

reduces the equation

$$-\frac{1}{2}\frac{d^2u}{dx^2} - \left(\frac{\nu + 1/2}{x} + cx\right)\frac{du}{dx} = \lambda u \tag{48}$$

to the Whittaker's form of the confluent hypergeometric equation (see Abramowitz and Stegun 1972, p. 505; Slater 1960, p. 9; Buchholz 1969, p. 11):

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right)w = 0, \tag{49}$$

$$\text{where} \quad m = -\frac{\nu}{2}, \quad k = \frac{\lambda}{2c} - \frac{\nu + 1}{2}, \quad z = cx^2. \tag{50}$$

The two Whittaker functions $M_{k,m}(z)$ and $W_{k,m}(z)$ are solutions of Eq. (49) and have the following asymptotic properties (Eq. (53) is valid for $m > 0$; $\Gamma(x)$ is the standard gamma function):

$$M_{k,m}(z) \sim z^{m+\frac{1}{2}} e^{-\frac{z}{2}} \quad \text{as} \quad z \to 0, \tag{51}$$

$$M_{k,m}(z) \sim \frac{\Gamma(2m+1)}{\Gamma\left(m-k+\frac{1}{2}\right)} z^{-k} e^{\frac{z}{2}} \quad \text{as} \quad z \to \infty, \tag{52}$$

$$W_{k,m}(z) \sim \frac{\Gamma(2m)}{\Gamma\left(m+k+\frac{1}{2}\right)} z^{-m+\frac{1}{2}} e^{-\frac{z}{2}} \quad \text{as} \quad z \to 0, \tag{53}$$

$$W_{k,m}(z) \sim z^{k} e^{-\frac{z}{2}} \quad \text{as} \quad z \to \infty. \tag{54}$$

From Eq. (51), for $\nu < 0$ the solution

$$\psi(x,\lambda) = x^{-\nu-1} e^{-\frac{cx^2}{2}} M_{\frac{\lambda}{2c}-\frac{\nu+1}{2},-\frac{\nu}{2}}(cx^2) \tag{55}$$

vanishes as $x \downarrow 0$, is square-integrable with $\mathfrak{m}$ near the origin, and is entire in λ since $M_{k,m}(z)$ is entire in k. The Whittaker function considered as a function of complex variable k, keeping $m > -1/2$ (in our case $m > 0$) and $z > 0$ fixed, has all its zeros concentrated along the positive real line. Moreover, all zeros are simple, occur in an infinite set $0 < k_1 < k_2 < \ldots, k_n \uparrow \infty$ as $n \uparrow \infty$, and are decreasing as the value of z increases (Buchholz 1969, pp. 185–186). Let $\{k_{n,m}(z)\}_{n=1}^{\infty}$ be the positive zeros of $M_{k,m}(z)$. From Eq. (50) the eigenvalues $\{\lambda_{n,y}\}_{n=1}^{\infty}$ are:

$$\lambda_{n,y} = 2ck_{n,-\frac{\nu}{2}}(cy^2) + c(\nu+1). \tag{56}$$

In general, to determine the zeros $\{k_{n,m}(z)\}_{n=1}^{\infty}$, the equation $M_{k,m}(z) = 0$ has to be solved numerically. However, a useful estimate can be obtained as follows. Slater (1960, p. 70) gives the following asymptotics as $k \to \infty$ (valid for complex k and z such that $\arg(kz) < 2\pi$):

$$M_{k,m}(z)=\Gamma(1+2m)z^{\frac{1}{4}}\pi^{-\frac{1}{2}}k^{-m-\frac{1}{4}}\cos\left(2\sqrt{kz}-\pi m-\frac{\pi}{4}\right)\left\{1+O(|k|^{-\frac{1}{2}})\right\}. \tag{57}$$

This gives a large-n estimate of the Whittaker function zeros:

$$k_{n,m}(z) \sim \frac{(n+m-1/4)^2\pi^2}{4z}. \tag{58}$$

From Eq. (56) we have the large-n eigenvalue asymptotics:

$$\lambda_{n,y} \sim \frac{(n-\frac{\nu}{2}-\frac{1}{4})^2\pi^2}{2y^2} + c(\nu+1). \tag{59}$$

Putting everything together, we arrive at the result for the generalized Bessel process with $\nu < 0$, $c > 0$ and with killing at 0 ($0 < x < y < \infty$):

$$\mathbb{P}_x(\mathcal{T}_y^R \le t) = \frac{\gamma(-\nu, cx^2)}{\gamma(-\nu, cy^2)} - \sum_{n=1}^{\infty} \frac{2c}{\lambda_{n,y}} e^{-\lambda_{n,y} t} \left(\frac{x}{y}\right)^{-\nu-1} e^{\frac{c}{2}(y^2-x^2)} \frac{M_{k_{n,y}, -\frac{\nu}{2}}(cx^2)}{\Delta_{n,y}^M}, \tag{60}$$

$$\text{where} \quad \Delta_{n,y}^M := - \left.\frac{\partial M_{k,-\frac{\nu}{2}}(cy^2)}{\partial k}\right|_{k=k_{n,y}} \quad \text{and} \quad k_{n,y} := k_{n,-\frac{\nu}{2}}(cy^2). \tag{61}$$

To calculate $\Delta_{n,y}^M$ we need the derivative of the Whittaker function $M_{k,m}(z)$ with respect to its first index k. Recall that $M_{k,m}(z)$ is related to the Kummer confluent hypergeometric function $M(a,b,z)$ as follows (Abramowitz and Stegun 1972, p. 505; this function is also often denoted ${}_1F_1(a;b;z)$ and is available in the *Mathematica* software package with the call Hypergeometric1F1[a,b,z]):

$$M_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} e^{-\frac{z}{2}} M(1/2 + \mu - \kappa, 1 + 2\mu, z).$$

The derivative of $M(a,b,z)$ with respect to its first index is:

$$\frac{\partial}{\partial a}\{M(a,b,z)\} = \sum_{k=0}^{\infty} \frac{(a)_k \psi(a+k)}{(b)_k\, k!} z^k - \psi(a) M(a,b,z),$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{k=0}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma$$

is the digamma function (Abramowitz and Stegun 1972). This derivative is available in *Mathematica* with the call Hypergeometric1F1$^{(1,0,0)}$[a,b,z].

To compute the first hitting time distribution for the CEV diffusion, we note that:

$$\mathbb{P}_{S_0}(\mathcal{T}_Y^{CEV} \le t) = \mathbb{P}_x(\mathcal{T}_y^R \le t), \tag{62}$$

where $\mathcal{T}_Y^{CEV}$ is the first time the CEV process $\{S_t, t \ge 0\}$ hits the level Y starting from S_0, $\mathcal{T}_y^R$ is the first time the generalized Bessel process $\{R_t^{(\nu,c)}, t \ge 0\}$ hits the level y starting from x, and

$$x = \frac{1}{\delta|\beta|} S_0^{-\beta}, \quad y = \frac{1}{\delta|\beta|} Y^{-\beta}. \tag{63}$$

The distribution of the maximum for the CEV diffusion is given by Eq. (7), and the standard lookback put and the call on the maximum pricing formulas (4) and (5) are computed by performing numerical integration in Y (see Sect. 5 below).

First hitting time down. To calculate the distribution of the minimum, we need to consider the case of hitting y from above, $\mathbb{P}_x(\mathcal{T}_y^R \le t)$ for $0 < y < x < \infty$. For $c > 0$ infinity is attracting natural. The function $\Phi^\infty(x,y)$ (38) is given by:

$$\Phi^\infty(x,y) = \frac{\Gamma(-\nu, cx^2)}{\Gamma(-\nu, cy^2)}, \tag{64}$$

where $\Gamma(a,x) = \int_x^\infty z^{a-1}e^{-z}dz$ is the complementary incomplete gamma function. Since $\Phi^\infty(x,y) \sim \frac{(cx^2)^{-\nu-1}e^{-cx^2}}{\Gamma(-\nu,cy^2)}$ as $x \uparrow \infty$, $\Phi^\infty(x,y)$ is square-integrable with $\mathfrak{m}$ on (y,∞). From Eq. (54), the solution

$$\phi(x,\lambda) = x^{-\nu-1}e^{-\frac{cx^2}{2}}W_{\frac{\lambda}{2c}-\frac{\nu+1}{2},-\frac{\nu}{2}}(cx^2) \tag{65}$$

has the asymptotics $\phi(x,\lambda) \sim e^{-cx^2}(cx^2)^{\frac{\lambda}{2c}-\frac{\nu+1}{2}}x^{-\nu-1}$ as $x \uparrow \infty$. Thus, it is square-integrable with $\mathfrak{m}$ on $[y,\infty)$, infinity is a non-oscillatory natural boundary for all real λ, and $\phi(x,\lambda)$ is entire in λ ($W_{k,m}(z)$ is entire in k). Let $\{k_{n,m}(z)\}_{n=1}^\infty$ be the positive zeros of $W_{k,m}(z)$, keeping $m > 0$ and $z > 0$ fixed. Then the eigenvalues $\{\lambda_{n,y}\}_{n=1}^\infty$ are given by Eq. (56). In general, to determine the zeros $\{k_{n,m}(z)\}_{n=1}^\infty$, the equation $W_{k,m}(z) = 0$ has to be solved numerically. However, a useful estimate can be obtained as follows. Slater (1960, p. 70) gives the following asymptotics for the Whittaker function $W_{k,m}(z)$ as $k \to \infty$ (valid for complex k and z such that $|\arg(k)| < \pi$ and $|\arg(kz)| < 2\pi$):

$$W_{k,m}(z) = \sqrt{2}z^{\frac{1}{4}}k^{-\frac{1}{4}}k^k e^{-k}\cos\left(2\sqrt{kz} - \pi k + \frac{\pi}{4}\right)\left\{1 + O(|k|^{-\frac{1}{2}})\right\}. \tag{66}$$

This gives a large-n estimate of the Whittaker function zeros:

$$k_{n,m}(z) \sim n - \frac{1}{4} + \frac{2z}{\pi^2} + \frac{2}{\pi}\sqrt{\left(n-\frac{1}{4}\right)z + \frac{z^2}{\pi^2}} \tag{67}$$

and a large-n estimate of the eigenvalues:

$$\lambda_{n,y} \sim 2c\left(n + \frac{2cy^2}{\pi^2} + \frac{1}{4} + \frac{\nu}{2} + \frac{2}{\pi}\sqrt{\left(n-\frac{1}{4}\right)cy^2 + \frac{c^2y^4}{\pi^2}}\right). \tag{68}$$

Putting everything together, we arrive at the result for the generalized Bessel process with $\nu < 0$, $c > 0$ in the form ($0 < y < x < \infty$):

$$\mathbb{P}_x(\mathcal{T}_y^R \le t) = \frac{\Gamma(-\nu, cx^2)}{\Gamma(-\nu, cy^2)} - \sum_{n=1}^\infty \frac{2c}{\lambda_{n,y}}e^{-\lambda_{n,y}t}\left(\frac{x}{y}\right)^{-\nu-1} e^{\frac{c}{2}(y^2-x^2)}\frac{W_{k_{n,y},-\frac{\nu}{2}}(cx^2)}{\Delta_{n,y}^W}, \tag{69}$$

$$\text{where } \Delta_{n,y}^W := -\left.\frac{\partial W_{k,-\frac{\nu}{2}}(cy^2)}{\partial k}\right|_{k=k_{n,y}} \text{ and } k_{n,y} := k_{n,-\frac{\nu}{2}}(cy^2). \tag{70}$$

To calculate $\Delta^W_{n,y}$ we need the derivative of the Whittaker function $W_{k,m}(z)$ with respect to its first index k. Recall that $W_{k,m}(z)$ is related to the Tricomi confluent hypergeometric function $U(a,b,z)$ as follows (Abramowitz and Stegun 1972, p. 505; this function is available in the *Mathematica* software package with the call HypergeometricU[a,b,z]):

$$W_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} e^{-\frac{z}{2}} U(1/2+\mu-\kappa, 1+2\mu, z).$$

The derivative of $U(a,b,z)$ with respect to its first index is:

$$\frac{\partial}{\partial a}\{U(a;b;z)\} = \frac{\Gamma(b-1)z^{1-b}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\psi(a-b+k+1)(a-b+1)_k z^k}{k!(2-b)_k}$$
$$+\frac{\Gamma(1-b)}{\Gamma(a-b+1)} \sum_{k=0}^{\infty} \frac{\psi(a+k)(a)_k z^k}{k!(b)_k} - \{\psi(a)+\psi(a-b+1)\}\, U(a,b,z).$$

This derivative is available in *Mathematica* with the call HypergeometricU$^{(1,0,0)}$[a,b,z].

The distribution of the minimum for the CEV diffusion is recovered by using Eqs. (7), (62)–(63). The standard lookback call and the put on the minimum pricing formulas (3) and (6) are computed by performing numerical integration with respect to Y (see Sect. 5 below). The difference with the case $x < y$ is that the eigenvalues (68) grow linearly with n, in contrast with the quadratic growth for the former case in Eq. (59). Thus, convergence of the spectral expansion is much slower in this case.

4.3 Zero Drift

In this section we study the zero-drift case $\mu = 0$ used to model futures ($r = q$). The problem of pricing lookback options under the CEV process with $\beta < 0$ and $\mu = 0$ reduces to the problem of calculating the first hitting time distribution for the Bessel process $\{R_t^{(\nu)}, t \geq 0\}$ with $\nu < 0$ and killed at 0. The scale and speed densities are as in Eq. (45) with $c = 0$.

First hitting time up. First consider the case $0 < x < y < \infty$. The function $\Phi_0(x,y)$ (12) takes the form:

$$\Phi_0(x,y) = \left(\frac{x}{y}\right)^{-2\nu}. \tag{71}$$

Notice that the substitution $u(x) = x^{-\nu} w(x\sqrt{2\lambda})$ reduces the equation

$$-\frac{1}{2}\frac{d^2u}{dx^2} - \frac{(\nu+1/2)}{x}\frac{du}{dx} = \lambda u \tag{72}$$

to the Bessel equation ($z = x\sqrt{2\lambda}$)

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\nu^2)w = 0. \tag{73}$$

From the asymptotics $J_\alpha(z) \sim \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}$ as $z \to 0$, the solution that vanishes at the origin and is square-integrable with $\mathfrak{m}(x) = 2x^{2\nu+1}$ near the origin is:

$$\psi(x,\lambda) = x^{-\nu} J_{-\nu}(x\sqrt{2\lambda}), \tag{74}$$

where $J_\alpha(z)$ is the Bessel function of the first kind. Let $\{j_{\alpha,n}\}_{n=1}^\infty$ be the positive zeros of the Bessel function $J_\alpha(z)$ (see Abramowitz and Stegun 1972, p. 370). Then the eigenvalues $\{\lambda_{n,y}\}_{n=1}^\infty$ are given by:

$$\lambda_{n,y} = \frac{j_{-\nu,n}^2}{2y^2}. \tag{75}$$

Using $J'_\alpha(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z)$, $\psi_\lambda(y,\lambda_{n,y})$ in Eq. (23) is given by: $\psi_\lambda(y,\lambda_{n,y}) = -\frac{y^{-\nu+2}}{j_{-\nu,n}} J_{-\nu+1}(j_{-\nu,n})$. Putting everything together, we arrive at the result for the Bessel process with $\nu < 0$ and killing boundary at 0 in the form of a Fourier-Bessel series ($0 < x < y < \infty$; the corresponding result for Bessel process with $\nu > 0$ is well known, e.g., Borodin and Salminen 1996, p. 369, Eq. (1.1.4)):

$$\mathbb{P}_x(\mathcal{T}_y^R \le t) = \left(\frac{x}{y}\right)^{-2\nu} - 2\left(\frac{x}{y}\right)^{-\nu} \sum_{n=1}^\infty \exp\left\{-\frac{j_{-\nu,n}^2 t}{2y^2}\right\} \frac{J_{-\nu}\left(\frac{x}{y} j_{-\nu,n}\right)}{j_{-\nu,n} J_{-\nu+1}(j_{-\nu,n})}. \tag{76}$$

When $\beta = -1$ (and, hence, $\nu = -1/2$), the process reduces to the Brownian motion killed at zero. Recalling that $J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z$, $j_{1/2,n} = n\pi$, and $J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z\right)$, the Fourier-Bessel series (76) reduces to the classical Fourier series for the first hitting time of y of Brownian motion killed at zero and starting at x, $0 < x < y$:

$$\mathbb{P}_x(\mathcal{T}_y \le t) = \frac{x}{y} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n} \exp\left\{-\frac{\pi^2 n^2 t}{2y^2}\right\} \sin\left(\frac{x\pi n}{y}\right). \tag{77}$$

First hitting time down. Infinity is non-attracting natural and $\Phi^\infty(x,y) = 1$. The speed density is integrable near infinity if and only if $\nu < -1$. Thus, when $0 < y < x < \infty$ the spectral expansion for the hitting time is valid only for $\nu < -1$. Using the Bessel function asymptotics

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \nu\pi/2 - \pi/4) \quad \text{as} \quad z \to \infty, \tag{78}$$

we see that infinity is an oscillatory boundary for all $\lambda > 0$ (solution $x^{-\nu} J_\nu(x\sqrt{2\lambda})$ has an infinite sequence of zeros increasing towards infinity). Thus, $+\infty$ is O-NO with cutoff $\Lambda = 0$ and we have continuous spectrum above 0. Recalling the

Wronskian $J_\nu(z)Y'_\nu(z) - J'_\nu(z)Y_\nu(z) = \frac{2}{\pi z}$, the solution $\psi(x,\lambda)$, $x \in [y,\infty)$, satisfying the initial conditions $\psi(y,\lambda) = 0$, $\psi'(y,\lambda) = 1$ is given by:

$$\psi(x,\lambda) = \frac{\pi y}{2}\left(\frac{x}{y}\right)^{-\nu}\left[Y_{-\nu}(x\sqrt{2\lambda})J_{-\nu}(y\sqrt{2\lambda}) - Y_{-\nu}(y\sqrt{2\lambda})J_{-\nu}(x\sqrt{2\lambda})\right], \tag{79}$$

where $Y_\nu(z)$ is the Bessel function of the second kind. The spectral function is absolutely continuous (see Titchmarsh 1962, p. 87, Eq. (4.10.3))

$$d\rho_y(\lambda) = \frac{2y^{-2\nu-2}}{\pi^2[J^2_{-\nu}(y\sqrt{2\lambda}) + Y^2_{-\nu}(y\sqrt{2\lambda})]}d\lambda, \tag{80}$$

and from Eqs. (14),(41) we have for $\nu < -1$, $0 < y < x < \infty$:

$$\mathbb{P}_x(\mathcal{T}_y^R \le t) = 1 - \frac{2}{\pi}\left(\frac{x}{y}\right)^{-\nu}\int_0^\infty e^{-\frac{s^2 t}{2}}\frac{[Y_{-\nu}(xs)J_{-\nu}(ys) - Y_{-\nu}(ys)J_{-\nu}(xs)]}{s[J^2_{-\nu}(ys) + Y^2_{-\nu}(ys)]}ds, \tag{81}$$

where we introduced a new integration variable $s = \sqrt{2\lambda}$. The integral in s has to be computed numerically. To avoid numerical integration, we can follow the recipe of Remark 1 in Sect. 3 and modify the problem by killing the CEV process at some level $\mathcal{E} \in (S_0,\infty)$. The spectrum of the modified problem is purely discrete and the eigenvalues grow as n^2 (Eq. (37)). The hitting time distribution for the process on $(0,\infty)$ can be estimated by picking a sufficiently large upper barrier $\mathcal{E}$ and computing the hitting time distribution of the process killed at $\mathcal{E}$ as an approximation to (81).

5 Numerical examples

Consider the following set of parameters: initial underlying price $S_0 = 100$, local volatility at this price level $\sigma_0 = \delta S_0^\beta = 0.25$ or 25% (for each elasticity β the scale parameter δ is selected so that $\delta = \sigma_0 S_0^{-\beta}$ and $\sigma(S) = \sigma_0(S/S_0)^\beta$ with $\sigma_0 = 0.25$), $T = 1/2$ and 2 years, $r = 0.1$ and $q = 0$ ($\mu = 0.1$). We consider the following values of the elasticity parameter $\beta = -1/2, -1, -2, -3, -4$. Empirical estimates of β implicit in S&P 500 index options are around -3.

For all calculations in this paper we used *Mathematica 5.0* running on a Pentium III PC. The steps needed to compute lookback option prices are as follows. We discretize the one-dimensional integrals in Eqs. (3–6) using the Romberg integration rule. For each node in the numerical integration formula, we compute the probabilities (2), (7) via the spectral expansions (60), (69). First, we determine the eigenvalues. We use the built-in numerical root finding function in *Mathematica* to determine each λ_n. *Mathematica* includes all required special functions as built-in functions (in particular, the Kummer and Tricomi confluent hypergeometric functions and their derivatives). We use the estimates (59), (68) to initialize the numerical root finding procedure. After the eigenvalues are determined, the eigenfunction expansions (60), (69) are calculated (we truncate the series after we achieve the desired error tolerance level). Due to the factors $e^{-\lambda_n t}$ in the eigenfunction expansions, the longer the time to expiration, the faster the convergence. For the standard lookback call and the put on the minimum, we need the distribution of the minimum and, hence, the distribution of the hitting time down (Eqs. (7), (69)). For the standard lookback put and the call on the maximum, we need the distribution of the maximum and, hence, the distribution of the hitting time up (Eqs. (7), (60)). From the estimates (59), (68), the eigenvalues for the hitting time up (down) increase as n^2 (n). Hence, the eigenfunction expansion for the distribution of the maximum/hitting time up converges much faster than the distribution of the minimum/hitting time down. We also note that from Eqs. (59), (63), (68), (44) the eigenvalues for the maximum/hitting time up (minimum/hitting time down) increase as β^2 $(|\beta|)$ as $|\beta|$ increases (the volatility skew gets steeper). This results in the faster convergence of the spectral expansion for steeper skews. This is in contrast with Monte Carlo, as we will discuss below.

In Table 3 we compute probabilities for the CEV process $\mathbb{P}_{S_0}(M_T \geq Y) = \mathbb{P}_{S_0}(\mathcal{T}_Y^{CEV} \leq T)$ for $Y = 120$, $T = 1/2$ and 2 and $\mathbb{P}_{S_0}(m_T \leq Z) = \mathbb{P}_{S_0}(\mathcal{T}_Z^{CEV} \leq T)$ for $Z = 90, T = 1/2$ and 2. The corresponding probabilities for the generalized Bessel process are $\mathbb{P}_x(\mathcal{T}_y^R \leq T)$ and $\mathbb{P}_x(\mathcal{T}_z^R \leq T)$. Parameters of the generalized Bessel process corresponding to our CEV process parameters are given in Table 1. Table 2 gives the eigenvalues of the SL problems for the CEV process on the intervals $(0, Y)$ and (Z, ∞) (correspondingly, for the generalized Bessel process on the intervals $(0, y)$ and (z, ∞)). The values in parenthesis are the estimates (59) and (68). We observe that the estimates approximate the exact eigenvalues very well even for lower n (exact eigenvalues are computed by numerically finding the roots of the Whittaker functions M and W). We also observe that, while in the case of hitting Y from below ($S_0 < Y$) the eigenvalues grow as n^2 with n and as β^2 with $|\beta|$, in the case of hitting Z from above ($Z < S_0$) the eigenvalues grow linearly with n and linearly with $|\beta|$ and, as a result, we need more terms in the eigenfunction expansions to compute the probabilities in the latter case (Table 3).

Table 4 gives prices of the standard lookback call, the standard lookback put, the call on the maximum, and the put on the minimum, as well as the corresponding hedge ratios (deltas). To compute lookback deltas, we take the analytical derivatives of Eqs. (3–6) with respect to the underlying price. There is no loss of accuracy in computing the delta. Two times to expiration are considered: $T = 1/2$ and $T = 2$. The longer the time expiration, the faster the eigenfunction expansions converge.

We now compare our results with alternative approaches in the literature. In the Introduction we have already mentioned the problem with the trinomial lattice approach for lookbacks (see Boyle et al. 1999, Sect. 1). The standard Monte Carlo simulation method also experiences difficulties in pricing continuously monitored lookbacks even under the lognormal assumption (Boyle et al. 1999, Sect. 3). When we discretize the continuous price process and simulate it at discrete points in time we lose information about the part of the path between the discrete points. As a result, the simulation is biased when estimating lookback prices. Andersen and Brotherton-Ratcliffe (1996) show that for a one year lookback this bias is

Table 1. Generalized Bessel process parameters corresponding to the CEV process parameters

β	ν	c	x	y	z
−1/2	−1	0.05	8	$8\sqrt{6/5}$	$12\sqrt{2/5}$
−1	−1/2	0.1	4	4.8	3.6
−3	−1/6	0.3	4/3	2.304	0.972

Table 2. Eigenvalues. Eigenvalues λ_n are obtained by numerical root finding. Estimates (59) and (68) are given in parenthesis. The number in the n-column indicates the eigenvalue number. CEV process parameters: $\sigma_0 = \delta S_0^\beta = \delta 100^\beta = 0.25$, $\mu = 0.1$

	$\beta = -0.5$		-1		-3	
n	Eigenvalues for hitting $Y = 120$ from below					
1	0.12625	(0.10040)	0.29608	(0.26418)	0.97218	(0.89557)
10	6.77796	(6.75082)	21.5068	(21.4684)	90.2394	(90.1390)
20	26.3758	(26.3487)	85.7620	(85.7236)	366.026	(365.926)
n	Eigenvalues for hitting $Z = 90$ from above					
1	0.23393	(0.24363)	0.38170	(0.38860)	0.89662	(0.91391)
10	1.37271	(1.37572)	2.50572	(2.50816)	6.76357	(6.77038)
50	5.79629	(5.79762)	11.0751	(11.0762)	31.5661	(31.5692)
250	26.7411	(26.7417)	52.3433	(52.3438)	153.347	(153.348)

Table 3. Hitting probabilities. The number in parenthesis next to each probability gives the number of terms in the spectral expansion required to achieve the accuracy of five decimals. CEV process parameters: $S_0 = 100$, $\sigma_0 = \delta S_0^\beta = \delta 100^\beta = 0.25$, $\mu = 0.1$

$\beta = -0.5$		-1		-2		-3		-4	
$\mathbb{P}_{100}(\mathcal{T}_{120}^{CEV} \le 1/2)$									
0.35968	(17)	0.35247	(10)	0.33451	(5)	0.31156	(4)	0.28361	(4)
$\mathbb{P}_{100}(\mathcal{T}_{120}^{CEV} \le 2)$									
0.73168	(8)	0.74184	(4)	0.76703	(2)	0.79598	(2)	0.81799	(2)
$\mathbb{P}_{100}(\mathcal{T}_{90}^{CEV} \le 1/2)$									
0.48380	(143)	0.47998	(73)	0.47200	(34)	0.46369	(27)	0.45523	(19)
$\mathbb{P}_{100}(\mathcal{T}_{90}^{CEV} \le 2)$									
0.65139	(40)	0.63307	(25)	0.60040	(14)	0.57197	(9)	0.54693	(7)

around 5% of the option price and suggest a procedure to correct it. Boyle et al. (1999) develop a simulation algorithm that incorporates Andersen and Brotherton-Ratcliffe (AB) (1996) bias correction method and uses the lookback price under the lognormal assumption as a control variate for the CEV lookback price. The resulting prices for lookbacks under the CEV model with $\beta = -1/2$ and -1 are in good agreement with the exact prices in Table 4. However, the AB bias correction method is only valid for the lognormal process ($\beta = 0$). Strictly speaking, their formula is not applicable for the CEV process with $\beta < 0$. When the volatility skew is not too steep ($|\beta|$ is not too large), the AB bias correction approach may be used as an approximation, and Boyle et al. (1999) report good experimental results for

Table 4. Standard lookback call, call on maximum, standard lookback put and put on minimum prices under the CEV process. Parameters: $S = 100$, $\sigma_0 = \delta S^\beta = \delta 100^\beta = 0.25$ (25%), $\mu = 0.1$, Time to expiration $T = 1/2, 2$. For calls on maximum and puts on minimum the strike price K is given in the left column and it is assumed that the options are newly-written and valued at time $t = 0$ ($m_0 = M_0 = S = 100$). For standard lookback calls (puts) the minimum-to-date (maximum-to-da te) are given in the right column. The case $m = M = S = 100$ corresponds to newly-written options. The cases $m = 90, 95$ and $M = 105$ correspond to seasoned options initiated prior to the valuation date. The value of the option delta is given in parenthesis underneath the corresponding option price

	$\beta = -0.5$	-1	-2	-3	-4
Min-to-date	Standard lookback call $T = 1/2$				
95	16.5674	16.8843	17.7709	19.1065	20.4229
	(0.3615)	(0.3063)	(0.1565)	(-0.0513)	(-0.2452)
100	15.8791	16.1691	17.0048	18.2922	19.5630
	(0.0955)	(0.0282)	(-0.1447)	(-0.3744)	(-0.5893)
Min-to-date	Standard lookback call $T = 2$				
90	35.3165	36.1895	38.2866	39.4057	39.6719
	(0.5193)	(0.4319)	(0.2393)	(0.1030)	(0.0106)
100	33.8189	34.5825	36.4818	37.4250	37.5332
	(0.2369)	(0.1246)	(-0.1139)	(-0.2917)	(-0.4225)
Strike K	Call on maximum $T = 1/2$				
100	16.6084	16.1395	15.3807	14.7988	14.3562
105	12.2588	11.7748	10.9824	10.3599	9.8669
Max-to-date	Standard lookback put $T = 1/2$				
100	11.7313	11.2624	10.5037	9.9217	9.4791
	(0.0465)	(-0.0208)	(-0.1390)	(-0.245)	(-0.3473)
105	12.1379	11.6538	10.8615	10.2390	9.7459
	(-0.1201)	(-0.1808)	(-0.2840)	(-0.3491)	(-0.4514)
Max-to-date	Standard lookback put $T = 2$				
100	18.0578	16.0364	13.4643	11.8492	10.5875
	(0.0527)	(-0.0651)	(-0.2551)	(-0.4013)	(-0.4626)
105	18.1883	16.1574	13.5574	11.9207	10.6556
	(-0.0011)	(-0.1150)	(-0.2932)	(-0.4309)	(-0.5217)
Strike K	Put on minimum $T = 1/2$				
95	6.9342	7.2510	8.1378	9.4733	10.7897
100	11.0020	11.2920	12.1277	13.4151	14.6859

$\beta = -1/2, -1$. However, empirically, the skew is very steep for stock indexes with estimates of β around -3. The steeper the skew, the less valid is the application of the AB bias correction method. Furthermore, for steep skews and longer times to expiration CEV lookback prices differ significantly from the lognormal prices, and the quality of lognormal prices as control variates deteriorates as well. Thus, the accuracy of the Monte Carlo method deteriorates as the volatility skew gets steeper and the time to expiration increases. In contrast, as we have discussed previously, numerical convergence of the eigenfunction expansion *improves* for steeper skews (larger $|\beta|$) and longer times to expiration.

The values in Table 4 are in agreement with the values in Table 2 in Davydov and Linetsky (2001) where the probabilities (7) were calculated by numerically inverting Laplace transforms via the Abate and Whitt algorithm. In this paper the probabilities are calculated using the eigenfunction expansions (60) and (69). This avoids the need for numerical Laplace inversion.

6 Conclusion

Based on the relationship between diffusion maximum and minimum and hitting times and the spectral decomposition of diffusion hitting times, this paper gives an analytical characterization of lookback option prices in terms of spectral expansions. In particular, solutions for lookback options under the CEV diffusion are obtained by reducing the CEV process to a generalized Bessel process and explicitly constructing spectral expansions for hitting times of the latter. We point out three specific advantages of spectral expansions: (1) the Greeks can be calculated by taking analytical derivatives without any loss of precision; (2) long-dated contracts are easy to value (the longer the time to expiration, the faster the spectral expansion converges); (3) in the case of the CEV model, the steeper the volatility skew, the faster the spectral expansion convergence. These three properties of the spectral expansion method are in contrast with the properties of Monte Carlo simulation. Finally, the results in this paper also have an independent interest for other problems involving diffusion maximum, minimum and hitting time distributions (for related results on hitting times of CIR and OU processes see Linetsky (2003)).

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