Applied Mathematical Sciences, Vol. 9, 2015, no. 121, 6021 - 6028
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.58511

On Razumikhin Approach for Partial Stability Problem of Retarded Systems

V.I. Vorotnikov

Ural Federal University
ul. Mira, 19, 620002 Ekaterinburg, Russia

Abstract

For nonlinear retarded functional differential systems the problem of asymptotic stability with respect to a part of the variables of a “partial” equilibrium position is considered. The condition of uniform asymptotic stability of this type is obtained in the context of the Razumikhin approach.

Mathematics Subject Classification: 34K20, 34K25

Keywords: nonlinear retarded functional differential systems, partial uniform asymptotic stability, Razumikhin approach

1 Introduction

In the problem of stability with respect to a part of the variables of the zero equilibrium position (Lyapunov-Rumyantsev partial stability problem) assumes the domain of initial perturbations to be a sufficiently small neighborhood of the zero equilibrium position. Along with this statement, the cases of arbitrary or large initial perturbations for a part of variables that are non-controlled when studying stability are investigated [1, 2]. For the stability problems of “partial” zero equilibrium positions naturally also assume [3] that initial perturbations of variables that do not define the given equilibrium position can be large with respect to one part of the variables and arbitrary with respect to their other part.

In this article the problem of uniform asymptotic stability with respect to a part of the variables of the “partial” equilibrium position is considered for nonlinear retarded systems of functional differential equations. Condition of this type stability is obtained in the context of the Razumikhin approach [4-7]; this condition supplemented a number of existing results [8-10].

2 Statement of the Problem

We assume $\tau > 0$ is a given real number, R^n is a linear real space of n-dimensional vectors $\mathbf{x}$ with the norm $|\mathbf{x}| = \max |x_i|$ (x_i – ith component of the vector $\mathbf{x}$), C is the Banach space of continuous functions $\boldsymbol{\varphi}$: $[-\tau, 0] \to R^n$ with standard norm $\|\boldsymbol{\varphi}\| = \sup |\boldsymbol{\varphi}(\theta)|$ ($\theta \in [-\tau, 0]$), and $R_+ = [0, +\infty)$. If $t_0, \beta \in R_+, \beta > t_0$, then for a continuous function $\mathbf{x}$: $[t_0-\tau, \beta) \to R^n$ we define a function $\mathbf{x}_t \in C$ by the relation $\mathbf{x}_t = \mathbf{x}(t + \theta)$ ($\theta \in [-\tau, 0]$); in what follows, $\mathbf{x}'(t)$ denotes the right-hand derivative.

We introduce the partition $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$ (T denotes transposition), where $\mathbf{y} \in R^m$, $\mathbf{z} \in R^{n-m}$ ($1 \le m \le n$). According to this partition, we set $C = C^{\mathbf{y}} \times C^{\mathbf{z}}$, where $C^{\mathbf{y}}$ and $C^{\mathbf{z}}$ are the Banach space of continuous functions $\boldsymbol{\varphi}_{\mathbf{y}}$: $[-\tau, 0] \to R^m$ and $\boldsymbol{\varphi}_{\mathbf{z}}$: $[-\tau, 0] \to R^{n-m}$ with the norms $\|\boldsymbol{\varphi}_{\mathbf{y}}\| = \sup |\boldsymbol{\varphi}_{\mathbf{y}}(\theta)|$ and $\|\boldsymbol{\varphi}_{\mathbf{z}}\| = \sup |\boldsymbol{\varphi}_{\mathbf{z}}(\theta)|$ ($\theta \in [-\tau, 0]$). For $\boldsymbol{\varphi} \in C$, we have $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_{\mathbf{y}}^{\mathrm{T}}, \boldsymbol{\varphi}_{\mathbf{z}}^{\mathrm{T}})^{\mathrm{T}}$ and $\|\boldsymbol{\varphi}\| = \max(\|\boldsymbol{\varphi}_{\mathbf{y}}\|, \|\boldsymbol{\varphi}_{\mathbf{z}}\|)$.

Let there be given a system of nonlinear functional differential equations with holdover [11, 12]

$$\mathbf{x}'(t) = \mathbf{X}(t, \mathbf{x}_t),$$

which, with the above partitions taken into account, can be represented as

$$\mathbf{y}'(t) = \mathbf{Y}(t, \mathbf{y}_t, \mathbf{z}_t), \quad \mathbf{z}'(t) = \mathbf{Z}(t, \mathbf{y}_t, \mathbf{z}_t). \tag{1}$$

In the space C, we consider the set $M = \{\boldsymbol{\varphi} \in C: \boldsymbol{\varphi}_{\mathbf{y}} = \mathbf{0}\}$. If $\mathbf{Y}[t, \boldsymbol{\varphi}] \equiv \mathbf{0}$ for $\boldsymbol{\varphi} \in M$, then the solution $\mathbf{x}(t_0, \boldsymbol{\varphi})$ of system (1) satisfies the condition $\|\mathbf{y}_t(t_0, \boldsymbol{\varphi})\| \equiv 0$. In other words, $M = \{\mathbf{x}: \mathbf{y} = \mathbf{0}\}$ is a "partial" equilibrium position of system (1).

To consider the problem of stability with respect to a part of the variables of the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$, we assume that $\mathbf{y} = (\mathbf{y}_1^{\mathrm{T}}, \mathbf{y}_2^{\mathrm{T}})^{\mathrm{T}}$ and we also represent the component $\boldsymbol{\varphi}_{\mathbf{y}}$ of the vector function $\boldsymbol{\varphi}$ as $\boldsymbol{\varphi}_{\mathbf{y}} = (\boldsymbol{\varphi}_{\mathbf{y}1}^{\mathrm{T}}, \boldsymbol{\varphi}_{\mathbf{y}2}^{\mathrm{T}})^{\mathrm{T}}$.

Let us assume that the operator $\mathbf{X}$: $R_+ \times C \to R^n$ determining the right-hand side of system (1) is completely continuous in the domain

$$G = \{t \ge 0, \quad \|\boldsymbol{\varphi}_{\mathbf{y}1}\| < h, \|\boldsymbol{\varphi}_{\mathbf{y}2}\| + \|\boldsymbol{\varphi}_{\mathbf{z}}\| < \infty\}, \tag{2}$$

We also assume that the Cauchy-Lipschitz condition is satisfied on each compact subset K in the domain (2). Then [11,12], for each point t_0, $\boldsymbol{\varphi}$ in the domain (2) there is a unique solution $\mathbf{x}(t_0, \boldsymbol{\varphi})$ of system (1) which can be continued to the boundary of the domain $\|\boldsymbol{\varphi}_{\mathbf{y}1}\| < h$, $\|\boldsymbol{\varphi}_{\mathbf{y}2}\| + \|\boldsymbol{\varphi}_{\mathbf{z}}\| < \infty$ and continuously depends on t_0, $\boldsymbol{\varphi}$, while the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ is an invariant set of this system.

Following [12], we let $\mathbf{x}(t) = \mathbf{x}(t, t_0, \boldsymbol{\varphi})$ denote the value of $\mathbf{x}(t_0, \boldsymbol{\varphi})$ at time t. We additionally assume [8-10] that the solutions are ($\mathbf{y}_2$, $\mathbf{z}$)-continuable, namely, the solutions of the system are determined for all $t \ge t_0$ such that $|\mathbf{y}_1(t, t_0, \boldsymbol{\varphi})| < h$.

We represent the component $\boldsymbol{\varphi}_{\mathbf{z}}$ of the vector function $\boldsymbol{\varphi}$ in the form $\boldsymbol{\varphi}_{\mathbf{z}} = [\boldsymbol{\varphi}_{\mathbf{z}1}^{\mathrm{T}}, \boldsymbol{\varphi}_{\mathbf{z}2}^{\mathrm{T}}]$ and let D_δ denote the domain $\boldsymbol{\varphi}$ such that $\|\boldsymbol{\varphi}_{\mathbf{y}}\| < \delta$, $\|\boldsymbol{\varphi}_{\mathbf{z}1}\| \le L$, $\|\boldsymbol{\varphi}_{\mathbf{z}2}\| < \infty$; the domain D_Δ is obtained by replacing δ by Δ.

Definition [8-10]**.** A "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ of system (1) is *uniformly* $\mathbf{y}_1$*-asymptotically stable for a large values of* $\boldsymbol{\varphi}_{\mathbf{z}1}$ *and on the whole with respect to* $\boldsymbol{\varphi}_{\mathbf{z}2}$, if :

1) for any $\varepsilon > 0$, $t_0 \ge 0$ and for any given number $L > 0$ there is $\delta(\varepsilon, L) > 0$ such that from $\boldsymbol{\varphi} \in D_\delta$ it follows that $|\mathbf{y}_1(t; t_0, \boldsymbol{\varphi})| < \varepsilon$ for all $t \ge t_0$;

2) there is $\Delta(L) > 0$ such that an arbitrary solution $\mathbf{x}(t_0, \boldsymbol{\varphi})$ of the system (1) with $\boldsymbol{\varphi} \in D_\Delta$ satisfies the limit relation $\lim |\mathbf{y}_1(t; t_0, \boldsymbol{\varphi})| = 0$, $t \to \infty$, uniformly in t_0 and $\boldsymbol{\varphi}$ in the domain $t_0 \ge 0$, $\boldsymbol{\varphi} \in D_\Delta$ (for each $\varepsilon > 0$, there can be found a number $T = T(\varepsilon, L) > 0$ such that $|\mathbf{y}_1(t, t_0, \boldsymbol{\varphi})| < \varepsilon$ for all $t \ge t_0 + T(\varepsilon, L)$, if $t_0 \ge 0$, $\boldsymbol{\varphi} \in D_\Delta$).

3 Main Results

We consider single-valued scalar continuously differentiable functions $V = V(t, \mathbf{x})$, $V(t, \mathbf{0}) \equiv 0$, defined in the domain

$$E = \{t \ge 0,\ |\mathbf{y}_1| < h,\ |\mathbf{y}_2| + |\mathbf{z}| < \infty\}. \tag{3}$$

The derivative V' of the V-function along the solutions of system (1) is functional which understood as (we denote by symbol $<\cdot>$ the scalar product)

$$V'(t, \boldsymbol{\varphi}) = \partial V(t, \boldsymbol{\varphi}(0))/\partial t + <\partial V(t, \boldsymbol{\varphi}(0))/\partial \mathbf{x} \cdot \mathbf{X}(t, \boldsymbol{\varphi})>.$$

To obtain the partial stability conditions, we also consider:

1) auxiliary scalar function $V^*(t, \mathbf{y}, \mathbf{z}_1)$ which is continuous in the domain (3) and auxiliary generally vector functions $\boldsymbol{\mu}(t, \mathbf{x})$ and $\mathbf{w}(\mathbf{x})$, which are continuously differentiable in the domain (3);

2) continuous monotonically increasing for $r \in R_+$ scalar functions $a_i(r)$ and $p(r)$, $a_i(0) = 0$, $p(0) = 0$, $i = 1,2,3$.

We associate functions $\boldsymbol{\mu}(t, \mathbf{x})$ in the domain (3) with functions $\boldsymbol{\mu}(t, \boldsymbol{\varphi})$ in the space G and let us define

$$\|\boldsymbol{\mu}(t, \boldsymbol{\varphi})\| = \sup |\boldsymbol{\mu}(t, \boldsymbol{\varphi}(\theta))|,\ \theta \in [-\tau, 0], \quad t \in R_+.$$

Let us assume that it is possible to represent V-function in the form

$$V(t, \mathbf{x}) = V^{**}(t, \mathbf{y}_1, \boldsymbol{\mu}(t, \mathbf{x}), \mathbf{y}_2, \mathbf{z}), \tag{4}$$

where V^{**} is continuously differentiable function in the domain (3).

We also denote

$$\Omega_{t,p}(V) = \{\boldsymbol{\varphi} \in G^*:\ V^{**}(t+\theta, \boldsymbol{\varphi}_{\mathbf{y}1}(\theta), \boldsymbol{\mu}(t+\theta, \boldsymbol{\varphi}(\theta)), \boldsymbol{\varphi}_{\mathbf{y}2}(0), \boldsymbol{\varphi}_{\mathbf{z}}(0))$$
$$\le p(V^{**}(t, \boldsymbol{\varphi}_{\mathbf{y}1}(0), \boldsymbol{\mu}(t, \boldsymbol{\varphi}(0)), \boldsymbol{\varphi}_{\mathbf{y}2}(0), \boldsymbol{\varphi}_{\mathbf{z}}(0))), \quad \theta \in [-\tau, 0], \quad t \in R_+\},$$
$$G^* = \|\boldsymbol{\varphi}_{\mathbf{y}1}\| + \|\boldsymbol{\mu}(t, \boldsymbol{\varphi})\| < h_1 < h, \quad \|\boldsymbol{\varphi}_{\mathbf{y}2}\| + \|\boldsymbol{\varphi}_{\mathbf{z}}\| < \infty.$$

Theorem. *Suppose that for system (1) along with a V-function, it is possible to find a vector function* $\boldsymbol{\mu}(t, \mathbf{x})$, $\boldsymbol{\mu}(t, \mathbf{0}) \equiv \mathbf{0}$, *such that:*

i) *it is possible to represent V-function in the form (4);*

ii) *in the domain*

$$t \geq 0, \quad |\mathbf{y}_1| + |\boldsymbol{\mu}(t, \mathbf{x})| < h_1 < h, \quad |\mathbf{y}_2| + |\mathbf{z}| < \infty \tag{5}$$

the conditions are satisfied:

$$a_1(|\mathbf{y}_1| + |\boldsymbol{\mu}(t, \mathbf{x})|) \leq V(t, \mathbf{x}) \leq a_2(|\mathbf{y}_1| + |\mathbf{w}(\mathbf{x})|), \tag{6}$$

$$V(t, \mathbf{x}) \leq V^*(\mathbf{y}, \mathbf{z}_1),\ V^*(\mathbf{0}, \mathbf{z}_1) \equiv 0; \tag{7}$$

iii) $V'(t, \boldsymbol{\varphi}) \leq -a_3(|\boldsymbol{\varphi}_{\mathbf{y}1}(0)| + |\mathbf{w}(\boldsymbol{\varphi}(0))|)$ *for all* $t \in R_+$, $\boldsymbol{\varphi} \in \Omega_{t,p}(V)$.

Then, the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ *of the functional differential system (1) is uniformly asymptotically* $\mathbf{y}_1$*-stable for a large values of* $\boldsymbol{\varphi}_{\mathbf{z}1}$ *and on the whole with respect to* $\boldsymbol{\varphi}_{\mathbf{z}2}$.

Proof. The uniform $\mathbf{y}_1$-stability for a large values of $\boldsymbol{\varphi}_{\mathbf{z}1}$ and on the whole with respect to $\boldsymbol{\varphi}_{\mathbf{z}2}$ of the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ of system (1) follows from [9]: for any $\varepsilon > 0$, $t_0 \geq 0$ and for any given number $L > 0$ there is $\delta(\varepsilon, L) > 0$ such that from $\boldsymbol{\varphi} \in D_\delta$ it follows that $|\mathbf{y}_1(t, t_0, \boldsymbol{\varphi})| < \varepsilon$ for all $t \geq t_0$.

For the given number h_1 let us choose $\Delta(h_1, L) > 0$, so that we have $b(\Delta, L) = \sup V^*(\boldsymbol{\varphi}_{\mathbf{y}}, \boldsymbol{\varphi}_{\mathbf{z}1}) < a_1(h_1)$ under $\|\boldsymbol{\varphi}_{\mathbf{y}}\| \leq \Delta$, $\|\boldsymbol{\varphi}_{\mathbf{z}1}\| \leq L$. We introduce the functional

$$W(t, \boldsymbol{\varphi}) = \sup V^{**}(t + \theta, \boldsymbol{\varphi}_{\mathbf{y}1}(\theta), \boldsymbol{\mu}(t + \theta, \boldsymbol{\varphi}(\theta)), \boldsymbol{\varphi}_{\mathbf{y}2}(0), \boldsymbol{\varphi}_{\mathbf{z}}(0)),$$
$$\theta \in [-\tau, 0]$$

for all $t \in R_+$, $\boldsymbol{\varphi} \in \Omega_t(V)$. For any given number $L > 0$, it follows from the continuity of the function V (functional W), the condition $V(t, \mathbf{0}) \equiv 0$ (the condition $W(t, \mathbf{0}) \equiv 0$), and the conditions (7) (the conditions $W(t, \boldsymbol{\varphi}) \leq V^*(t, \boldsymbol{\varphi}_{\mathbf{y}}, \boldsymbol{\varphi}_{\mathbf{z}1})$, $V^*(t, \mathbf{0}, \boldsymbol{\varphi}_{\mathbf{z}1}) \equiv 0$) that there is $\Delta(h_1, L) > 0$ such that from $\boldsymbol{\varphi} \in D_\Delta$ it follows that $W(t_0, \boldsymbol{\varphi}) < a_1(h_1)$.

The same arguments as in [9] shows that for the arbitrary solution $\mathbf{x}(t_0, \boldsymbol{\varphi})$, $\boldsymbol{\varphi} \in D_\Delta$ of the system (1) the function $W_0(t) = W(t, \mathbf{x}_t(t_0, \boldsymbol{\varphi}))$ is a non-increasing in t, and the condition $\boldsymbol{\varphi} \in D_\Delta$ implies $|\mathbf{y}_1(t, t_0, \boldsymbol{\varphi})| + |\boldsymbol{\mu}(t, \mathbf{x}(t, t_0, \boldsymbol{\varphi}))| < h_1$ for all $t \geq t_0$.

The uniform asymptotic $\mathbf{y}_1$-stability will be proved if, for each $\varepsilon > 0$, there can be found a number $T = T(\varepsilon, L) > 0$ such that $|\mathbf{y}_1(t, t_0, \boldsymbol{\varphi})| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon, L)$, if $t_0 \geq 0$, $\boldsymbol{\varphi} \in D_\Delta$. We shall prove following the scheme proposed in Hale [11].

From the properties of the function $p(r)$, for any $\varepsilon < \Delta$ there is a number $\beta(\varepsilon, L) > 0$ such that $p(r) - r > \beta$ for $a_1(\varepsilon) \leq r \leq b(\Delta, L)$. Let $N = N(\varepsilon, L)$ be the first integer such that $a_1(\varepsilon) + N\beta \geq b(\Delta, L)$, and let $\gamma = \inf a_3(r)$ $(\varepsilon \leq r \leq h_1)$ and

$$T = N\, b(\Delta, L))/\gamma.$$

We now show that $V(t, \mathbf{x}(t)) \le a_1(\varepsilon)$ for all $t \ge t_0 + T(\varepsilon, L)$, if $t_0 \ge 0$, $\boldsymbol{\varphi} \in D_\Delta$. First, we show that $V(t, \mathbf{x}(t)) \le a_1(\varepsilon) + (N-1)\beta$ for $t \ge t_0 + b(\Delta, L)/\gamma$. If $a_1(\varepsilon) + (N-1)\beta < V(t, \mathbf{x}(t))$ for $t_0 - r \le t < t_0 + b(\Delta, L)/\gamma$, then, since $V(t, \mathbf{x}(t)) \le b(\Delta, L)$ for all $t \ge t_0 - \tau$, it follows that

$$p(V(t, \mathbf{x}(t)) > V(t, \mathbf{x}(t)) + \beta \ge a_1(\varepsilon) + N\beta \ge b(\Delta, L) \ge V(t+\theta, \mathbf{x}(t+\theta)),$$
$$t_0 - r \le t < t_0 + b(\Delta, L)/\gamma,\ \theta \in [-\tau, 0],.$$

Condition iii) implies

$$V'(t, \mathbf{x}(t)) \le -a_3(|\mathbf{y}_1(t)| + |\mathbf{w}(\boldsymbol{\varphi}(\mathbf{x}(t))|) \le -\gamma$$

for $t_0 \le t < t_0 + b(\Delta, L)/\gamma$. Consequently, we have $V(t, \mathbf{x}(t)) \le V(t_0, \mathbf{x}(t_0)) - \gamma(t - t_0)$ on the same interval (we also take in the mind, that $V(t, \mathbf{x}) > a_1(\varepsilon)$ implies $|\mathbf{y}_1(t)| + |\mathbf{w}(\boldsymbol{\varphi}(\mathbf{x}(t))| > a_2^{-1}(a_1(\varepsilon)))$.

In result, for all $t \ge t_0$ we have

$$V(t, \mathbf{x}(t) \le \max\{V(t_0, \mathbf{x}(t_0)) - \gamma(t - t_0),\ a_1(\varepsilon) + (N-1)\beta\}$$
$$\le \max\{b(\Delta, L) - \gamma(t - t_0),\ a_1(\varepsilon) + (N-1)\beta\}.$$

Take in the mind that $V(t, \mathbf{x}(t) \ge 0$, it follows that $V(t, \mathbf{x}(t) \le a_1(\varepsilon) + (N-1)\beta$ for $t > t_0 + b(\Delta, L)/\gamma$.

In the case $N \ge 2$ let us replace t_0 on $t_1 = t_0 + b(\Delta, L)/\gamma + \tau$ and let us to use relation $V(t, \mathbf{x}(t)) \le a_1(\varepsilon) + (N-1)\beta$, $t_1 - r \le t \le t_1$, which we have already proved. With the same arguments we can to prove that $V(t, \mathbf{x}(t)) \le a_1(\varepsilon) + (N-2)\beta$ for $t \ge t_1 + b(\Delta, L)/\gamma = t_0 + 2b(\Delta, L)/\gamma + \tau$. We finish this proof, if we will to continue this consideration.

Corollary. *Let us assume system (1) have the equilibrium position* $\mathbf{x} = \mathbf{0}$. *Suppose that along with a V-function, it is possible to find a vector function* $\boldsymbol{\mu}(t, \mathbf{x})$, $\boldsymbol{\mu}(t, \mathbf{0}) \equiv \mathbf{0}$, *such that in the domain (5) conditions (6) and (7) are fulfilled.*

Then, the equilibrium position $\mathbf{x} = \mathbf{0}$ *of functional differential system (1) is uniformly asymptotically* $\mathbf{y}_1$*-stable (in the sense [2]).*

Remark 1. Theorem and Corollary are generalizations of the Razumikhin and Krasovskii theorems [4-7] and of the corresponding results obtained in [2, 3, 9]. For comparison, the stability problem with respect to all of the variables of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (1) was considered in [4-7]. The asymptotic stability with respect to a part of the variables ($\mathbf{y}_1$-stability) of the zero equilibrium position $\mathbf{x} = \mathbf{0}$ of the system (1) was considered in [2] under the assumptions $V(t, \mathbf{x}) = V^{**}(t, \mathbf{y}_1, \boldsymbol{\mu}(t, \mathbf{x}))$ and $\|\boldsymbol{\varphi}\| < \delta$. Stability (non asymptotic) of "partial" equilibrium position in sense Definition of this paper was considered in [9]. The aftereffect was not taken into account when these problems were analyzed in [3].

Remark 2. The stability with respect to the all of the variables of the "partial" equilibrium position $\mathbf{y} = \mathbf{0}$ of the system (1) was considered in [13] in the context of the method of Krasovskii functionals under the assumptions $V(t, \boldsymbol{\varphi}) \leq a_3(\|\boldsymbol{\varphi}_\mathbf{y}\|)$ and $\|\boldsymbol{\varphi}_\mathbf{y}\| < \delta$, $\|\boldsymbol{\varphi}_\mathbf{z}\| < \infty$.

Remark 3. The V-function and the derivative of the V-function in Theorem and Corollary are generally sign-alternating in the domain (see also [14])

$$t \geq 0, \quad |\mathbf{y}_1| < h_1 < h, \quad |\mathbf{y}_2| + |\mathbf{z}| < \infty. \tag{8}$$

Example. We assume that system (1) has the form [8,9]

$$\begin{aligned} y_1'(t) &= -(t+4)y_1(t) + y_1(t-\tau) + y_2^2(t-\tau)z_1(t-\tau), \\ y_2'(t) &= y_2(t)[1 + \mathrm{e}^t y_1(t) + y_2^2(t)z_1(t)], \\ z_1'(t) &= -2[2 + \mathrm{e}^t y_1(t)]z_1(t), \quad z_2'(t) = \mathrm{e}^t y_1(t)z_2(t). \end{aligned} \tag{9}$$

We consider an auxiliary functions V and μ_1 of the form

$$V(\mathbf{x}) = ½(y_1^2 + y_2^4 z_1^2), \quad \mu_1 = y_2^2 z_1, \tag{10}$$

satisfying the conditions (i) and (ii) of the Theorem.

On the set

$$\Omega_{t,p}(V) = \{\boldsymbol{\varphi} \in G^*: |\varphi_{y1}(\theta)| \leq q|\varphi_1(0)|, |\mu_1(\boldsymbol{\varphi}(\theta))| \leq q|\mu_1(\boldsymbol{\varphi}((0))|, \theta \in [-\tau, 0]\},$$

$$G^* = \|\varphi_{y1}\| + \|\mu_1(\boldsymbol{\varphi})\| < h_1 < h, \|\varphi_{y2}\| + \|\boldsymbol{\varphi}_\mathbf{z}\| < \infty, \quad \mu_1(\boldsymbol{\varphi}(\theta)) = \varphi_{y2}^2(\theta)\varphi_{z1}(\theta),$$

where $q = \text{const} > 1$, the derivative V' due to system (9) can be estimated as

$$\begin{aligned} V'(t, \boldsymbol{\varphi}) &= -(t+2)\varphi_{y1}^2(0) + \varphi_{y1}(0)\varphi_{y1}(-\tau) + \varphi_{y1}(0)\mu_1(-\tau) - \mu_1^2(0) + 2\mu_1^3(0) \\ &\leq -(t+2)\varphi_{y1}^2(0) + |\varphi_{y1}(0)||\varphi_{y1}(-\tau)| + |\varphi_{y1}(0)||\mu_1(-\tau)| - \mu_1^2(0) + 2\mu_1^3(0) \\ &\leq -(t+2)\varphi_{y1}^2(0) + q|\varphi_{y1}(0)||\varphi_{y1}(0)| + q|\varphi_{y1}(0)||\mu_1(0)| - \mu_1^2(0) + 2\mu_1^3(0) \\ &\leq -\gamma[\varphi_{y1}^2(0) + \mu_1^2(0)], \quad \gamma = \text{const} > 0, \\ &\mu_1(0) = \varphi_{y2}^2(0)\varphi_{z1}(0), \quad \mu_1(-\tau) = \varphi_{y2}^2(-\tau)\varphi_{z1}(-\tau), \end{aligned}$$

and condition (iii) of the Theorem also is satisfied under $p(r) = q^2 r$.

It follows from Theorem that the "partial" equilibrium position $y_1 = y_2 = 0$ of the system (9) is uniformly y_1-asymptotically stable for a large values of φ_{z1} and on the whole with respect to φ_{z2}.

Let us note that, in the domain (8), the derivative of the chosen V-function due to the system (9) is sign-alternating.

We also note that, when using *V*-function (10) but *V*-functional from [9] one can assume that delay τ is variable.

References

[1] V.I. Vorotnikov, Partial stability and control: the state of the art and developing prospects, *Autom. Remote Control*, **66** (2005), 511 - 561. http://dx.doi.org/10.1007/s10513-005-0099-9

[2] V.I. Vorotnikov, *Partial Stability and Control*, Birkhauser, Boston, 1998.

[3] V.I. Vorotnikov and Yu.G. Martyshenko, On the partial stability of nonlinear dynamical systems, *J. Comput. Syst. Sci. Int.*, **49** (2010), 702 - 709. http://dx.doi.org/10.1134/S1064230710050047

[4] B.S. Razumikhin, On stability of systems with delay, *Prikl. Mat. Mekh*., **20** (1956), 500 - 512. (in Russian)

[5] N.N. Krasovskii, On asymptotic stability of retarded systems, *Prikl. Mat. Mekh.*, **20** (1956), 513 - 518. (in Russian)

[6] N.N. Krasovskii, *On Stability of Motion*, Stanford University Press, Stanford, 1963.

[7] B.S. Razumikhin, *The Stability of Hereditary Systems*, Nauka, Moscow, 1988. (in Russian)

[8] V.I. Vorotnikov and Yu.G. Martyshenko, Stability in a part of variables of "partial" equilibria of systems with aftereffect, *Mathematical Notes*, **96** (2014), 477 - 483. http://dx.doi.org/10.1134/s0001434614090223

[9] V.I. Vorotnikov, On partial stability of retarded functional differential systems, *Applied Mathematical Sciences*, **9** (2015), 279 - 284. http://dx.doi.org/10.12988/ams.2015.411905

[10] V.I. Vorotnikov, On uniform partial asymptotic stability of retarded functional differential systems, *Applied Mathematical Sciences*, **9** (2015), 4149 - 4154. http://dx.doi.org/10.12988/ams.2015.55375

[11] J. Hale, *Theory of Functional Differential Equations.* Springer-Verlag, New York, 1977. http://dx.doi.org/10.1007/978-1-4612-9892-2

[12] T.A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.

[13] S.R. Bernfeld, C. Corduneanu, and A.O. Ignatyev, On the stability of invariant sets of functional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **55** (2003), 641 - 656.
http://dx.doi.org/10.1016/j.na.2003.08.002

[14] V.I. Vorotnikov, Problems of stability with respect to part of the variables, *J. Appl. Math. Mech.* **63** (1999), 695-703.
http://dx.doi.org/10.1016/S0021-8928(99)00088-X

Received: August 23, 2015; Published: September 30, 2015