The follower location problem with attraction thresholds*

Rafael Suárez-Vega, Dolores R. Santos-Peñate, Pablo Dorta-González

University of Las Palmas de Gran Canaria, Facultad de Ciencias Económicas y Empresariales, Campus Universitario de Tafira, 35017, Las Palmas de Gran Canaria, Spain
(e-mail: rsuarez@dmc.ulpgc.es)

Received: 23 February 2005 / Accepted: 8 December 2005

Abstract. In this research note, the follower location problem where the customers' choice is based on an attraction function is analysed. The attraction function depends on both the distance between customers and facilities, and the characteristics (quality) of the facilities. Customers at each node impose a minimum level of attraction in order to patronise a facility and then they share their buying power among the facilities that pass this threshold. The amount of demand captured by each of these facilities is proportional to the attraction perceived by the customers. In this case, a discretisation of this network problem is proved.

JEL classification: L10, R30

Key words: competitive location, proportional preferences, consumer choice, medianoid, follower problem

1 Introduction

In this article, we investigate the follower competitive location problem in the leader-follower model. In this model, a firm, the leader, operates in the market with $p \geq 1$ facilities and a competitor, the follower, wants to enter the market opening

* This article has been partially supported by The Ministerio de Educación y Ciencia (Spain) and FEDER (grants BFM2002-04525-C02-01 and MTM2005-09362-C03-03), and by The University of Las Palmas de Gran Canaria (grant UNI2003/25).

$r \geq 1$ new facilities. If the market is represented by a network where any point is a candidate location for a facility, the resulting follower problem is the $(r|X_p)$-medianoid problem formalised by Hakimi (1983).

Many competitive location models studied in the field of Operations Research assume that customers choose the facilities taking the distance as the unique criterion. Nevertheless, this criterion is not very convincing when qualitative differences between the facilities exist and the transport is not difficult. In some papers, such as Eiselt and Laporte (1989b), Eiselt et al. (1989), Peeters and Plastria (1998), Suárez-Vega et al. (2004), and Thill (1992, 2000), the models involve other aspects related to the characteristics of the facilities in addition to the distance between them and the customers, who take a choice according to the attraction or utility that they perceive from the facilities. Useful reviews on competitive location models can be found in Friesz et al. (1988), Eiselt and Laporte (1989a, 1996), Serra and ReVelle (1995), and Plastria (2001).

The attraction perceived by the customers from the facilities has been represented mathematically by an *attraction function* that is increasing of certain attributes such as the facility size, and decreasing of the distance. Different attraction functions, sometimes derived from utility functions, have been defined: multiplicative functions (Huff 1964; Nakanishi and Cooper 1974; Eiselt and Laporte 1988a,b, 1989b; Eiselt et al. 1989; Drezner 1994b, 1998; Plastria 1997), additive functions (Drezner 1994a) and exponential functions (Hodgson 1981). Attraction functions are used to define the customer choice rule, which represents the customer's behaviour and the flows of consumers in the market. Assuming certain customer's preferences, the firms, whose natural objective is the maximisation of the market share or the profit, make location and facility attribute decisions, influencing with their actions the results and strategies of their competitor. This movement of individuals in the market leads to spatial interaction which can be broadly defined as movement or communication over space that results from a decision process (Fotheringham and O'Kelly 1989). A recent review of spatial interaction modelling can be found in Roy and Thill (2004). Previous papers such as those by Williams et al. (1990), and Williams and Kim (1990a,b), explore different location-spatial interaction models.

The first spatial interaction models were gravity models, which assume analogies between the human behaviour and the Newtonian gravity laws. The basic gravity formulation, in which the movement of individuals between two points is inversely proportional to the distance separating these two points, was applied by Reilly (1929, 1931) and Converse (1949) to analyse retail market areas. Later, Huff (1964) proposed an alternative model to overcome certain limitations of the Reilly-type approach. According to this new model, the probability that a customer at i buys at a facility j is given by:

$$P_{ij} = \frac{\frac{a_j}{d_{ij}^{\lambda}}}{\sum_{k=1}^{n} \frac{a_k}{d_{ik}^{\lambda}}},$$

where a_j represents the size of a service centre j, d_{ij} is the distance (or travel time) from demand point i to facility j and λ is a parameter which reflects the effect of the distance on the consumer's behaviour, and whose value is estimated empirically. The quotient a_j / d_{ij}^{λ} can be interpreted as an attraction function representing that the attraction felt by a customer at point i towards facility j is directly proportional to the size of the facility, and inversely proportional to a power of the distance between them. Different probabilistic choice models can be derived from different assumptions about the sampling design, and about the size distribution of large enough errors (Leonardi and Papageorgiou 1992).

The most frequent customer choice rules treated in the related literature are binary and proportional preferences. Customers show binary preferences when they patronise only the most attractive facility. Proportional preferences mean that customers purchase the product from all the facilities operating in the market, and the amount of buying power captured by each facility depends on the attraction exerted on the customers.

The assumption imposing that all the customers behave according to the same choice rule, that is everybody patronise the most attractive facility or everybody shares their buying power among all the facilities in the market, seems too restrictive. To deal with a more realistic model, we propose the proportional choice model with attraction threshold. In this case, we suppose that customers at each demand node have associated a minimum level of attraction in order to patronise a facility, and then they share their buying power among the facilities that pass this threshold. This customer choice rule implies that different preferences can appear in the same scenario. For example, when the attraction threshold is very low, this rule is similar to that of the proportional preferences model. For higher threshold values this procedure coincides with the binary preferences model. For intermediate threshold values, the customers' choices may be different to both the above cases. Note that when different attraction thresholds exist in the same node, the problem can be easily solved defining artificial nodes with their corresponding demand. The attraction threshold term utilised in this article is in some way similar to the constrained choice-set concept used by Thill (1992, 1997, 2000). Since the thresholds reflect the customers' preferences, their values would be estimated by means of a sample survey of the customers' tastes.

In this research note, we analyse a competitive location model on networks assuming proportional preferences with attraction thresholds. Among other results, we prove an optimality node property. The remainder of the note is structured as follows. The model is defined in Sect. 2. Discretisation results for the network location problem are presented in Sect. 3. In Sect. 4, the problem assuming different attraction thresholds at the same node is solved. Section 5 includes some concluding remarks.

2 The model

Let $N(V, E)$ be a weighted network with node set $V = \{v_i\}_{i=1}^{n}$ and edge set E, where each node v has associated a weight $w(v)$ (≥ 0) and each edge $e = [v_i, v_j] \in E$, with

v_i, $v_j \in V$, has associated a length $l(e)$ (≥ 0). Given x_1, $x_2 \in [v_i, v_j]$, the closed segment $[x_1, x_2]$ is the subset of points of $[v_i, v_j]$ between and including x_1 and x_2. The open segment $]x_1, x_2[$ is the set $[x_1, x_2] \backslash \{x_1, x_2\}$ and the semi-open segments $]x_1, x_2[$ and $[x_1, x_2[$ are the sets $]x_1, x_2] = [x_1, x_2] \backslash \{x_1\}$ and $[x_1, x_2[= [x_1, x_2] \backslash \{x_2\}$, respectively. It is assumed that $N(V, E)$ represents a market where $w(v)$ is the demand (or buying power) at node v and $l(e)$ represents the unitary transportation cost along the edge e. For points x, $y \in N(V, E)$, $d(x, y)$ is the length of a shortest path joining x and y.

The attraction felt by customers at node v towards a facility j at $x_j \in N(V, E)$ with quality level a_j is given by:

$$a_{vj} = \frac{a_j}{f_{vj}},$$

where $f_{vj} = f_v(d(v, x_j))$, with $f_v: \Re_0^+ \rightarrow \Re$ as an increasing concave function.

Let $Y_{\mathrm{r}} = (x_1, x_2, \ldots, x_r)$ and $X_p = (x_{r+1}, x_{r+2}, \ldots, x_{r+p})$ be the locations of facilities belonging to the entry firm, F_Y, and the existing firm F_X, respectively, with quality levels $A_r = (a_1, a_2, \ldots, a_r)$ and $A_p = (a_{r+1}, a_{r+2}, \ldots, a_{r+p})$. Let $X_{p+r} = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+p})$ and $A_{p+r} = (a_1, \ldots, a_r, a_{r+1}, \ldots, a_{r+p})$ Demand at v captured by a facility at x_j is denoted by $w_j(v)$, $\forall v \in V$, $0 \leq j \leq p + r$. Suppose that values for A_r, X_{p}, and A_p are given, then the demand captured by the entry firm is given by:

$$W(Y_r) = \sum_{v \in V} w_Y(v),$$

where:

$$w_Y(v) = \sum_{j=1}^{r} w_j(v).$$

For each node v, an attraction threshold τ_v exists such that customers at v patronise facility j at x_j only if $a_j / f_{vj} \geq \tau_v$.

Example 1. Consider the market in Fig. 1 where there exist a demand node v and three facilities with different sizes (inside the boxes), and the digit next to the

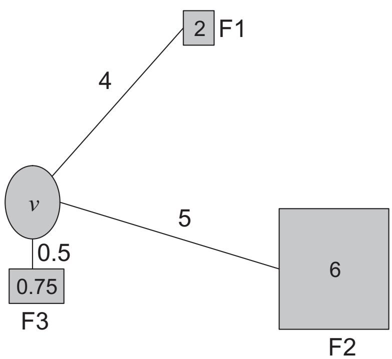

Fig. 1. Network for Example 1

edges shows its length. Consider $f_v(d(v, x_j)) = 1 + d(v, x_j)$, then $a_{v1} = 0.4$, $a_{v2} = 1$, and $a_{v3} = 0.5$. If customers at v are very demanding, for example if $0.5 < \tau_v \leq 1$, they only patronise facility 2 (binary preferences), but if they are not strict, such as $\tau_v \leq 0.4$, then they patronise the three facilities (proportional preferences). But for intermediate thresholds, $0.4 < \tau_v \leq 0.5$, customers at v patronise facility 3, small but the closest, and facility 2, distant but very big. Of course, if at a certain node, customers with different preferences exist, the problem can be solved defining several artificial nodes with their corresponding demand.

Let $\Upsilon = \{\tau_v\}_{v\in V}$ then:

$$S(v, \tau_v, Y_r, A_r) = S_{Y_r}(v) = \left\{ x_j : \frac{a_j}{f_{vj}} \geq \tau_v, 1 \leq j \leq r \right\},$$

is the set of the locations belonging to Y_r that capture demand at v. Thus, the set:

$$S^{-1}(\Upsilon, Y_r, A_r) = S^{-1}_{Y_r} = \left\{ v \in V : \exists\, j \in \{1, \ldots r\} \quad \text{with} \quad x_j \in S_{Y_r}(v) \right\}$$

contains the nodes whose customers patronise a facility belonging to Y_r. These sets are defined in a similar way for X_p and A_p, and X_{p+r} and A_{p+r}. In this model, $w(v)$ is shared among the facilities belonging to the set $S_{X_{p+r}}(v)$, and the market share captured by F_Y is:

$$W(Y_r) = \sum_{v\in S^{-1}_{Y_r}} w(v) \frac{\displaystyle\sum_{x_j \in S_{Y_r}(v)} \frac{a_j}{f_{vj}}}{\displaystyle\sum_{x_j \in S_{X_{p+r}}(v)} \frac{a_j}{f_{vj}}}.$$

3 Discretisation of the problem

In this section we prove that this network problem can be solved evaluating a finite set of network points. To obtain this result the following definition is necessary.

Definition 1. Given a quality level a_j, a point $x \in N(V, E)$ is a (v, τ_v, a_j)-threshold point if:

$$f_v(d(v, x)) = \frac{a_j}{\tau_v}.$$

Note that under the conditions imposed on f_v, this equality is equivalent to:

$$d(v, x) = f_v^{-1}\left(\frac{a_j}{\tau_v}\right).$$

Let $T(v, \tau_v, a_j) = \{x \in N(V, E) : x \text{ is a } (v, \tau_v, a_j) - \text{threshold point}\}$, $T(a_j) = \cup_{v\in V} T(v, \tau_v, a_j)$, and

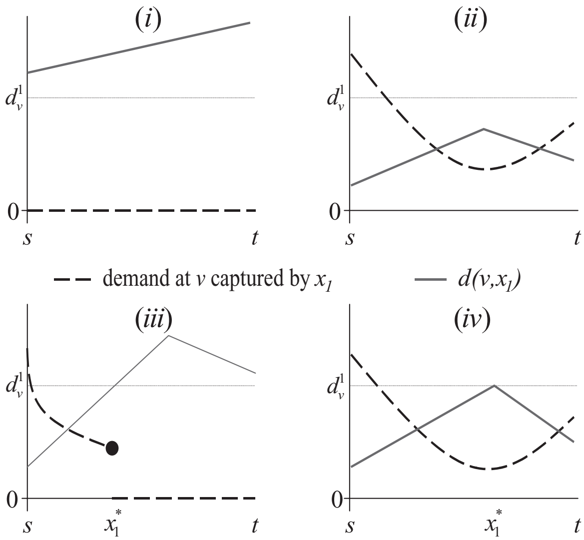

Fig. 2. Market share captured by x_1 along an edge (zero or one threshold points)

$$T = \bigcup_{j=1}^{r} T(a_j).$$

Each element in T is called a threshold point.

Consider the case where F_Y wants to enter the market with only one new facility, $Y_1 = x_1$, with a given quality level a_1. The facility x_1 will capture demand from node v if:

$$f_{v1} \leq \frac{a_1}{\boldsymbol{\tau}_v}.$$

Suppose x_1 is located along the edge $[s,t] \in E$. The amount of demand at node v captured by x_1 along this edge is shown in Figs. 2 and 3.

Let $d_v^1 = f_v^{-1}(a_1/\tau_v)$, then $w_1(v)$ along $[s,t]$ is given by:

$$w_1(v) = \begin{cases} w(v)\dfrac{\dfrac{a_1}{f_{v1}}}{\dfrac{a_1}{f_{v1}} + K_v^1} & \text{if} \quad d(v, x_1) \leq d_v^1, \\ 0 & \text{if} \quad d(v, x_1) > d_v^1, \end{cases}$$

with $K_v^1 = \sum_{x_j \in S_{X_p}(v)} a_j / f_{vj}$

The following cases are possible:

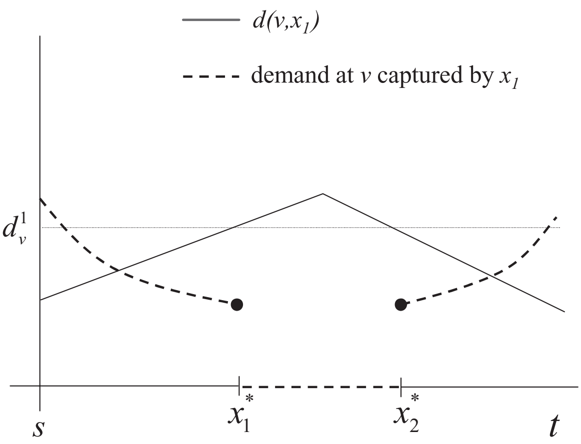

Fig. 3. Market share captured by x_1 along an edge (two threshold points)

1. $d_v^1 < \min\{d(v,s), d(v,t)\}$.
   In this case, x_1 never captures demand from v because $d(v,x_1) > d_v^1, \forall x_1 \in [s,t]$ (Fig. 2 (*i*)).
2. $d_v^1 \geq \min\{d(v,s), d(v,t)\}$
   Suppose that $\min\{d(v,s), d(v,t)\} = d(v,s)$ (the case $\min\{d(v,s), d(v,t)\} = d(v,t)$ is similar). In this case, depending on the number of (v, τ_v, a_1)-threshold points on $[s,t]$, three situations may occur.
   (a) There is no (v, τ_v, a_1)-threshold point on $[s,t]$.
   As $d(v,x_1) < d_v^1, \forall x_1 \in [s,t]$, every point $x_1 \in [s,t]$ captures demand at v (Fig. 2 (*ii*)).
   (b) A unique (v, τ_v, a_1)-threshold point exists on $[s,t]$.
   Let $x_1^* \in [s,t]$, such that $d\left(v, x_1^*\right) = d_v^1$.
   i. If x_1^* is not a bottleneck, every point $x_1 \in [s, x_1^*]$ captures demand at v (Fig. 2 (*iii*)). The capture for $x \in]x_1^*, t]$ is null.
   ii. If x_1^* is a bottleneck, every point $x_1 \in [s,t]$ captures demand at v (Fig. 2 (*iv*)).
   (c) Two (v, τ_v, a_1)-threshold points exist on $[s,t]$.
   If x_1^* and x_2^* are the threshold points, then points $x_1 \in [s, x_1^*] \cup [x_2^*, t]$ capture demand at v (Fig. 3).

For each edge $[s,t] \in E$, consider the partition generated by s, t and the (v, τ_v, a_1)-threshold points on $]s,t[$ with $v \in V$. The demand captured by the facility at x_1 is a sum of functions, as are the ones plotted in Figs. 2 and 3, and then it is a convex function when x_1 moves along each subinterval of the partition. Therefore the maximum capture is reached at a subinterval endpoint, that is to say, at a node or at a threshold point.

Proposition 2. *Given X_p, A_p, and A_r, let $Y_r = (x_1, \dots, x_j, \dots, x_r)$. Then the set $S_{Y_r}^{-1}$ is constant when x_j varies on $]s,t[$ with $]s,t[\cap T(a_j) = \emptyset$ and $\{s,t\} \subset V \cup T(a_j)$.*

Proof. Suppose, without loss of generality, that $j = 1$. If $S^{-1}(x_1) = S_{Y_r}^{-1}$ is not constant when x_1 varies on $]s,t[$, then there exist $x_1', x_1'' \in]s,t[$ such that $S^{-1}(x_1') \neq S^{-1}(x_1'')$. Thus $v \in S^{-1}(x_1') \setminus S^{-1}(x_1'')$ exists (or $v \in S^{-1}(x_1'') \setminus S^{-1}(x_1')$), therefore:

$$\frac{a_1}{f_v(d(v, x_1''))} < \tau_v < \frac{a_1}{f_v(d(v, x_1'))}.$$

As $f_v(d(v, x_1))$ is a continuous and increasing function of x_1 on $]s, t[$, it follows that:

$$d(v, x_1') < f_v^{-1}\left(\frac{a_1}{\tau_v}\right) < d(v, x_1''),$$

and then there exists a point $z \in]x_1', x_1''[\subseteq]s,t[$ such that $d(v, z) = f_v^{-1}(a_1/\tau_v)$ which means that z is a (v, τ_v, a_1)-threshold point, and this is not possible.

Lemma 3. *If g is non-decreasing and (strictly) concave and f is concave, then the composite function $g \circ f$ is (strictly) concave.*

Lemma 4. *If g is non-increasing and (strictly) convex and f is concave, then the composite function $g \circ f$ is (strictly) convex.*

Proposition 5. *Given X_p, A_p, A_r and the edge $[s,t]$, let $\{y_k\}_{k=1}^{q} =]s,t[\cap T$, ordered by increasing value of the distance to node s. Let $y_0 = s$ and $Y_{q+1} = t$. Then the function $W(x_{j_0}) = W(Y_r)$ with $Y_r = (x_1, \dots, x_0, \dots, X_r)$, is convex on $]y_k, y_{k+1}[$, $k = 0, 1, \dots, q$.*

Proof. Suppose, without loss of generality, that $j_0 = 1$ and $x_1 \in]y_k, y_{k+1}[$. The demand captured by firm F_Y is:

$$W(Y_r) = \sum_{v \in S_{Y_r}^{-1}} w(v) \frac{\sum_{x_j \in S_{Y_r}(v)} \frac{a_j}{f_{vj}}}{\sum_{x_j \in S_{Y_r}(v)} \frac{a_j}{f_{vj}} + \sum_{x_j \in S_{X_p}(v)} \frac{a_j}{f_{vj}}}.$$

Let $K_{vX} = \sum_{x_j \in S_X(v)} \frac{a_j}{f_{vj}}$, and $K'_{vX} = \sum_{x_j \in (S_X(v) \setminus \{x_1\})} \frac{a_j}{f_{vj}}$, $\forall v \in V$, $X = X_p, Y_r$, then, $W(Y_r) = \sum_{v \in S_{Y_r}^{-1}} w(v) \frac{K_{vY_r}}{K_{vY_r} + K_{vX_p}}$.

Let $K_{vX} = \sum_{x_j \in S_X(v)} \frac{a_j}{f_{vj}}$, and $K'_{vX} = \sum_{x_j \in (S_X(v) \setminus \{x_1\})} \frac{a_j}{f_{vj}}$, $\forall v \in V, X = X_p, Y_r$, then, $W(Y_r) = \sum_{v \in S^{-1}_{Y_r}} w(v) \frac{K_{vY_r}}{K_{vY_r} + K_{vX_p}}$.

If $x_1 \notin S_{Y_r}(v)$, then $\frac{K_{vY_r}}{K_{vY_r} + K_{vX_p}}$ does not depend on x_1.

If $x_1 \in S_{Y_r}(v)$, then $\frac{K_{vY_r}}{K_{vY_r} + K_{vX_p}} = \frac{\frac{a_1}{f_{v1}} + K'_{vY_r}}{\frac{a_1}{f_{v1}} + K'_{vY_r} + K_{vX_p}} = \frac{a_1 + f_{v1} K'_{vY_r}}{a_1 + f_{v1}\left(K'_{vY_r} + K_{vX_p}\right)}$.

Since $d(v, x_1)$ is a concave function of x_1 on $]y_k, y_{k+1}[$ and f_v is increasing and concave, from Lemma 3, $f_{v1} = f_v(d(v, x_1))$ is concave, and $K_{vY_r} / K_{vY_r} + K_{vX_p}$ is convex $\left(\text{applying Lemma 4 to } (g \circ f_{v1}) \text{ with } g(x) = \left(a_1 + xK'_{vY_r}\right) / a_1 + x\left(K'_{vY_r} + K_{vX_p}\right)\right)$. Therefore, $W(Y_r)$ is a convex function of x_1 on $]y_k, y_{k+1}[$.

Proposition 6. *Given X_p, let A_p, A_r, let $Y_r = (x_1, \ldots, x_j, \ldots, x_r)$ and consider the function $W(x_j) = W(Y_r)$ with $x_j \in [s,t]$, such that $]s,t[\cap T(a_j) = \emptyset$ and $\{s,t\} \subset V \cup T(a_j)$, then the maximum of $W(x_j)$* on $[s,t]$ *is reached at either s or t.*

Proof. Suppose, without loss of generality, that $j = 1$. Three different cases may occur:

1. If $\{s,t\} \cap T(a_1) = \emptyset$, function $W(x_1)$ is continuous on $[s,t]$ and as $W(x_1)$ is convex on $]s,t[$, it follows that:
$$\max_{x_1 \in [s,t]} W(x_1) = \max\{W(s), W(t)\}.$$
2. If $s \in T(a_1)$ (the case for t is analogous), then there exists $v_0 \in V$ such that s is a $\left(v_0, \tau_{v_0}, a_1\right)$ – threshold point, that is $d(v_0, s) = f^{-1}_{v_0}\left(a_1 / f_{v_0 1}\right)$. Let $H = \{v \in V :$ $\exists j = 1 \ldots, r$ with s a (v, τ_v, a_1) – threshold point$\}$.

As $\{s,t\} \cap T(a_1) = \emptyset$, for each node $v_0 \in H$ only two situations can occur:
   (a) If $d(v_0, s) > d(v_0, s + \delta)$, with $\delta > 0$ (sufficiently small), then function $w_1(v_0)$ (the amount of $w_1(v_0)$ captured by x_1) is continuous in $[s,t]$ because node v_0 patronises facility x_1, $\forall x_1 \in [s,t]$.
   (b) If $d(v_0, s) < d(v_0, s + \delta)$, with $\delta > 0$ (sufficiently small), then $w_1(v_0)$ presents a discontinuity at point s such that customers at node v_0 patronise facility x_1 if it is located at point s but not to the right of s.

Let H define the following subsets:

$$H^{>} = \{v_0 \in H : d(v_0, s) > d(v_0, s+\delta), \delta > 0\},$$
$$H^{<} = \{v_0 \in H : d(v_0, s) < d(v_0, s+\delta), \delta > 0\}.$$

If $H^{<} = \emptyset$ then $W(x_1)$ is continuous and convex on $[s,t]$ and this situation is similar to case (i). Otherwise, $W(x_1)$ is convex on $]s,t]$ and it holds that $W(s) > W(s+\delta)$ with $\delta > 0$ sufficiently small, since $x_1 = s + \delta$ losses all the capture from nodes belonging to $H^{<}$. Therefore:

$$\max_{x_1 \in [s,t]} W(x_1) = \max\{W(s), W(t)\}.$$

Proposition 7. *Given* X_p, A_p *and* A_r, *there exists an r-tupla* $Y_r^* = \left(x_1^*, x_2^*, \ldots, x_r^*\right) \in N(V, E)^r$, *with* $x_j^* \in V \cup T\left(a_j\right)$, *such that*:

$$W\left(Y_r^*\right) = \max_{Y_r \in N(V,E)^r} W(Y_r).$$

Proof. It follows from Proposition 6.

As on each edge, at most two (v, τ_v, a_j)-threshold points for each node v exist, the maximum number of candidates to locate a new facility is $2|V||A| + |V|$. If:

$$\tau_v \le \min\left\{\frac{a_j}{f_{ik}} : a_j \in A_{p+r}, v_i, v_k \in V\right\}, \forall v \in V,$$

then all points belonging to x_{p+r} capture part of the demand at v, $\forall v \in V$. In this case, the model coincides with the proportional $(r|X_p)$-medianoid problem studied by Hakimi (1990).

4 Considering nodes with different threshold levels

In this section, we present an example that shows the possible applications of this model to the case where nodes with different attraction thresholds exist. Suppose that there exists a market where the leader has a facility operating, and the follower wants to enter the market with a new centre. This new facility must compete with the existing one to maximise the market share captured. Each facility, new or existing, is characterised by its quality level, and exerts an attraction to the demand nodes that is directly proportional to its quality level, and inversely proportional to an increasing function of the distance between facility and customers. In this note, we suppose that customers at each demand node have associated a minimum level of attraction in order to patronise a facility, and then they share their buying power among all the facilities that pass this threshold.

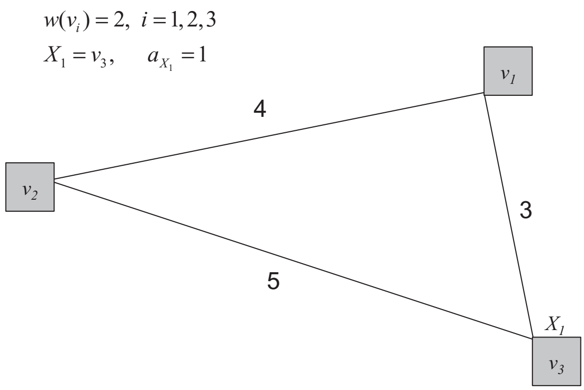

Fig. 4. Network used in Example 2

In general, the following assumption can be considered:

1. Each node v_i contains customers with K different threshold levels, τ_{ik}, $k = 1 \ldots,$ K, such that $w(v_i) = \Sigma_{k=1}^{K} w^k(v_i)$, with $w^k(v_i)$ the buying power of customers located at v_i with threshold level τ_{ik}.
2. To solve the problem for every node v_i, K artificial nodes v_{ik} with buying power $w^k(v_i)$, are considered.

Example 2. Consider the market proposed in Fig. 4. In this market there exist three demand points located at the nodes of the network with a buying power of two units ($w(v_i) = 2$, $i = 1, 2, 3$). An existing facility is located at node v_3 with a quality level equal to one, that is $X_1 = v_3$, $a_{X_1} = 1$. Suppose that there exist two types of customers with different threshold levels at each node, half of them with a threshold value of 1/6 and the other half with a threshold value of 1/4.

Suppose now, that a competing firm wants to enter a new facility into the market whose quality level is equal to one $(a_{Y_1} = 1)$, and it has to decide the facility location that maximises the market share. To take into account the different threshold levels, three artificial nodes are considered; see Fig. 5 for the correspondence between real and artificial nodes. Note that when only the leader is operating, its facility captures all the demand apart from the one belonging to v_5 (the most demanding customers at v_2). As pointed out in Proposition 7, a solution to the network problem of locating the follower's facility can be found evaluating the nodes and the threshold points of the network. In this case, the threshold points obtained for customers at node v_1 with threshold 1/6 when $a_{Y_1} = 1$ are:

$$T\left(v_1, \frac{1}{6}, 1\right) = \left\{x \in N(V, E) : \frac{1}{1 + d(v_1, x)} = \frac{1}{6}\right\} = \{x \in N(V, E) : d(v_1, x) = 5\} = \{x_1, x_2\}.$$

See Fig. 5 for the physical location of these points. Note that, at most, two threshold points may exist in the arc $[v_i, v_j]$ because $d(v_1, x) = 5$ can be reached both

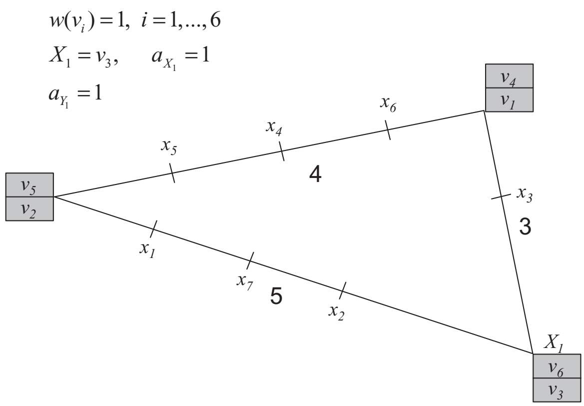

Fig. 5. Network for Example 2 with the artificial nodes

Table 1. Data for the nodes of Example 2

Node	$w(v_i)$	τ_{v_i}	$T\left(v_i, \tau_{v_i}, 1\right)$
v_1	1	1/6	$\{x_1, x_2\}$
v_2	1	1/6	$\{v_3, x_3\}$
v_3	1	1/6	$\{v_2, x_4\}$
v_4	1	1/4	$\{v_3, x_5\}$
v_5	1	1/4	$\{x_2, x_6\}$
v_6	1	1/4	$\{v_1, x_7\}$

through v_i and v_j. The rest of the threshold points are calculated in the same way. Table 1 summarises the buying power, the threshold level, and the threshold points for each demand node. Fig. 5 shows the potential locations for the problem solution. Evaluating these points, we obtain the solution that maximises the market share to be $Y_1 = x_6$, with $W(Y_1) = 3.1$. The market share obtained at each possible location, and the nodes that would patronise the new facility in each case are presented in Table 2.

Fig. 6 shows the demand distribution pattern for the problem solution. Note that customers at node v_6 do not patronise facility Y_1, whereas customers at v_5 only patronise the new facility. The rest of the nodes share their demand, proportionally to the attraction perceived, between the two facilities operating in the market.

5 Conclusions

In this research note, the follower location problem on networks with attraction threshold is analysed. The amount of node demand that is captured by a facility depends on the attraction that the facility exerts towards customers at that node.

Table 2. Results obtained for the potential locations

Location	$W(Y_1)$	Captured nodes
v_1	2.545	v_1, v_2, v_3, v_4, v_6
v_2	2.746	v_1, v_2, v_4, v_5
v_3	2.500	v_1, v_2, v_3, v_4, v_6
x_1	2.316	v_1, v_2, v_3, v_5
x_2	2.433	v_1, v_2, v_3, v_5, v_6
x_3	2.264	v_1, v_2, v_3, v_4, v_6
x_4	2.952	v_1, v_2, v_3, v_4, v_5
x_5	2.750	v_1, v_2, v_4, v_5
x_6	3.100	v_1, v_2, v_3, v_4, v_5
x_7	2.066	v_2, v_3, v_5, v_6

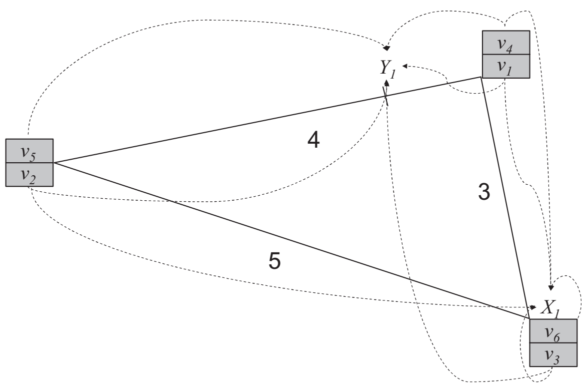

Fig. 6. Demand distribution pattern

This attraction is a function of the facility quality level, and the distance between facility and demand node. While most of the related models consider the same choice rule for all the customers in the market, in this article a differentiation among the consumers' preferences is included. In this case, it is assumed that customers located at each node impose a minimum level of attraction in order to patronise a facility, and then they share their buying power among the facilities that pass this threshold. This implies that customers at certain nodes may present binary preferences (high attraction thresholds), proportional preferences (low attraction thresholds) or a behaviour that combines these two preferences (intermediate thresholds).

A discretisation result for the proposed model has been proved. Taking into account this result, an example has been solved to illustrate the applicability of this customer choice rule. Although a single facility problem is studied in the example, the theoretical results presented in the article can be used to solve the multi-facility problem. Nevertheless, when the number of new facilities increases, the complexity of the problem may hinder the exhaustive search of the solution and the use of combinatorial tools would be necessary.

References

Converse PD (1949) New laws of retail gravitation. *Journal of Marketing* 14: 379–384

Drezner T (1994a) Locating a single new facility among existing unequally attractive facilities. *Journal of Regional Science* 34(2): 237–252

Drezner T (1994b) Optimal continuous location of a retail facility, facility attractiveness, and market share: an interactive model. *Journal of Retailing* 70(1): 49–64

Drezner T (1998) Location of multiple retail facilities with limited budget constraints in continuous space. *Journal of Retailing and Consumer Services* 5(3): 173–184

Eiselt HA, Laporte G (1988a) Location of a new facility in the presence of weights. *Asia-Pacific Journal of Operational Research* 5: 160–165

Eiselt HA, Laporte G (1988b) Trading areas of facilities with different sizes. *Recherche Opérationnelle/Operations Research* 22(1): 33–44

Eiselt HA, Laporte G (1989a) Competitive spatial models. *European Journal of Operational Research* 39: 231–242

Eiselt HA, Laporte G (1989b) The Maximum Capture Problem in a weighted network. *Journal of Regional Science* 29(3): 433–439

Eiselt HA, Laporte G (1996) Sequential location problems. *European Journal of Operational Research* 96: 217–231

Eiselt HA, Laporte G, Pederzoli G (1989) Optimal sizes of facilities on a linear market. *Mathematical and Computer Modelling* 12(1): 97–103

Fotheringham AS, O'Kelly ME (1989) Spatial interaction models: Formulations and applications. Kluwer, Dordrecht

Friesz TL, Miller T, Tobin RL (1988) Competitive network facility location models: a survey. *Papers of the Regional Science Association* 65: 47–57

Hakimi SL (1983) On locating new facilities in a competitive environment. *European Journal of Operational Research* 12: 29–35

Hakimi SL (1990) Location with spatial interactions: Competitive locations and games. In Mirchandani, PB, Francis RL (eds) *Discrete Location Theory*. John Wiley & Sons, New York

Hodgson MJ (1981) A location-allocation model maximizing consumers' welfare, *Regional Studies* 15(6): 493–506

Huff DL (1964) Defining and estimating a trading area. *Journal of Marketing* 28: 34–38

Leonardi G, Papageorgiou YY (1992) Conceptual foundations of spatial choice models. *Environment and Planning A* 24: 1393–1408

Nakanishi M, Cooper LG (1974) Parameter estimation for a Multiplicative Compeitive Interaction Model-Least Squares approach. *Journal of Marketing Research* 11: 303–311

Peeters PH, Plastria F (1998) Discretization results for the Huff and Pareto-Huff competitive location models on networks. *Top* 6(2): 247–260

Plastria F (1997) Profit maximising single competitive facility location in the plane. *Studies in Locational Analysis* 11: 115–126

Plastria F (2001) Static competitive facility location: An overview of optimization approaches. *European Journal of Operational Research* 129: 461–470

Reilly WJ (1929) *The law of retail gravitation*. New York, Knickerbocker Press

Reilly WJ (1931) *The law of retail gravitation*. New York, Knickerbocker Press

Roy JR, Thill JC (2004) Spatial interaction modeling. *Papers in Regional Science* 83: 339–361

Serra D, ReVelle C (1995) Competitive location in discrete space. In Drezner Z (ed) *Facility location: A survey of applications and methods*. Springer, Berlín

Suárez-Vega R., Santos-Peñate DR, Dorta-González P. (2004) Discretization and resolution of the $(r|X_p)$-medianoid problem involving quality criteria. *Top* 12(1): 111–133

Thill JC (1992) Competitive strategies for multi-establishment firms. *Economic Geography* 68(3): 290–309

Thill JC (1997) Multi-outlet firms, competition and market segmentation strategies *Regional Science and Urban Economics* 27: 67–86

Thill JC (2000) Network competition and branch differentiation with consumer heterogeneity. *The Annals of Regional Science* 34: 451–468

Williams HCWL, Kim KS (1990a) Location-spatial interaction models: 2 Competition between independent firms. Environment and Planning A 22: 1155–1168

Williams HCWL, Kim KS (1990b) Location-spatial interaction models: 3. Competition between organizations. Environment and Planning A 22: 1281–1290

Williams HCWL, Kim KS, Martin D (1990) Location-spatial interaction models: 1. Benefit-maximizating configurations of services. Environment and Planning A 22: 1079–1089