disturbance. Here the reliance has been placed on its most plausible feature, namely its independence of input. This yields a set of assumptions in excess of the minimal requirement and an endeavor has been made to exploit this excess to reduce the sum of squares of estimation errors.

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Model Reduction and Control System Design by Shifted Legendre Polynomial Functions

Rong-Yeu Chang[1] and Maw-Ling Wang[1]

A method of model reduction for reducing a higher order transfer function to its lower order model is developed based on the shifted Legendre function approximation. The expansion coefficients of the shifted Legendre series which represents the approximate responses of transfer functions are computed by the recursive formula via operational matrix approach. The significances of applying shifted Legendre function to model reduction problem are that the method is simple, straightforward and the computational results obtained are accurate as well as the final time of the control system can be adjustable without any restriction. Based on the model reduction technique, a new algebraic method is proposed for the design of a feedback control system to satisfy specifications. Illustrative examples are given and satisfactory results are obtained.

Introduction

Model reduction has been receiving great attention in the field of process analysis and synthesis with the last twenty years. The purpose of model reduction is to provide a lower order model which is computationally simpler than the original higher order system. Typical methods for approaching model reduction problems are error minimization [1], retaining dominant eigenvalues [2], moment matching [1], continued fractions [3], stability equation method [4], and Páde and Routh approximation [5]. Recently, the model reduction problems have also been investigated by block pulse functions [6] and Laguerre functions [7] and Chebyshev function [8]. Satisfactory examples are given to illustrate the effectiveness of those methods.

In this paper, an effective method of shifted Legendre functions is employed to approach the problems of model reduction. The operational matrix for the integration of the shifted Legendre polynomial vectors whose elements are shifted Legendre function are first developed. Using the developed operational matrix, the expansion coefficients of the shifted Legendre series which represents the approximate responses of transfer functions are computed by the recursive formula. The significance of the present research is that the present method is simple, straightforward and the computational results are accurate as well as the final time of the system can be adjustable without any restriction. Based on the model reduction technique, the design of a feedback control system to satisfy the prescribed specifications is studied by the proposed new algebraic method. Satisfactory examples are given to illustrate the method.

Properties of Shifted Legendre Functions

The shifted Legendre function, $\bar{P}_n(t)$ is related to the well-known Legendre function $P_n(\tau)$ by transforming the independent variable as $\tau = 2(t/T_f) - 1$.

One of the properties of shifted Legendre polynomial functions is,

$$\frac{2(2n+1)}{T_f}\bar{P}_n(t) = \bar{P}'_{n+1}(t) - \bar{P}'_{n-1}(t) \quad (1)$$

Thus, the integration of $\bar{P}_n(t)$ with respect to t can be ob-

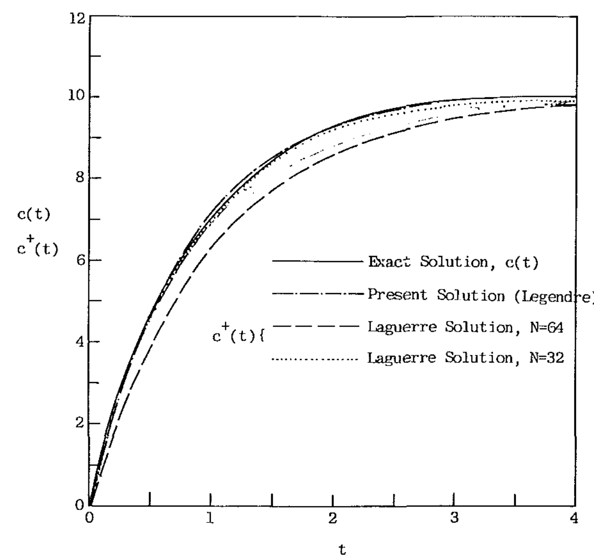

Fig. 1 Step response of original and reduced models

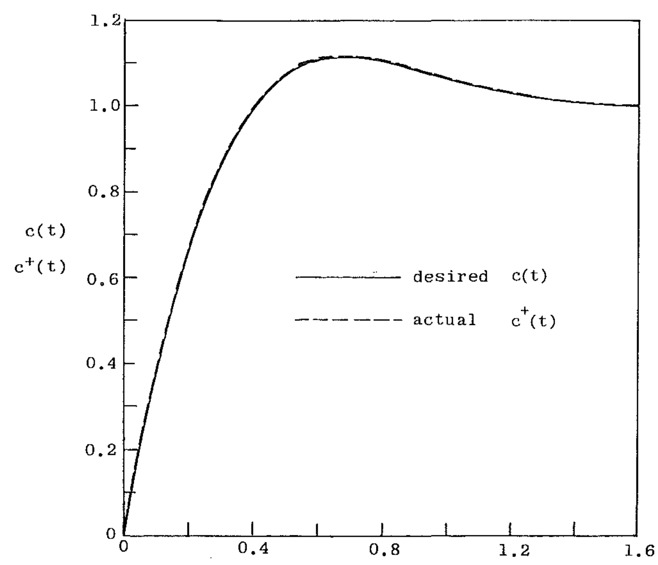

Fig. 2 Comparison of the desired and actual response

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tained directly from the above equation. Generally, the general form of matix integration can be written as,

$$\int_0^t \bar{\mathbf{P}}(t)\,dt = \mathbf{H}\,\bar{\mathbf{P}}(t) \qquad (2)$$

Where

$$\bar{\mathbf{P}}(t) = [\bar{P}_0(t), 3\bar{P}_1(t), 5\bar{P}_2(t)\ \ldots\ (2m-1)\bar{P}_{m-1}(t)]^T \qquad (3)$$

$$\mathbf{H} = T_f \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{10} & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & \frac{1}{14} & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} & 0 & \cdots\cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & \frac{-1}{2(2m-5)} & 0 & \frac{1}{2(2m-1)} \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & \frac{-1}{2(2m-3)} & 0 \end{bmatrix} \qquad (4)$$

The matrix **H** as shown by equation (4) is called operational matrix for the integration of the shifted Legendre vector. In the following section, the reduction of higher order transfer function to its lower order model is studied.

Model Reduction

Consider a linear time-invariant system characterized by the rational transfer function,

$$\frac{C(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \qquad (5)$$

Where $C(s)$ and $U(s)$ are the Laplace transforms of the output variable, $c(t)$, and the input variable, $u(t)$, respectively. Cross multiplication of equation (5) and on dividing s^n give,

$$C(s) + \sum_{i=1}^{n} a_i s^{-i} C(s) = \sum_{i=1}^{n} b_i s^{-i} U(s) \qquad (6)$$

Let

$$C_i(s) = s^{-1} C_{i-1}(s),\ i = 1,2,\ldots n \qquad (7)$$

and

$$U_i(s) = s^{-1} U_{i-1},\ i = 1,2,\ldots n \qquad (8)$$

With $C_0(s) = C(s)$, and $U_0(s) = U(s)$, then equation (6) becomes

$$C_0(s) + \sum_{i=1}^{n} a_i C_i(s) = \sum_{i=1}^{n} b_i U_i(s) \qquad (9)$$

The shifted Legendre series approximation of $c_i(t)$ and $u_i(t)$ are, respectively, represented by

$$c_i(t) = \sum_{j=0}^{\infty} \alpha_{ij}(2j+1)\bar{P}_j = \mathbf{\Lambda}_i^T\,\bar{\mathbf{P}}(t) \qquad (10)$$

$$u_i(t) = \sum_{j=0}^{\infty} \gamma_{ij}(2j+1)\bar{P}_j(t) = \mathbf{\Gamma}_i^T\,\bar{\mathbf{P}}(t) \qquad (11)$$

Where the shifted Legendre spectrum vectors $\mathbf{\Lambda}_i$ and $\mathbf{\Gamma}_i$ are,

$$\mathbf{\Lambda}_i = [\alpha_{i0}, \alpha_{i1}, \ldots\ \alpha_{i,m-1}]^T \qquad (12)$$

$$\mathbf{\Gamma}_i = [\gamma_{i0}, \gamma_{i1}, \ldots\ \gamma_{i,m-1}]^T \qquad (13)$$

From equations (7) and (8), we have

$$c_i(t) = \int_0^t c_{i-1}(t')\,dt' \qquad (14)$$

$$u_i(t) = \int_0^t u_{i-1}(t')\,dt' \qquad (15)$$

Using equations (2), (4), (14), and (15), it can be easily shown that,

$$\mathbf{\Lambda}_i = \mathbf{H}^T \mathbf{\Lambda}_{i-1} \qquad (16)$$

or

$$\alpha_{ij} = \frac{T_f}{2(2j+1)}(\alpha_{i-1,j-1} - \alpha_{i-1,j+1}) \qquad (17)$$

and

$$\left.\begin{aligned} &\mathbf{\Gamma}_i = \mathbf{H}^T \mathbf{\Gamma}_{i-1} \\ &\gamma_{i,j} = \frac{T_f}{2(2j+1)}(\gamma_{i-1,j-1} - \gamma_{i-1,j+1}); \\ &i = 1,2,3,\ldots n \\ &j = 0,1,2,\ldots m-1 \end{aligned}\right\} \qquad (18)$$

Where $\alpha_{i,-1} = \alpha_{i,0}$, $\gamma_{i,-1} = \gamma_{i,0}$ and $\alpha_{i,m} = \gamma_{i,m} = 0$. Substituting equations (10) and (11) into equation (9) and equating the coefficients of $\bar{\mathbf{P}}(t)$, we obtain,

$$\left(\mathbf{\Lambda}_0 + \sum_{i=1}^{n} a_i\,\mathbf{\Lambda}_i\right) = \sum_{i=1}^{n} b_i\,\mathbf{\Gamma}_i \qquad (19)$$

or

$$\left(\mathbf{I} + \sum_{i=1}^{n} a_i\,\mathbf{H}^{T^i}\right)\mathbf{\Lambda}_0 = \sum_{i=1}^{n} b_i\,\mathbf{\Gamma}_i \qquad (20)$$

Therefore, $\mathbf{\Lambda}_0$ can be obtained from above equation,

$$\Lambda_0 = \left(\mathbf{I} + \sum_{i=1}^{n} a_i \mathbf{H}^{T^i}\right)^{-1} \left(\sum_{i=1}^{n} b_i \Gamma_i\right) \quad (21)$$

As indicated in equation (4), **H** is a tridiagonal matrix. The elements of the diagonal are zero except $H(1, 1)$. Therefore, the summation of the polynomial matrix of **H** as shown in equation (21) is very easily to be calculated. Once Λ_0 are calculated, Λ_i can be obtained from the recursive algorithm which is shown in equation (17).

Let a reduced model of the given system as shown in equation (5) be

$$\frac{C^+(s)}{U(s)} = G^+(s) = \frac{b_1^+ s^{r-1} + b_2^+ s^{r-2} + \ldots + b_{r-1}^+ s + b_r^+}{s^r + a_1^+ s^{r-1} + \ldots + a_{r-1}^+ s + a_r^+}; \quad r<n \quad (22)$$

Where a_i^+ and b_i^+ are unknown parameters to be determined.

Using the previous method, equation (22) can be simplified as,

$$\Lambda_0 + \sum_{i=1}^{r} a_i^+ \Lambda_i = \sum_{i=1}^{r} b_i^+ \Gamma_i \quad (23)$$

or

$$\Omega\,\Theta = \Lambda_0 \quad (24)$$

Where the parameter vector Θ is,

$$\Theta = [a_1^+, a_2^+, a_3^+ \ldots a_r^+, b_1^+, b_2^+, b_3^+ \ldots b_r^+]^T \quad (25)$$

and Ω is an $m \times 2r$ matrix given by

$$\Omega = [-\Lambda_1, -\Lambda_2, -\Lambda_3 \ldots -\Lambda_r, \Gamma_1, \Gamma_2, \Gamma_3 \ldots \Gamma_r] \quad (26)$$

Following the argument by Hwang and Shih [7], the parameter a_r^+ and b_r^+ must be chosen to satisfy

$$\frac{b_r^+}{a_r^+} = \frac{b_n}{a_n} \quad (27)$$

which can be written in a matrix form

$$\mathbf{V}^T \Theta = 0 \quad (28)$$

Where **V** is a $2r$ vector as follows:

$$\mathbf{V} = [0, 0, \ldots, b_n, 0, 0, \ldots, -a_n]^T \quad (29)$$

In order to get the unknown coefficient vector, Θ, for best representing the original model, the sum of squares of the algebraic equation errors of equation (24) is a minimum subject to the constraint of equation (28), i.e.,

$$\text{minimize } J = [\Omega\,\Theta - \Lambda_0]^T [\Omega\,\Theta - \Lambda_0] \quad (30)$$

with the adjustable parameter vector Θ and the constraint of equation (28), we have

$$\tilde{J} = (\Omega\,\Theta - \Lambda_0)^T (\Omega\,\Theta - \Lambda_0) + \lambda \mathbf{V}\,\Theta \quad (31)$$

The necessary condition to mimimize J is,

$$\frac{\partial \tilde{J}}{\partial \Theta} = \mathbf{0}$$

which leads to

$$2\Omega^T(\Omega\,\Theta - \Lambda_0) + \lambda \mathbf{V} = 0 \quad (32)$$

Equation (32) together with equation (28) forms a set of linear simultaneous equations which can be solved to give the unknown parameter vector Θ and λ. Thus, a_i^+ and b_i^+, $i = 1, 2, \ldots, r$ are determined.

Example 1: Consider a fourth order system [10]

$$\frac{C(s)}{U(s)} = G(s) = \frac{28s^3 + 496s^2 + 1800s + 2400}{2s^4 + 36s^3 + 204s^2 + 360s + 240} \quad (33)$$

Table 1 a_i^+ **and** b_i^+ **of the reduced model**

$T_f = 5$

	a_1^+	a_2^+	b_1^+	b_2^+
$m=10$	1.42828	0.270364	12.39181	2.70364
$m=15$	1.42490	0.265935	12.39243	2.65935
$m=20$	1.42492	0.265961	12.39243	2.65961

$T_f = 10$

	a_1^+	a_2^+	b_1^+	b_2^+
$m=10$	1.86883	0.859908	12.25560	8.59908
$m=20$	1.86972	0.861142	12.25508	8.61142
$m=32$	1.86971	0.861121	12.25508	8.61121

$T_f = 10$

	a_1^+	a_2^+	b_1^+	b_2^+
$m^*=16$	15.2964	16.8514	17.4487	168.514
$m^*=32$	24.7809	30.2187	12.0232	302.187
$m^*=64$	55.7868	54.5533	12.9676	545.533

*Data were obtained from the work of Hwang and Shih [7].

Table 2 Square root of impulse response energy

Method		$\left[\int_0^\infty g(t)^2 dt\right]^{1/2}$
Shifted Legendre matching	$T_f=5$	7.95111
	$T_f=7.5$	7.95078
	$T_f=10$	7.94930
Laguerre [7] spectra matching	$m=16$	8.06442
	$m=32$	7.99303
	$m=64$	7.99788
Routh approximation	7.63800	
original model	8.03335	

Table 3 Parameters of designing control system

	K_c	τ_i	τ_d
Shifted Legendre method			
$T_f=2.0$, N=10	13.58053	1.32689	0.18438
$T_f=3.0$, N=20	13.68791	1.35229	0.18053
$T_f=4$, N=15	13.75941	1.36651	0.17742
$T_f=5$, N=20	13.80771	1.37507	0.17493
$T_f=5$, N=25	13.80771	1.37507	0.17493
$T_f=10$, N=40	13.90792	1.39033	0.16768
Laguerre method [7]	13.6079	1.3398	0.1841

Let the reduced model be a second-order,

$$\frac{C^+(s)}{U(s)} = G^+(s) = \frac{b_1^+ s + b_2^+}{s^2 + a_1^+ s + a_2^+} \quad (34)$$

Equations (32) and (28) give the results for several values of m and T_f which are shown in Table 1. The results obtained from the Laguerre functions is also shown in Table 1 for comparison.

The comparison of $c(t)$ and $c^+(t)$ with respect to a unit step input, $u(t)$, is shown in Fig. 1. The frequency responses of the original and reduced model are shown in Fig. 2. Other comparisons of the impulse response energy of the original and reduced model and those of the Routh approximation [10] are shown in Table 2. The agreement is very satisfactory, especially for the present proposed method.

Control System Design

The control system design problems are frequently investigated through the techniques of model reduction [6, 9]. Because of the availability of industrial specifications and the easiness to construct the low-order model, the industrial specifications are used to specify a lower order reference model. In this section, a set of PID controller parameters are found by employing the matching of information-bearing shifted Legendre spectra of the response of the reference model and those of the actual system with respect to a specified input. In general, the set of industrial specifications which are used by Chen and Shieh [9] for closed loop system is velocity error constant, k_v, cross-over frequency, ω_c, and damping ratio, ξ. Usually a second order system,

$$G=\frac{4.061s+16.83}{s^2+5.745s+16.83} \tag{35}$$

is able to satisfy these specifications. The above equation is obtained for $k_v = 10$, $\omega_c=5$, and $\xi=0.7$ [7, 9].

The transfer functions of the process, $G_p(s)$, and the proportional-integral derivative controller, $G_c(s)$, are respectively,

$$G_p(s)=\frac{1.6}{s^2+2.8s+1.6} \tag{36}$$

$$G_c(s)=K_c\left(1+\frac{1}{\tau_i s}+\tau_d s\right) \tag{37}$$

Where K_c, τ_i and τ_d are to be determined to meet the specifications. The overall closed-loop transfer function of the control system is given as

$$\frac{C(s)}{U(s)}=G(s)$$

$$=\frac{1.6K_c\left(\tau_d s^2+s+\dfrac{1}{\tau_i}\right)}{s^3+(2.8+1.6K_c\tau_d)s^2+1.6(1+K_c)s+\dfrac{1.6K_c}{\tau_i}} \tag{38}$$

For $u(t)=1$, the series representation of the above equation is,

$$\Lambda_0+(2.8+1.6K_c\tau_d)\Lambda_1+1.6(1+K_c)\Lambda_2 + 1.6\frac{K_c}{\tau_i}\Lambda_3=1.6K_c(\tau_d\Gamma_1+\Gamma_2+1/\tau_i\Gamma_3) \tag{39}$$

K_c, τ_i, and τ_d are then determined by least squares estimate. The computational results are shown in Table 3. The results are very close to the values obtained by Laguerre vector method [7].

Results and Discussion

The main advantage of using shifted Legendre function to approach to model reduction problem is its stable characteristics of the Legendre series. As shown in Table 1, the values of a_1^+, a_2^+, b_1^+, and b_2^+ obtained by the present method approach to a finite value, respectively. In addition, the computational results are insensitive to the value of m chosen. For example, $a_1^+=1.42490$ for $m=15$ and $a_1^+=1.42492$ for $m=20$ by using $T_f=5$. However, one probably cannot receive such good computational results for using Laguerre function method [7]. It is obviously to see the results in Fig. 1, the reduced model for $m=32$ will be better than that of $m=64$ by employing the Hwang and Shih's approach [7].

The other significance of using Legendre function is that the final time of the system can be arbitrarily chosen. Typical results for various values of T_f are also shown in Table 1.

Comparing the frequency responses of the reduced model as shown in Fig. 2, the present proposed reduced model is better than that of the Routh approximation [10]. The computational values obtained from the present method receive a goodness of fit in frequency responses.

In the present paper, a recursive algorithm for computing the expansion coefficients of the shifted Legendre function which is similar to the Laguerre function [7], is introduced. Therefore much computational time is reduced. In addition to calculation of the parameters are simple and straightforward. Therefore, the shifted Legendre function provides an effective tool in the reduction of models.

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