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Research Article

Multilinear Singular and Fractional Integral Operators on Weighted Morrey Spaces

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We will study the boundedness properties of multilinear Calderón-Zygmund operators and multilinear fractional integrals on products of weighted Morrey spaces with multiple weights.

1. Introduction and Main Results

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [1] and was later systematically studied by Grafakos and Torres in [2–4]. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the set of all tempered distributions on $\mathbb{R}^n$. Let $m \ge 2$ and T be an m-linear operator initially defined on the m-fold product of Schwartz spaces, and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \quad (1)$$

Following [2], for given $\vec{f} = (f_1, \ldots, f_m)$, we say that T is an m-linear Calderón-Zygmund operator if, for some $q_1, \ldots, q_m \in [1, \infty)$ and $q \in (0, \infty)$ with $1/q = \sum_{k=1}^{m} 1/q_k$, it extends to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ and if there exists a kernel function $K(x, y_1, \ldots, y_m)$ in the class m-$CZK(A, \varepsilon)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ such that

$$\begin{aligned} T(\vec{f})(x) &= T(f_1, \ldots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m, \end{aligned} \quad (2)$$

whenever $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin \bigcap_{k=1}^{m} \operatorname{supp} f_k$. We say that $K(x, y_1, \ldots, y_m)$ is a kernel in the class m-$CZK(A, \varepsilon)$ if it satisfies the size condition

$$|K(x, y_1, \ldots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}, \quad (3)$$

for some $A > 0$ and all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_k$ for some $1 \le k \le m$. Moreover, for some $\varepsilon > 0$, it satisfies the regularity condition that

$$\begin{aligned} &|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \\ &\quad \le \frac{A \cdot |x - x'|^{\varepsilon}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}} \end{aligned} \quad (4)$$

whenever $|x - x'| \le (1/2)\max_{1 \le k \le m}|x - y_k|$ and also that, for each fixed k with $1 \le k \le m$,

$$\begin{aligned} &|K(x, y_1, \ldots, y_k, \ldots, y_m) - K(x, y_1, \ldots, y_k', \ldots, y_m)| \\ &\quad \le \frac{A \cdot |y_k - y_k'|^{\varepsilon}}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon}} \end{aligned} \quad (5)$$

whenever $|y_k - y_k'| \le (1/2)\max_{1 \le i \le m}|x - y_i|$. In recent years, many authors have been interested in studying the boundedness of these operators on function spaces; see, for example, [5–8]. In 2009, the weighted strong and weak type estimates of multilinear Calderón-Zygmund singular integral operators were established in [9] by Lerner et al. New more refined multilinear maximal function was defined and used in [9] to characterize the class of multiple $A_{\vec{P}}$ weights.

Theorem A (see [9]). *Let $m \geq 2$ and T be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in (1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$ and $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P}}$ condition, then there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|T\left(\vec{f}\right)\right\|_{L^p(\nu_{\vec{w}})} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \tag{6}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem B (see [9]). *Let $m \geq 2$, and let T be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in [1,\infty)$, $\min\{p_1, \ldots, p_m\} = 1$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P}}$ condition, then there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|T\left(\vec{f}\right)\right\|_{WL^p(\nu_{\vec{w}})} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \tag{7}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Let $m \geq 2$, and let $0 < \alpha < mn$. For given $\vec{f} = (f_1, \ldots, f_m)$, the m-linear fractional integral operator is defined by

$$\begin{aligned} I_\alpha\left(\vec{f}\right)(x) &= I_\alpha\left(f_1, \ldots, f_m\right)(x) \\ &= \int_{(\mathbb{R}^n)^m} \frac{f_1\left(y_1\right)\cdots f_m\left(y_m\right)}{\left|\left(x-y_1, \ldots, x-y_m\right)\right|^{mn-\alpha}} dy_1 \cdots dy_m. \end{aligned} \tag{8}$$

For the boundedness properties of multilinear fractional integrals on various function spaces, we refer the reader to [10–16]. In 2009, Moen [17] considered the weighted norm inequalities for multilinear fractional integral operators and constructed the class of multiple $A_{\vec{P},q}$ weights (see also [18]).

Theorem C (see [17, 18]). *Let $m \geq 2$, $0 < \alpha < mn$, and let I_α be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in (1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$ and $1/q = 1/p - \alpha/n$, and $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P},q}$ condition, then there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|I_\alpha\left(\vec{f}\right)\right\|_{L^q((\nu_{\vec{w}})^q)} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}, \tag{9}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

Theorem D (see [17, 18]). *Let $m \geq 2$, $0 < \alpha < mn$, and let I_α be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in [1,\infty)$, $\min\{p_1, \ldots, p_m\} = 1$, $1/p = \sum_{k=1}^m 1/p_k$ and $1/q = 1/p - \alpha/n$, and $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P},q}$ condition, then there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|I_\alpha\left(\vec{f}\right)\right\|_{WL^q((\nu_{\vec{w}})^q)} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}, \tag{10}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

On the other hand, the classical Morrey spaces $\mathscr{L}^{p,\lambda}$ were originally introduced by Morrey in [19] to study the local behavior of solutions to second-order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator, and the Calderón-Zygmund singular integral operator on these spaces, we refer the reader to [20–22]. For the properties and applications of classical Morrey spaces, one can see [23–25] and the references therein.

In 2009, Komori and Shirai [26] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces and studied the boundedness of the above classical operators in Harmonic Analysis on these weighted spaces. Recently, in [27–34], we have established the continuity properties of some other operators and their commutators on the weighted Morrey spaces $L^{p,\kappa}(w)$.

The main purpose of this paper is to establish the boundedness properties of multilinear Calderón-Zygmund operators and multilinear fractional integrals on products of weighted Morrey spaces with multiple weights. We now formulate our main results as follows.

Theorem 1. *Let $m \geq 2$, and let T be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in (1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$ and $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ with $w_1, \ldots, w_m \in A_\infty$, then for any $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|T(\vec{f})\right\|_{L^{p,\kappa}(\nu_{\vec{w}})} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)}, \tag{11}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem 2. *Let $m \geq 2$ and T be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in [1,\infty)$, $\min\{p_1, \ldots, p_m\} = 1$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ with $w_1, \ldots, w_m \in A_\infty$, then for any $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\|T\left(\vec{f}\right)\right\|_{WL^{p,\kappa}(\nu_{\vec{w}})} \leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)}, \tag{12}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem 3. *Let $m \geq 2$, let $0 < \alpha < mn$, and let I_α be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in (1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q_k = 1/p_k - \alpha/mn$ and $1/q = \sum_{k=1}^m 1/q_k = 1/p - \alpha/n$, and $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P},q}$ with*

$w_1^{q_1},\dots,w_m^{q_m} \in A_\infty$, then for any $0<\kappa<p/q$, there exists a constant $C>0$ independent of $\vec{f}=(f_1,\dots,f_m)$ such that

$$\left\|I_\alpha(\vec{f})\right\|_{L^{q,\kappa q/p}((\nu_{\vec{w}})^q)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}, \quad (13)$$

where $\nu_{\vec{w}}=\prod_{i=1}^m w_i$.

Theorem 4. *Let $m\ge 2$, let $0<\alpha<mn$, and let I_α be an m-linear fractional integral operator. If $p_1,\dots,p_m\in[1,\infty)$, $\min\{p_1,\dots,p_m\}=1$, $1/p=\sum_{k=1}^m 1/p_k$, $1/q_k=1/p_k-\alpha/mn$ and $1/q=\sum_{k=1}^m 1/q_k=1/p-\alpha/n$, and $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{P},q}$ with $w_1^{q_1},\dots,w_m^{q_m}\in A_\infty$, then, for any $0<\kappa<p/q$, there exists a constant $C>0$ independent of $\vec{f}=(f_1,\dots,f_m)$ such that*

$$\left\|I_\alpha(\vec{f})\right\|_{WL^{q,\kappa q/p}((\nu_{\vec{w}})^q)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}, \quad (14)$$

where $\nu_{\vec{w}}=\prod_{i=1}^m w_i$.

2. Notations and Definitions

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [35]. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$; $B=B(x_0,r_B)$ denotes the ball with the center x_0 and radius r_B. For $1<p<\infty$, a weight function w is said to belong to A_p if there is a constant $C>0$ such that, for every ball $B\subseteq\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1}\le C, \quad (15)$$

where $|B|$ denotes the Lebesgue measure of B. For the case $p=1$, $w\in A_1$, if there is a constant $C>0$ such that for every ball $B\subseteq\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B w(x)\,dx \le C\cdot \operatorname*{ess\,inf}_{x\in B} w(x). \quad (16)$$

A weight function $w\in A_\infty$ if it satisfies the A_p condition for some $1<p<\infty$. We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [36]. A weight function w belongs to $A_{p,q}$ for $1<p<q<\infty$ if there is a constant $C>0$ such that, for every ball $B\subseteq\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{1/p'}\le C. \quad (17)$$

When $p=1$, w is in the class $A_{1,q}$ with $1<q<\infty$ if there is a constant $C>0$ such that, for every ball $B\subseteq\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{1/q}\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)}\right)\le C. \quad (18)$$

Now let us recall the definitions of multiple weights. For m exponents $p_1,\dots,p_m$, we will write $\vec{P}$ for the vector $\vec{P}=(p_1,\dots,p_m)$. Let $p_1,\dots,p_m\in[1,\infty)$, and let $p\in(0,\infty)$ with $1/p=\sum_{k=1}^m 1/p_k$. Given $\vec{w}=(w_1,\dots,w_m)$, set $\nu_{\vec{w}}=\prod_{i=1}^m w_i^{p/p_i}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if it satisfies

$$\sup_B\left(\frac{1}{|B|}\int_B \nu_{\vec{w}}(x)\,dx\right)^{1/p}\prod_{i=1}^m\left(\frac{1}{|B|}\int_B w_i(x)^{1-p_i'}dx\right)^{1/p_i'}<\infty. \quad (19)$$

When $p_i=1$, $((1/|B|)\int_B w_i(x)^{1-p_i'}\,dx)^{1/p_i'}$ is understood as $(\inf_{x\in B}w_i(x))^{-1}$.

Let $p_1,\dots,p_m\in[1,\infty)$, let $1/p=\sum_{k=1}^m 1/p_k$, and let $q>0$. Given $\vec{w}=(w_1,\dots,w_m)$, set $\nu_{\vec{w}}=\prod_{i=1}^m w_i$. We say that $\vec{w}$ satisfies the $A_{\vec{P},q}$ condition if it satisfies

$$\sup_B\left(\frac{1}{|B|}\int_B \nu_{\vec{w}}(x)^q\,dx\right)^{1/q}\times\prod_{i=1}^m\left(\frac{1}{|B|}\int_B w_i(x)^{-p_i'}dx\right)^{1/p_i'}<\infty. \quad (20)$$

When $p_i=1$, $((1/|B|)\int_B w_i(x)^{-p_i'}dx)^{1/p_i'}$ is understood as $(\inf_{x\in B}w_i(x))^{-1}$.

Given a ball B and $\lambda>0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and the weighted measure of E by $w(E)$, where $w(E)=\int_E w(x)dx$.

Lemma 5 (see [37]). *Let $w\in A_p$ with $1\le p<\infty$. Then, for any ball B, there exists an absolute constant $C>0$ such that*

$$w(2B)\le Cw(B). \quad (21)$$

Lemma 6 (see [38]). *Let $w\in A_\infty$. Then for all balls $B\subseteq\mathbb{R}^n$, the following reverse Jensen inequality holds:*

$$\int_B w(x)\,dx\le C|B|\cdot\exp\left(\frac{1}{|B|}\int_B\log w(x)\,dx\right). \quad (22)$$

Lemma 7 (see [37]). *Let $w\in A_\infty$. Then, for all balls B and all measurable subsets E of B, there exists $\delta>0$ such that*

$$\frac{w(E)}{w(B)}\le C\left(\frac{|E|}{|B|}\right)^\delta. \quad (23)$$

Lemma 8 (see [9]). *Let $p_1,\dots,p_m\in[1,\infty)$, and let $1/p=\sum_{k=1}^m 1/p_k$. Then $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{P}}$ if and only if*

$$\nu_{\vec{w}}\in A_{mp},\qquad w_i^{1-p_i'}\in A_{mp_i'},\quad i=1,\dots,m, \quad (24)$$

where $\nu_{\vec{w}}=\prod_{i=1}^m w_i^{p/p_i}$ and the condition $w_i^{1-p_i'}\in A_{mp_i'}$ in the case $p_i=1$ is understood as $w_i^{1/m}\in A_1$.

Lemma 9 (see [17, 18]). *Let* $0 < \alpha < mn$, *and* $p_1, \dots, p_m \in [1,\infty)$, *let* $1/p = \sum_{k=1}^m 1/p_k$, *and let* $1/q = 1/p - \alpha/n$. *If* $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P},q}$, *then*

$$\begin{gathered}(\nu_{\vec{w}})^q \in A_{mq}, \\ w_i^{-p_i'} \in A_{mp_i'}, \quad i = 1, \dots, m,\end{gathered} \tag{25}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

Given a weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is defined as the set of all functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty. \tag{26}$$

We also denote by $WL^p(w)$ the weighted weak space consisting of all measurable functions f such that

$$\|f\|_{WL^p(w)} = \sup_{\lambda>0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty. \tag{27}$$

In 2009, Komori and Shirai [26] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 \le p < \infty$. In order to deal with the multilinear case $m \ge 2$, we will define $L^{p,\kappa}(w)$ for all $0 < p < \infty$.

Definition 10. Let $0 < p < \infty$, let $0 < \kappa < 1$, and let w be a weight function on $\mathbb{R}^n$. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \left\{f \in L^p_{\mathrm{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\right\}, \tag{28}$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x)\,dx\right)^{1/p} \tag{29}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 11. Let $0 < p < \infty$, let $0 < \kappa < 1$, and let w be a weight function on $\mathbb{R}^n$. Then the weighted weak Morrey space is defined by

$$WL^{p,\kappa}(w) = \left\{f \text{ measurable} : \|f\|_{WL^{p,\kappa}(w)} < \infty\right\}, \tag{30}$$

where

$$\|f\|_{WL^{p,\kappa}(w)} = \sup_B \sup_{\lambda>0} \frac{1}{w(B)^{\kappa/p}} \lambda \cdot w(\{x \in B : |f(x)| > \lambda\})^{1/p}. \tag{31}$$

Furthermore, in order to deal with the fractional order case, we need to consider the weighted Morrey spaces with two weights.

Definition 12. Let $0 < p < \infty$ and $0 < \kappa < 1$. Then for two weights u and v, the weighted Morrey space is defined by

$$L^{p,\kappa}(u,v) = \left\{f \in L^p_{\mathrm{loc}}(u) : \|f\|_{L^{p,\kappa}(u,v)} < \infty\right\}, \tag{32}$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x)\,dx\right)^{1/p}. \tag{33}$$

Throughout this paper, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Moreover, we will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$.

3. Proofs of Theorems 1 and 2

Before proving the main theorems of this section, we need to establish the following lemma.

Lemma 13. *Let* $m \ge 2$, *let* $p_1, \dots, p_m \in [1,\infty)$, *and let* $p \in (0,\infty)$ *with* $1/p = \sum_{k=1}^m 1/p_k$. *Assume that* $w_1, \dots, w_m \in A_\infty$ *and* $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$; *then, for any ball B, there exists a constant* $C > 0$ *such that*

$$\prod_{i=1}^m \left(\int_B w_i(x)\,dx\right)^{p/p_i} \le C \int_B \nu_{\vec{w}}(x)\,dx. \tag{34}$$

Proof. Since $w_1, \dots, w_m \in A_\infty$, then, by using Lemma 6, we have

$$\begin{aligned}&\prod_{i=1}^m \left(\int_B w_i(x)\,dx\right)^{p/p_i} \\ &\quad\le C \prod_{i=1}^m \left(|B| \cdot \exp\left(\frac{1}{|B|}\int_B \log w_i(x)\,dx\right)\right)^{p/p_i} \\ &\quad= C \prod_{i=1}^m \left(|B|^{p/p_i} \cdot \exp\left(\frac{1}{|B|}\int_B \log w_i(x)^{p/p_i}\,dx\right)\right) \\ &\quad= C \cdot (|B|)^{\sum_{i=1}^m p/p_i} \cdot \exp\left(\sum_{i=1}^m \frac{1}{|B|}\int_B \log w_i(x)^{p/p_i} dx\right).\end{aligned} \tag{35}$$

Note that $\sum_{i=1}^m p/p_i = 1$ and $\nu_{\vec{w}}(x) = \prod_{i=1}^m w_i(x)^{p/p_i}$. Then, by Jensen inequality, we obtain

$$\begin{aligned}&\prod_{i=1}^m \left(\int_B w_i(x)\,dx\right)^{p/p_i} \\ &\quad\le C \cdot |B| \cdot \exp\left(\frac{1}{|B|}\int_B \log \nu_{\vec{w}}(x)\,dx\right) \\ &\quad\le C \int_B \nu_{\vec{w}}(x)\,dx.\end{aligned} \tag{36}$$

We are done. □

Proof of Theorem 1. For any ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and letting $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{2B}$, $i = 1, \ldots, m$, and χ_{2B} denotes the characteristic function of $2B$, then we write

$$\begin{aligned}\prod_{i=1}^m f_i(y_i) &= \prod_{i=1}^m \left(f_i^0(y_i) + f_i^\infty(y_i)\right) \\ &= \sum_{\alpha_1,\ldots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m) \\ &= \prod_{i=1}^m f_i^0(y_i) + \sum{}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),\end{aligned} \tag{37}$$

where each term of $\sum'$ contains at least one $\alpha_i \neq 0$. Since T is an m-linear operator, then we have

$$\begin{aligned}&\frac{1}{\nu_{\vec{w}}(B)^{\kappa/p}}\left(\int_B |T(f_1,\ldots,f_m)(x)|^p \nu_{\vec{w}}(x)\,dx\right)^{1/p} \\ &\quad\leq \frac{1}{\nu_{\vec{w}}(B)^{\kappa/p}}\left(\int_B \left|T\left(f_1^0,\ldots,f_m^0\right)(x)\right|^p \nu_{\vec{w}}(x)dx\right)^{1/p} \\ &\qquad+ \sum{}' \frac{1}{\nu_{\vec{w}}(B)^{\kappa/p}}\left(\int_B \left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right|^p \nu_{\vec{w}}(x)\,dx\right)^{1/p} \\ &\quad= I^0 + \sum{}' I^{\alpha_1,\ldots,\alpha_m}.\end{aligned} \tag{38}$$

In view of Lemma 8, we have that $\nu_{\vec{w}} \in A_{mp}$. Applying Theorem A and Lemmas 13 and 5, we get

$$\begin{aligned}I^0 &\leq C \cdot \frac{1}{\nu_{\vec{w}}(B)^{\kappa/p}} \prod_{i=1}^m \left(\int_{2B} |f_i(x)|^{p_i} w_i(x)\,dx\right)^{1/p_i} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \frac{\prod_{i=1}^m w_i(2B)^{\kappa/p_i}}{\nu_{\vec{w}}(B)^{\kappa/p}} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \frac{\nu_{\vec{w}}(2B)^{\kappa/p}}{\nu_{\vec{w}}(B)^{\kappa/p}} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)}.\end{aligned} \tag{39}$$

For the other terms, let us first consider the case when $\alpha_1 = \cdots = \alpha_m = \infty$. By the size condition, for any $x \in B$, we obtain

$$\begin{aligned}&\left|T\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\ &\quad\leq C\int_{(\mathbb{R}^n)^m \setminus (2B)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{\left(|x-y_1|+\cdots+|x-y_m|\right)^{mn}} dy_1\cdots dy_m \\ &\quad\leq C\sum_{j=1}^\infty \int_{(2^{j+1}B)^m \setminus (2^jB)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{\left(|x-y_1|+\cdots+|x-y_m|\right)^{mn}} dy_1\cdots dy_m \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^m \int_{2^{j+1}B\setminus 2^jB} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^m \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |f_i(y_i)|\,dy_i,\end{aligned} \tag{40}$$

where we have used the notation $E^m = E \times \cdots \times E$. Furthermore, by using Hölder's inequality, the multiple $A_{\vec{P}}$ condition, and Lemma 13, we deduce that

$$\begin{aligned}&\left|T\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^m \frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B} |f_i(y_i)|^{p_i} w_i(y_i)\,dy_i\right)^{1/p_i} \\ &\qquad\times \left(\int_{2^{j+1}B} w_i(y_i)^{1-p_i'} dy_i\right)^{1/p_i'} \\ &\quad\leq C\sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^m} \cdot \frac{|2^{j+1}B|^{1/p+\sum_{i=1}^m (1-1/p_i)}}{\nu_{\vec{w}}(2^{j+1}B)^{1/p}} \\ &\qquad\times \prod_{i=1}^m \left(\|f_i\|_{L^{p_i,\kappa}(w_i)} w_i\left(2^{j+1}B\right)^{\kappa/p_i}\right) \\ &\quad\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \sum_{j=1}^\infty \left(\frac{\prod_{i=1}^m w_i\left(2^{j+1}B\right)^{\kappa/p_i}}{\nu_{\vec{w}}(2^{j+1}B)^{1/p}}\right) \\ &\quad\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \sum_{j=1}^\infty \nu_{\vec{w}}\left(2^{j+1}B\right)^{(\kappa-1)/p}.\end{aligned} \tag{41}$$

Since $\nu_{\vec{w}} \in A_{mp} \subset A_\infty$, then it follows directly from Lemma 7 that

$$\frac{\nu_{\vec{w}}(B)}{\nu_{\vec{w}}(2^{j+1}B)} \leq C\left(\frac{|B|}{|2^{j+1}B|}\right)^\delta. \tag{42}$$

Hence,

$$\begin{aligned}I^{\infty,\ldots,\infty} &\leq \nu_{\vec{w}}(B)^{(1-\kappa)/p}\left|T\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \sum_{j=1}^\infty \frac{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}{\nu_{\vec{w}}(2^{j+1}B)^{(1-\kappa)/p}} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)} \cdot \sum_{j=1}^\infty \left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)/p} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa}(w_i)},\end{aligned} \tag{43}$$

where the last inequality holds since $0 < \kappa < 1$ and $\delta > 0$. We now consider the case where exactly ℓ of the α_i are ∞ for some $1 \leq \ell < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. Using the size condition again, we deduce that, for any $x \in B$,

$$\begin{aligned}&\left|T\left(f_1^\infty,\ldots,f_\ell^\infty,f_{\ell+1}^0,\ldots,f_m^0\right)(x)\right| \\ &\quad\leq C\int_{(\mathbb{R}^n)^\ell \setminus (2B)^\ell} \int_{(2B)^{m-\ell}} \frac{|f_1(y_1)\cdots f_m(y_m)|}{\left(|x-y_1|+\cdots+|x-y_m|\right)^{mn}} \\ &\qquad\times dy_1\cdots dy_m\end{aligned}$$

$$
\begin{aligned}
&\le C\prod_{i=\ell+1}^{m}\int_{2B}|f_i(y_i)|\,dy_i \\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^m} \\
&\quad\quad\times\int_{(2^{j+1}B)^\ell\setminus(2^jB)^\ell}|f_1(y_1)\cdots f_\ell(y_\ell)|\,dy_1\cdots dy_\ell \\
&\le C\prod_{i=\ell+1}^{m}\int_{2B}|f_i(y_i)|\,dy_i \\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^m}\prod_{i=1}^{\ell}\int_{2^{j+1}B\setminus 2^jB}|f_i(y_i)|\,dy_i \\
&\le C\sum_{j=1}^{\infty}\prod_{i=1}^{m}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i,
\end{aligned}
\tag{44}
$$

and we arrive at the expression considered in the previous case. So for any $x\in B$, we also have

$$
\begin{aligned}
&\left|T\left(f_1^\infty,\ldots,f_\ell^\infty,f_{\ell+1}^0,\ldots,f_m^0\right)(x)\right| \\
&\quad\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\sum_{j=1}^{\infty}\nu_{\vec{w}}\left(2^{j+1}B\right)^{(\kappa-1)/p}.
\end{aligned}
\tag{45}
$$

Therefore, by the inequality (42) and the above pointwise inequality, we have

$$
\begin{aligned}
&I^{\alpha_1,\ldots,\alpha_m} \\
&\quad\le\nu_{\vec{w}}(B)^{(1-\kappa)/p}\left|T\left(f_1^\infty,\ldots,f_\ell^\infty,f_{\ell+1}^0,\ldots,f_m^0\right)(x)\right| \\
&\quad\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\sum_{j=1}^{\infty}\frac{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}{\nu_{\vec{w}}(2^{j+1}B)^{(1-\kappa)/p}} \\
&\quad\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\sum_{j=1}^{\infty}\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)/p} \\
&\quad\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}.
\end{aligned}
\tag{46}
$$

Combining the above estimates and then taking the supremum over all balls $B\subseteq\mathbb{R}^n$, we complete the proof of Theorem 1. □

Proof of Theorem 2. For any ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$ and decomposing $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{2B}$, $i=1,\ldots,m$, then, for any given $\lambda>0$, we can write

$$
\begin{aligned}
&\nu_{\vec{w}}(\{x\in B:|T(f_1,\ldots,f_m)|>\lambda\})^{1/p} \\
&\quad\le\nu_{\vec{w}}\left(\left\{x\in B:\left|T\left(f_1^0,\ldots,f_m^0\right)\right|>\frac{\lambda}{2^m}\right\}\right)^{1/p} \\
&\quad\quad+\sum{}'\nu_{\vec{w}}\left(\left\{x\in B:\left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)\right|>\frac{\lambda}{2^m}\right\}\right)^{1/p} \\
&\quad=I_*^0+\sum{}'I_*^{\alpha_1,\ldots,\alpha_m},
\end{aligned}
\tag{47}
$$

where each term of $\sum'$ contains at least one $\alpha_i\neq 0$. By Lemma 8 again, we know that $\nu_{\vec{w}}\in A_{mp}$ with $1\le mp<\infty$. Applying Theorem B and Lemmas 13 and 5, we have

$$
\begin{aligned}
I_*^0&\le\frac{C}{\lambda}\prod_{i=1}^{m}\left(\int_{2B}|f_i(x)|^{p_i}w_i(x)\,dx\right)^{1/p_i} \\
&\le\frac{C\cdot\prod_{i=1}^{m}w_i(2B)^{\kappa/p_i}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)} \\
&\le\frac{C\cdot\nu_{\vec{w}}(2B)^{\kappa/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)} \\
&\le\frac{C\cdot\nu_{\vec{w}}(B)^{\kappa/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}.
\end{aligned}
\tag{48}
$$

In the proof of Theorem 1, we have already showed the following pointwise estimate (see (40) and (44)). Consider

$$
\left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right|\le C\sum_{j=1}^{\infty}\prod_{i=1}^{m}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i.
\tag{49}
$$

Without loss of generality, we may assume that $p_1=\cdots=p_\ell=\min\{p_1,\ldots,p_m\}=1$ and $p_{\ell+1},\ldots,p_m>1$. Using Hölder's inequality, the multiple $A_{\vec{P}}$ condition, and Lemma 13, we obtain

$$
\begin{aligned}
&\left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right| \\
&\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{\ell}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i \\
&\quad\quad\times\prod_{i=\ell+1}^{m}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f_i(y_i)|\,dy_i \\
&\quad\le C\sum_{j=1}^{\infty}\prod_{i=1}^{\ell}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f_i(y_i)|\,w_i(y_i)\,dy_i \\
&\quad\quad\times\left(\inf_{y_i\in 2^{j+1}B}w_i(y_i)\right)^{-1} \\
&\quad\quad\times\prod_{i=\ell+1}^{m}\frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B}|f_i(y_i)|^{p_i}w_i(y_i)\,dy_i\right)^{1/p_i} \\
&\quad\quad\times\left(\int_{2^{j+1}B}w_i(y_i)^{1-p_i'}dy_i\right)^{1/p_i'} \\
&\quad\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\sum_{j=1}^{\infty}\nu_{\vec{w}}\left(2^{j+1}B\right)^{(\kappa-1)/p}.
\end{aligned}
\tag{50}
$$

Observe that $\nu_{\vec{w}}\in A_{mp}$ with $1\le mp<\infty$. Thus, it follows from the inequality (42) that, for any $x\in B$,

$$
\begin{aligned}
&\left|T\left(f_1^{\alpha_1},\ldots,f_m^{\alpha_m}\right)(x)\right| \\
&\quad=C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\frac{1}{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}\sum_{j=1}^{\infty}\frac{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}{\nu_{\vec{w}}(2^{j+1}B)^{(1-\kappa)/p}}
\end{aligned}
$$

$$\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\frac{1}{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}\sum_{j=1}^{\infty}\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta(1-\kappa)/p}$$
$$\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\frac{1}{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}. \tag{51}$$

If $\{x\in B:|T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)|>\lambda/2^m\}=\emptyset$, then the inequality

$$I_*^{\alpha_1,\dots,\alpha_m}\leq\frac{C\cdot\nu_{\vec{w}}(B)^{\kappa/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)} \tag{52}$$

holds trivially. Now, if instead we suppose that $\{x\in B:|T(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)|>\lambda/2^m\}\neq\emptyset$, then, by the pointwise inequality (51), we have

$$\lambda<C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}\cdot\frac{1}{\nu_{\vec{w}}(B)^{(1-\kappa)/p}}, \tag{53}$$

which is equivalent to

$$\nu_{\vec{w}}(B)^{1/p}\leq\frac{C\cdot\nu_{\vec{w}}(B)^{\kappa/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}. \tag{54}$$

Therefore,

$$I_*^{\alpha_1,\dots,\alpha_m}\leq\nu_{\vec{w}}(B)^{1/p}\leq\frac{C\cdot\nu_{\vec{w}}(B)^{\kappa/p}}{\lambda}\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}. \tag{55}$$

Summing up all the above estimates and then taking the supremum over all balls $B\subseteq\mathbb{R}^n$ and all $\lambda>0$, we complete the proof of Theorem 2. □

By using Hölder's inequality, it is easy to check that if each w_i is in A_{p_i}, then

$$\prod_{i=1}^{m}A_{p_i}\subset A_{\vec{P}}, \tag{56}$$

and this inclusion is strict (see [9]). Thus, as direct consequences of Theorems 1 and 2, we immediately obtain the following.

Corollary 14. *Let $m\geq 2$, and let T be an m-linear Calderón-Zygmund operator. If $p_1,\dots,p_m\in(1,\infty)$ and $p\in(0,\infty)$ with $1/p=\sum_{k=1}^{m}1/p_k$ and $\vec{w}=(w_1,\dots,w_m)\in\prod_{i=1}^{m}A_{p_i}$, then, for any $0<\kappa<1$, there exists a constant $C>0$ independent of $\vec{f}=(f_1,\dots,f_m)$ such that*

$$\left\|T\left(\vec{f}\right)\right\|_{L^{p,\kappa}(\nu_{\vec{w}})}\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}, \tag{57}$$

where $\nu_{\vec{w}}=\prod_{i=1}^{m}w_i^{p/p_i}$.

Corollary 15. *Let $m\geq 2$ and let T be an m-linear Calderón-Zygmund operator. If $p_1,\dots,p_m\in[1,\infty)$, $\min\{p_1,\dots,p_m\}=1$ and $p\in(0,\infty)$ with $1/p=\sum_{k=1}^{m}1/p_k$, and $\vec{w}=(w_1,\dots,w_m)\in\prod_{i=1}^{m}A_{p_i}$, then, for any $0<\kappa<1$, there exists a constant $C>0$ independent of $\vec{f}=(f_1,\dots,f_m)$ such that*

$$\left\|T\left(\vec{f}\right)\right\|_{WL^{p,\kappa}(\nu_{\vec{w}})}\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa}(w_i)}, \tag{58}$$

where $\nu_{\vec{w}}=\prod_{i=1}^{m}w_i^{p/p_i}$.

4. Proofs of Theorems 3 and 4

Following along the same lines as those of Lemma 13, we can also show the following result, which plays an important role in our proofs of Theorems 3 and 4.

Lemma 16. *Let $m\geq 2$, $q_1,\dots,q_m\in[1,\infty)$ and $q\in(0,\infty)$ with $1/q=\sum_{k=1}^{m}1/q_k$. Assume that $w_1^{q_1},\dots,w_m^{q_m}\in A_\infty$ and $\nu_{\vec{w}}=\prod_{i=1}^{m}w_i$; then, for any ball B, there exists a constant $C>0$ such that*

$$\prod_{i=1}^{m}\left(\int_B w_i^{q_i}(x)\,dx\right)^{q/q_i}\leq C\int_B\nu_{\vec{w}}(x)^q dx. \tag{59}$$

Proof of Theorem 3. Arguing as in the proof of Theorem 1, fix a ball $B=B(x_0,r_B)\subseteq\mathbb{R}^n$ and decompose $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{2B}$, $i=1,\dots,m$. Since I_α is an m-linear operator, then we have

$$\frac{1}{\nu_{\vec{w}}^q(B)^{\kappa/p}}\left(\int_B\left|I_\alpha\left(f_1,\dots,f_m\right)(x)\right|^q\nu_{\vec{w}}(x)^q dx\right)^{1/q}$$
$$\leq\frac{1}{\nu_{\vec{w}}^q(B)^{\kappa/p}}\left(\int_B\left|I_\alpha\left(f_1^0,\dots,f_m^0\right)(x)\right|^q\nu_{\vec{w}}(x)^q dx\right)^{1/q}$$
$$+\sum{}'\frac{1}{\nu_{\vec{w}}^q(B)^{\kappa/p}}\left(\int_B\left|I_\alpha\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right|^q\nu_{\vec{w}}(x)^q dx\right)^{1/q}$$
$$=J^0+\sum{}'J^{\alpha_1,\dots,\alpha_m}, \tag{60}$$

where each term of $\sum'$ contains at least one $\alpha_i\neq 0$. In view of Lemma 9, we can see that $(\nu_{\vec{w}})^q\in A_{mq}$. Using Theorem C and Lemmas 16 and 5, we get

$$J^0\leq C\cdot\frac{1}{\nu_{\vec{w}}^q(B)^{\kappa/p}}\prod_{i=1}^{m}\left(\int_{2B}\left|f_i(x)\right|^{p_i}w_i(x)^{p_i}dx\right)^{1/p_i}$$
$$\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}\cdot\frac{\prod_{i=1}^{m}w_i^{q_i}(2B)^{\kappa q/pq_i}}{\nu_{\vec{w}}^q(B)^{\kappa/p}}$$
$$=C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}\cdot\frac{\left(\prod_{i=1}^{m}w_i^{q_i}(2B)^{q/q_i}\right)^{\kappa/p}}{\nu_{\vec{w}}^q(B)^{\kappa/p}}$$
$$\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}\cdot\frac{\nu_{\vec{w}}^q(2B)^{\kappa/p}}{\nu_{\vec{w}}^q(B)^{\kappa/p}}$$
$$\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}. \tag{61}$$

For the other terms, let us first deal with the case when $\alpha_1 = \cdots = \alpha_m = \infty$. By the definition of I_α, for any $x \in B$, we obtain

$$\begin{aligned}
&\left|I_\alpha\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\
&\quad = \int_{(\mathbb{R}^n)^m\setminus(2B)^m} \frac{\left|f_1(y_1)\cdots f_m(y_m)\right|}{\left(\left|x-y_1\right|+\cdots+\left|x-y_m\right|\right)^{mn-\alpha}} dy_1\cdots dy_m \\
&\quad = \sum_{j=1}^{\infty} \int_{(2^{j+1}B)^m\setminus(2^jB)^m} \frac{\left|f_1(y_1)\cdots f_m(y_m)\right|}{\left(\left|x-y_1\right|+\cdots+\left|x-y_m\right|\right)^{mn-\alpha}} \\
&\qquad \times dy_1\cdots dy_m \\
&\quad \le C\sum_{j=1}^{\infty}\prod_{i=1}^{m} \int_{2^{j+1}B\setminus 2^jB} \frac{\left|f_i(y_i)\right|}{\left|x-y_i\right|^{n-\alpha/m}} dy_i \\
&\quad \le C\sum_{j=1}^{\infty}\prod_{i=1}^{m} \frac{1}{\left|2^{j+1}B\right|^{1-\alpha/mn}} \int_{2^{j+1}B} \left|f_i(y_i)\right| dy_i.
\end{aligned} \tag{62}$$

Moreover, by using Hölder's inequality, the multiple $A_{\vec{P},q}$ condition, and Lemma 16, we deduce that

$$\begin{aligned}
&\left|I_\alpha\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\
&\quad \le C\sum_{j=1}^{\infty}\prod_{i=1}^{m} \frac{1}{\left|2^{j+1}B\right|^{1-\alpha/mn}} \\
&\qquad \times \left(\int_{2^{j+1}B} \left|f_i(y_i)\right|^{p_i} w_i(y_i)^{p_i} dy_i\right)^{1/p_i} \\
&\qquad \times \left(\int_{2^{j+1}B} w_i(y_i)^{-p_i'} dy_i\right)^{1/p_i'} \\
&\quad \le C\sum_{j=1}^{\infty} \frac{1}{\left|2^{j+1}B\right|^{m-\alpha/n}} \cdot \frac{\left|2^{j+1}B\right|^{1/q+\sum_{i=1}^m(1-1/p_i)}}{\nu_{\vec{w}}^q\left(2^{j+1}B\right)^{1/q}} \\
&\qquad \times \prod_{i=1}^{m} \left(\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} w_i^{q_i}\left(2^{j+1}B\right)^{\kappa q/p q_i}\right) \\
&\quad \le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} \\
&\qquad \cdot \sum_{j=1}^{\infty} \left[\frac{\left(\prod_{i=1}^m w_i^{q_i}\left(2^{j+1}B\right)^{q/q_i}\right)^{\kappa/p}}{\nu_{\vec{w}}^q\left(2^{j+1}B\right)^{1/q}}\right] \\
&\quad \le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^{\infty} \nu_{\vec{w}}^q\left(2^{j+1}B\right)^{\kappa/p-1/q}.
\end{aligned} \tag{63}$$

Since $(\nu_{\vec{w}})^q \in A_{mq} \subset A_\infty$, then it follows immediately from Lemma 7 that

$$\frac{\nu_{\vec{w}}^q(B)}{\nu_{\vec{w}}^q(2^{j+1}B)} \le C\left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta'}. \tag{64}$$

Hence,

$$\begin{aligned}
J^{\infty,\ldots,\infty} &\le \nu_{\vec{w}}^q(B)^{1/q-\kappa/p}\left|I_\alpha\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| \\
&\le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^{\infty} \frac{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}}{\nu_{\vec{w}}^q\left(2^{j+1}B\right)^{1/q-\kappa/p}} \\
&\le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^{\infty} \left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta'(1/q-\kappa/p)} \\
&\le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})},
\end{aligned} \tag{65}$$

where in the last inequality we have used the fact that $0 < \kappa < p/q$ and $\delta' > 0$. We now consider the case where exactly ℓ of the α_i are ∞ for some $1 \le \ell < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. Using the definition of I_α again, we can see that, for any $x \in B$,

$$\begin{aligned}
&\left|I_\alpha\left(f_1^\infty,\ldots,f_\ell^\infty,f_{\ell+1}^0,\ldots,f_m^0\right)(x)\right| \\
&\quad = \int_{(\mathbb{R}^n)^\ell\setminus(2B)^\ell}\int_{(2B)^{m-\ell}} \frac{\left|f_1(y_1)\cdots f_m(y_m)\right|}{\left(\left|x-y_1\right|+\cdots+\left|x-y_m\right|\right)^{mn-\alpha}} \\
&\qquad \times dy_1\cdots dy_m \\
&\quad \le C\prod_{i=\ell+1}^{m}\int_{2B}\left|f_i(y_i)\right|dy_i \\
&\qquad \times \sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{m-\alpha/n}} \\
&\qquad \times \int_{(2^{j+1}B)^\ell\setminus(2^jB)^\ell}\left|f_1(y_1)\cdots f_\ell(y_\ell)\right|dy_1\cdots dy_\ell \\
&\quad \le C\prod_{i=\ell+1}^{m}\int_{2B}\left|f_i(y_i)\right|dy_i \\
&\qquad \times \sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{m-\alpha/n}}\prod_{i=1}^{\ell}\int_{2^{j+1}B\setminus 2^jB}\left|f_i(y_i)\right|dy_i \\
&\quad \le C\sum_{j=1}^{\infty}\prod_{i=1}^{m}\frac{1}{\left|2^{j+1}B\right|^{1-\alpha/mn}}\int_{2^{j+1}B}\left|f_i(y_i)\right|dy_i,
\end{aligned} \tag{66}$$

and we arrive at the expression considered in the previous case. Thus, for any $x \in B$, we also have

$$\begin{aligned}
&\left|I_\alpha\left(f_1^\infty,\ldots,f_\ell^\infty,f_{\ell+1}^0,\ldots,f_m^0\right)(x)\right| \\
&\quad \le C\prod_{i=1}^{m}\left\|f_i\right\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i},w_i^{q_i})} \\
&\qquad \cdot \sum_{j=1}^{\infty}\nu_{\vec{w}}^q\left(2^{j+1}B\right)^{\kappa/p-1/q}.
\end{aligned} \tag{67}$$

Therefore, by the inequality (64) and the above pointwise inequality, we obtain

$$\begin{aligned} J^{\alpha_1,\dots,\alpha_m} &\leq \nu_{\vec{w}}^q(B)^{1/q-\kappa/p} \left|I_\alpha\left(f_1^\infty,\dots,f_\ell^\infty,f_{\ell+1}^0,\dots,f_m^0\right)(x)\right| \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^\infty \frac{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}}{\nu_{\vec{w}}^q(2^{j+1}B)^{1/q-\kappa/p}} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^\infty \left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta'(1/q-\kappa/p)} \\ &\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}. \end{aligned} \tag{68}$$

Summarizing the estimates derived above and then taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we finish the proof of Theorem 3. □

Proof of Theorem 4. As before, fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and split f_i into $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{2B}$, $i = 1,\dots,m$. Then for each fixed $\lambda > 0$, we can write

$$\begin{aligned} &\nu_{\vec{w}}^q(\{x \in B : |I_\alpha(f_1,\dots,f_m)| > \lambda\})^{1/q} \\ &\quad\leq \nu_{\vec{w}}^q\left(\left\{x \in B : \left|I_\alpha\left(f_1^0,\dots,f_m^0\right)\right| > \frac{\lambda}{2^m}\right\}\right)^{1/q} \\ &\qquad + \sum{}' \nu_{\vec{w}}^q\left(\left\{x \in B : \left|I_\alpha\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)\right| > \frac{\lambda}{2^m}\right\}\right)^{1/q} \\ &\quad= J_*^0 + \sum{}' J_*^{\alpha_1,\dots,\alpha_m}, \end{aligned} \tag{69}$$

where each term of $\sum'$ contains at least one $\alpha_i \neq 0$. By Lemma 9 again, we know that $(\nu_{\vec{w}})^q \in A_{mq}$ with $1 < mq < \infty$. Using Theorem D and Lemmas 16 and 5, we have

$$\begin{aligned} J_*^0 &\leq \frac{C}{\lambda}\prod_{i=1}^m \left(\int_{2B} |f_i(x)|^{p_i} w_i(x)^{p_i}dx\right)^{1/p_i} \\ &\leq \frac{C \cdot \prod_{i=1}^m w_i^{q_i}(2B)^{\kappa q/pq_i}}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \\ &\leq \frac{C \cdot \nu_{\vec{w}}^q(2B)^{\kappa/p}}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \\ &\leq \frac{C \cdot \nu_{\vec{w}}^q(B)^{\kappa/p}}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})}. \end{aligned} \tag{70}$$

In the proof of Theorem 3, we have already proved the following pointwise estimate (see (62) and (66)). Consider

$$\begin{aligned} &\left|I_\alpha\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^m \frac{1}{|2^{j+1}B|^{1-\alpha/mn}} \int_{2^{j+1}B} |f_i(y_i)|\,dy_i. \end{aligned} \tag{71}$$

Without loss of generality, we may assume that $p_1 = \cdots = p_\ell = \min\{p_1,\dots,p_m\} = 1$ and $p_{\ell+1},\dots,p_m > 1$. By using Hölder's inequality, the multiple $A_{\vec{P},q}$ condition, and Lemma 16, we obtain

$$\begin{aligned} &\left|I_\alpha\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^\ell \frac{1}{|2^{j+1}B|^{1-\alpha/mn}} \int_{2^{j+1}B} |f_i(y_i)|\,dy_i \times \prod_{i=\ell+1}^m \frac{1}{|2^{j+1}B|^{1-\alpha/mn}} \int_{2^{j+1}B} |f_i(y_i)|\,dy_i \\ &\quad\leq C\sum_{j=1}^\infty \prod_{i=1}^\ell \frac{1}{|2^{j+1}B|^{1-\alpha/mn}} \times \int_{2^{j+1}B} |f_i(y_i)|\,w_i(y_i)\,dy_i \left(\inf_{y_i\in 2^{j+1}B} w_i(y_i)\right)^{-1} \\ &\qquad\times \prod_{i=\ell+1}^m \frac{1}{|2^{j+1}B|^{1-\alpha/mn}} \left(\int_{2^{j+1}B} |f_i(y_i)|^{p_i} w_i(y_i)^{p_i}dy_i\right)^{1/p_i} \times \left(\int_{2^{j+1}B} w_i(y_i)^{-p_i'}dy_i\right)^{1/p_i'} \\ &\quad\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \sum_{j=1}^\infty \nu_{\vec{w}}^q\left(2^{j+1}B\right)^{\kappa/p-1/q}. \end{aligned} \tag{72}$$

Note that $(\nu_{\vec{w}})^q \in A_{mq}$ with $1 < mq < \infty$. Hence, it follows from the inequality (64) that, for any $x \in B$,

$$\begin{aligned} &\left|I_\alpha\left(f_1^{\alpha_1},\dots,f_m^{\alpha_m}\right)(x)\right| \\ &\quad= C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \frac{1}{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}} \sum_{j=1}^\infty \frac{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}}{\nu_{\vec{w}}^q(2^{j+1}B)^{1/q-\kappa/p}} \\ &\quad\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \frac{1}{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}} \sum_{j=1}^\infty \left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta'(1/q-\kappa/p)} \\ &\quad\leq C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \frac{1}{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}}. \end{aligned} \tag{73}$$

If $\{x \in B : |I_\alpha(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)| > \lambda/2^m\} = \emptyset$, then the inequality

$$J_*^{\alpha_1,\dots,\alpha_m} \leq \frac{C \cdot \nu_{\vec{w}}^q(B)^{\kappa/p}}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \tag{74}$$

holds trivially. Now, if instead we assume that $\{x \in B : |I_\alpha(f_1^{\alpha_1},\dots,f_m^{\alpha_m})(x)| > \lambda/2^m\} \neq \emptyset$, then, by the pointwise inequality (73), we get

$$\lambda < C\prod_{i=1}^m \|f_i\|_{L^{p_i,\kappa p_i q/pq_i}(w_i^{p_i},w_i^{q_i})} \cdot \frac{1}{\nu_{\vec{w}}^q(B)^{1/q-\kappa/p}}, \tag{75}$$

which in turn gives that

$$\nu_{\vec{w}}^{q}(B)^{1/q} \le \frac{C \cdot \nu_{\vec{w}}^{q}(B)^{\kappa/p}}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i}, w_i^{q_i})}. \quad (76)$$

Therefore,

$$J_{*}^{\alpha_1,\ldots,\alpha_m} \le \nu_{\vec{w}}^{q}(B)^{1/q} \le \frac{C \cdot \nu_{\vec{w}}^{q}(B)^{\kappa/p}}{\lambda} \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i}, w_i^{q_i})}. \quad (77)$$

Collecting all the above estimates and then taking the supremum over all balls $B \subseteq \mathbb{R}^n$ and all $\lambda > 0$, we conclude the proof of Theorem 4. □

By using Hölder's inequality, it is easy to verify that if $1 \le p_i < q_i$, $1/q = \sum_{k=1}^{m} 1/q_k$, and each w_i is in A_{p_i,q_i}, then we have

$$\prod_{i=1}^{m} A_{p_i,q_i} \subset A_{\vec{P},q}, \quad (78)$$

and this inclusion is strict (see [17]). Also, recall that $w \in A_{p,q}$ if and only if $w^q \in A_{1+q/p'} \subset A_\infty$ (see [36]). Thus, as straightforward consequences of Theorems 3 and 4, we finally obtain the following.

Corollary 17. *Let $m \ge 2$, let $0 < \alpha < mn$, and let I_α be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in (1, \infty)$, $1/p = \sum_{k=1}^{m} 1/p_k$, $1/q_k = 1/p_k - \alpha/mn$ and $1/q = \sum_{k=1}^{m} 1/q_k = 1/p - \alpha/n$, and $\vec{w} = (w_1, \ldots, w_m) \in \prod_{i=1}^{m} A_{p_i,q_i}$, then, for any $0 < \kappa < p/q$, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\| I_\alpha(\vec{f}) \right\|_{L^{q,\kappa q/p}((\nu_{\vec{w}})^q)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i}, w_i^{q_i})}, \quad (79)$$

where $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i$.

Corollary 18. *Let $m \ge 2$, let $0 < \alpha < mn$, and let I_α be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in [1, \infty)$, $\min\{p_1, \ldots, p_m\} = 1$, $1/p = \sum_{k=1}^{m} 1/p_k$, $1/q_k = 1/p_k - \alpha/mn$ and $1/q = \sum_{k=1}^{m} 1/q_k = 1/p - \alpha/n$, and $\vec{w} = (w_1, \ldots, w_m) \in \prod_{i=1}^{m} A_{p_i,q_i}$, then, for any $0 < \kappa < p/q$, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that*

$$\left\| I_\alpha(\vec{f}) \right\|_{WL^{q,\kappa q/p}((\nu_{\vec{w}})^q)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\kappa p_i q/p q_i}(w_i^{p_i}, w_i^{q_i})}, \quad (80)$$

where $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i$.

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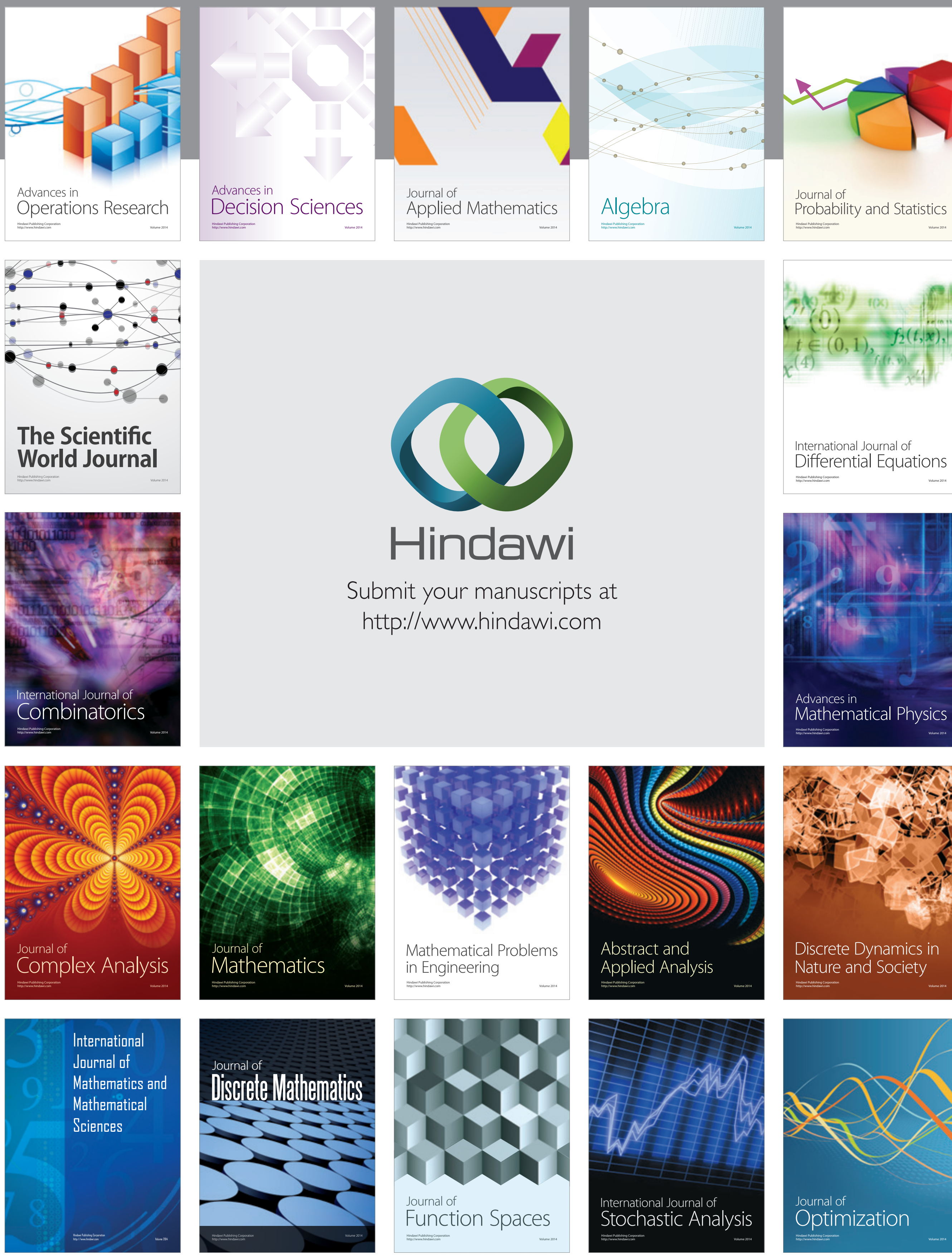
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