Multigrid Analysis of Scattering by Large Planar Structures

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Abstract

Fast iterative analysis of two-dimensional scattering by a large but finite array of perfectly conducting strips requires efficient evaluation of the electric field. We present a novel multigrid algorithm that carries out this task in CN computer operations, where C depends logarithmically on the desired accuracy in the field, and N is the number of spatial gridpoints. Numerical results are presented, and extensions of the algorithm are discussed.

Key terms: computational electromagnetics, fast multilevel summation, oscillatory kernels.

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1 Introduction

Wave scattering by periodic structures has been extensively treated by many researchers since the pioneering works of Lord Rayleigh. However, scattering by finite periodic and quasi-periodic geometries has received little attention, especially, in terms of numerically rigorous analysis. Examples of such structures are Fresnel lenses and planar reflector antennas as well as realistic finite Frequency Selective Surfaces (FSS) and patch antenna arrays.

Scattering by planar structures can be formulated in the integral equation form, which is conventionally discretized using the Method of Moments (MoM) [11]. The computational cost of solving matrix equations poses the main limitation on the electrical size of scattering problems that can be analyzed using MoM. The direct solution of the MoM matrix equations for electrically large geometries is impractical due to $O(N^3)$ complexity of direct solvers, N being the number of unknowns. The solution of very large linear systems is usually facilitated via iterative solvers, whose cost depends on the cost of matrix vector multiplication representing discretized evaluation of a field produced by a given current distribution. Recently, several fast direct and iterative algorithms for the solution of the planar scattering problem have been presented in [8, 7, 10].

In this paper, we propose an alternative iterative solution based on the general multilevel approach for fast evaluation of integral transforms with oscillatory kernels presented in [2]. We consider two-dimensional scattering by a

large but finite array of perfectly conducting strips. The problem requires the solution of a one-dimensional (1D) electric field integral equation. In the proposed approach, the integral equation is solved iteratively, and the main computational task is the repeated evaluation of the electric fields produced by the candidate solutions for the current distribution. Thus, we focus here on fast evaluation of the field. To this end, the 1D oscillatory kernel is represented as a linear combination of two "directional" kernels. Each such directional kernel is not oscillatory, but is asymptotically smooth: it is singular at short distances, but gets increasingly smoother at larger distances. As a result, it can be further decomposed into a local part (whose contribution to the total field is local and inexpensively computed), and a smooth part, which can be efficiently recovered from its values on a coarser grid. The task of evaluating the original field over N nodes is thus replaced by the task of evaluating the contribution of the smooth part of the kernel on a coarser resolution of about $N/2$ nodes, which may still be too large to compute directly. Consequently, further coarsening is applied recursively until a grid is reached on which the task can be computed directly in $O(N)$ operations. This implies that the original field evaluation can be carried out in only $O(N)$ computer operations, thereby reducing the $O(N^2)$ complexity required for a direct evaluation.

1.1 Problem specification

Consider a problem of Two-Dimensional (2D) Transverse Magnetic (TM) scattering from a finite planar perfectly conducting structure. The geometry and excitation are assumed uniform in the z direction. For clarity, the following formulation is presented for a strip of unit width depicted in Fig. 1. Generalizing the proposed method to geometries comprising multiple strips is relatively straightforward and will be discussed in §3. The strip is illuminated by a z-polarized incident field $\mathrm{E}^{\mathrm{inc}}$ with a harmonic time dependence $e^{i\omega t}$, which is assumed and suppressed throughout the paper. The scattering from the strip is analyzed using the Method of Moments (MoM). An Electric Field Integral Equation (EFIE) is constructed in terms of a z-directed electric current J. The EFIE, which requires the total electric field along the perfect conductors to vanish, yields

$$\mathrm{E}^{\mathrm{inc}}(x) = \frac{\eta k}{4} \int_0^1 \mathrm{H}_0^{(2)}(k|x-y|)\mathrm{J}(y)dy, \qquad \forall x \in [0,1], \tag{1}$$

where $\mathrm{H}_0^{(2)}$ denotes the zero-order Hankel function of the second kind, η is the free-space impedance, and $k = 2\pi/\lambda$, λ being the free-space wavelength. Conventionally, the strip is subdivided into N segments of width h, which satisfies $h < \lambda/10$. In the following, $\{x_m = y_m = (m-\frac{1}{2})h\}_{m=1}^N$ will represent mid-points of the N segments, although J and E can be discretized in general on different grids $\{x_j\}_j, \{y_m\}_m$. Discretization of (1) based on pulse basis functions, delta testing functions, and approximate integration [11] leads to

a set of linear equations

$$\mathrm{E}^{\mathrm{inc}}(x_j) = Z\mathrm{J}(x_j) + \frac{\eta kh}{4}\mathrm{E}(x_j), \qquad j = 1, \ldots, N, \tag{2}$$

where

$$Z := \frac{\eta kh}{4}\left(1 - \frac{2i}{\pi}\ln(\frac{\gamma kh}{4e})\right), \qquad \gamma = 1.781072418\ldots \tag{3}$$

and

$$\mathrm{E}(x_j) = \sum_{j=1, j\neq m}^{N} \mathrm{H}_0^{(2)}(x_j - y_m)\mathrm{J}(y_m). \tag{4}$$

Iterative solution of the linear system (2) calls for repeated evaluation of its (4), whose direct summation requires $O(N^2)$ computer operations. Therefore, we will concentrate on the task of evaluating (4) in $O(N)$ operations.

2 Fast evaluation of $\mathbf{E}(x)$

The task (4) can be reformulated as follows:

$$\mathrm{E}(x_j) = \sum_{m=1}^{N} G(|y_m - x_j|)e^{-ik|y_m - x_j|}\mathrm{J}(y_m), \quad j = 1, \ldots, N, \tag{5}$$

where

$$G(r) := \begin{cases} \mathrm{H}_0^{(2)}(kr)e^{ikr}, & r > 0, \\ 0, & r = 0. \end{cases} \tag{6}$$

Our evaluation algorithm is a straightforward application of the general multilevel approach for fast evaluation of integral transforms, which was first introduced in [5], and extended to the case of oscillatory kernels in [2]. Since then it has been applied to accelerate integral transform computations in

different fields (see for example [6, 9, 12]). We also incorporate the technical modification suggested in [12] (extended tables for the local corrections) that improves the work-accuracy relation of multilevel summation algorithms toward optimal performance.

2.1 Separation of directions

In order to utilize the smoothness properties of $G(r)$, we first decompose the sum (5) into a sum of two "directional" fields [2, §5]

$$\mathrm{E}(x_j) := e^{-ikx_j}\mathrm{E}_-(x_j) + e^{ikx_j}\mathrm{E}_+(x_j), \quad j = 1, \ldots, N, \tag{7}$$

where

$$\mathrm{E}_\pm(x_j) := \sum_{m=1}^{N} G_\pm(y_m - x_j)\mathrm{J}_\pm(y_m), \quad j = 1, \ldots, N, \tag{8}$$

$$G_-(r) := \begin{cases} 0, & r > 0, \\ G(-r), & r \le 0, \end{cases} \qquad G_+(r) := \begin{cases} G(r), & r > 0, \\ 0, & r \le 0, \end{cases} \tag{9}$$

and

$$\mathrm{J}_\pm(y_m) := e^{\mp iky_m}\mathrm{J}(y_m), \quad m = 1, \ldots, N. \tag{10}$$

Note that

$$G(r) \sim \frac{i}{2\pi}\log(r), \qquad r \to 0; \qquad\qquad G(r) \sim r^{-\frac{1}{2}}, \qquad r \to \infty. \tag{11}$$

Moreover, the kernel $G(r)$ is *asymptotically smooth* in the sense of [2, §4], that is, becomes increasingly smoother for larger r. Clearly, this property is inherited by $G_\pm$. Hence, $\mathrm{E}_\pm$ can be evaluated by using *twice* the multilevel evaluation algorithm for asymptotically smooth kernels, which is described in

§2.2–§2.7. The *entire algorithm* for evaluating (5) thus starts by computing (10); then evaluates (8), and finally computes (7).

2.2 Kernel softening

For simplicity we consider only the case of G_+ (the case of G_- is symmetric and is treated in a symmetric manner). The asymptotically smooth kernel $G_+(r)$ can be decomposed as in [2, 6, 9, 12] into

$$G_+(r) = G_S(r) + G_{\text{local}}(r), \tag{12}$$

such that

1. $G_S(r) = G_+(r)$ (or $G_{\text{local}}(r) = 0$) for all $r \notin [0, S]$.
2. G_S is *suitably smooth* on the scale S, namely, for any $\varepsilon > 0$ there exists $p = O(\log(1/\varepsilon)) \in \mathbb{N}$ such that G_S can be uniformly approximated to an accuracy ε by a p-order interpolation from its values on any uniform grid with a meshsize comparable with S [6].

As in traditional multilevel algorithms [2, 6], we use a polynomial softened kernel

$$G_S(r) := \begin{cases} \sum_{\nu=0}^{2p-1} a_\nu(S) r^\nu, & r \in [0, S], \\ G_+(r), & \text{elsewhere}, \end{cases} \tag{13}$$

that has $p-1$ continuous derivatives at $r = 0, S$ (see Fig. 2). The coefficients $\{a_\nu\}_\nu$ are found by solving a $(2p) \times (2p)$ linear system, whose left-hand side's inverse can be computed before the main evaluation algorithm. The relative

error ε_I in approximating the scale-S softened kernel $G_S(r)$ by a p-order central interpolation from its values on a meshsize-h uniform grid satisfies

$$\varepsilon_I \lesssim C \left(\frac{ph}{2eS} \right)^p, \tag{14}$$

as explained in [9, Appendix A]. In fact, (14) should hold only for $r \notin [0, S]$, since errors for $r \in [0, S]$ are eliminated by the feature described in §2.4).

2.3 Derivation of the algorithm

The main idea of the algorithm is to reduce the evaluation of (5) on the fine grids $\{x_j\}_j, \{y_m\}_m$ (which will be denoted level 0) to approximate evaluations on progressively coarser grids $\{X_J^l\}_J, \{Y_M^l\}_M$ (levels $l = 1, 2, \ldots$). The entire algorithm consists of repeated coarsening steps, followed by a direct summation on the coarsest level (denoted level t). We describe a coarsening step, starting by decomposing

$$\mathrm{E}_+(x_j) = \mathrm{E}_S^0(x_j) + \mathrm{E}_{\mathrm{local}}^0(x_j), \qquad j = 1, \ldots, N, \tag{15}$$

where

$$\mathrm{E}_S^0(x_j) := \sum_{m=1}^{N} G_S(y_m - x_j)\mathrm{J}_+(y_m), \qquad j = 1, \ldots, N \tag{16}$$

and

$$\mathrm{E}_{\mathrm{local}}^0(x_j) := \sum_{m:|y_m - x_j| \le S} (G_+(y_m - x_j) - G_S(y_m - x_j))\mathrm{J}_+(y_m), \quad j = 1, \ldots, N. \tag{17}$$

The sum (17) extends over $O(s)$ points y_m, if we choose $S = sh$. The softened kernel can be represented as

$$G_S(y_m - x_j) = \sum_{M \in \sigma_m} \omega_{mM} G_S(Y_M^1 - x_j) + O(\varepsilon_I). \tag{18}$$

Here $\{Y_M^1\}_{M=1}^{N_1}$ is a uniform grid with meshsize $H := ch$ over $[y_1, y_N]$, which may include $O(p)$ points to the left of y_1 and to the right of y_N to keep the interpolation central; $c = H/h$ is the coarsening ratio; p is the interpolation order, which is assumed hereafter to be even; $\sigma_m := \{M : |Y_M^1 - y_m| < pH/2\}$, ω_{mM} are the weights of interpolation from the gridpoints Y_M^1 to y_m, and ε_I is bounded by (14). It follows that

$$\mathrm{E}_S^0(x_j) = \sum_{m=1}^{N_1} \sum_{M \in \sigma_m} \omega_{mM} G_S(Y_M^1 - x_j) \mathrm{J}_+(y_m) + O(\varepsilon_I) =: \bar{E}_S^0(x_j) + O(\varepsilon_I), \quad (19)$$

where

$$\bar{E}_S^0(x_j) := \sum_{M=1}^{N_1} G_S(Y_M^1 - x_j) \mathrm{J}^1(Y_M^1), \qquad j = 1, \ldots N \quad (20)$$

and

$$\mathrm{J}^1(Y_M^1) := \sum_{m \in \tau_M} \omega_{mM} \mathrm{J}_+(y_m), \quad \tau_M := \{m : M \in \sigma_m\}, \quad M = 1, \ldots, N_1. \quad (21)$$

Note that the sums in (21) extend over $O(p)$ points; hence, they are *local*. $\{\mathrm{J}^1(Y_M)\}_M$ is simply the restriction of $\{\mathrm{J}_+(y_m)\}_m$ to the coarse level $l = 1$, a procedure referred to as *anterpolation* in [2], since it is the adjoint of interpolation. Similarly, we can use the smoothness of $G_S(y-x)$ in x to write

$$G_S(Y_M^1 - x_j) = \sum_{J \in \bar{\sigma}_j} \bar{\omega}_{jJ} G_S(Y_M^1 - X_J^1) + O(\varepsilon_I), \quad j = 1, \ldots, N, \quad (22)$$

for all $M = 1, \ldots, N_1$, where $\bar{\sigma}_j := \{J : |X_J^1 - x_j| < pH/2\}$, $\bar{\omega}_{jJ}$ are the x-interpolation weights, and $\{X_J^1\}_{J=1}^{N_1}$ is a uniform grid with meshsize H over $[x_1, x_N]$ (extending $O(p)$ points to the left of x_1 and to the right of x_N, e.g. $X_J = Y_J$). By (20) and (22),

$$\bar{E}_S^0(x_j) = \sum_{J \in \bar{\sigma}_j} \bar{\omega}_{jJ} \mathrm{E}_S^1(X_J^1), \qquad j = 1, \ldots, N, \quad (23)$$

holds up to an $O(\varepsilon_I)$ error, where

$$\mathrm{E}_S^1(X_J^1) := \sum_{M=0}^{N_1} G_S(Y_M^1 - X_J^1)\mathrm{J}^1(Y_M^1), \qquad J = 1, \ldots, N_1. \tag{24}$$

The sums in (23) are again over local sets; (24) is a uniform coarser version of (5). We have therefore reduced the evaluation of E at the fine level ($l = 0$) to the evaluation of the coarse E_S^1 at level $l = 1$. In order to keep the evaluation of (17) inexpensive, the coarsening ratio $c = H/h$ cannot be too large (e.g. $c = 2$ [6], which will be used hereafter), and s should not increase with N. To sum up, the multi-summation (5) is replaced by the following:

1. *Anterpolation:* calculate the "aggregated" $\{\mathrm{J}^1(Y_M)\}_M$ from (21).
2. *Coarse grid summation:* carry out the task (24).
3. *Interpolation:* interpolate $\{\mathrm{E}_S^1(X_J^1)\}_J$ to $\{\bar{E}_S^0(x_j)\}_j$ using (23).
4. *Local corrections:* add $\mathrm{E}_{\mathrm{local}}^0(x_j)$ defined by (17) to $\bar{E}_S^0(x_j)$.

The number of nodes at level 1 is roughly $N/2$, which may still be too large to calculate directly. Instead, the task (24) can be further reduced to summation at level $l = 2$ on twice as coarse (meshsize $2H$) x- and y-grids, using the same algorithm [(i)–(iv)]: decomposition of G_S into G_{2S} plus a local part, anterpolation of J^1 to level 2, level 2 summation, interpolation of E_{2S}^2 to level 1, and the addition of the local corrections. This procedure can be repeated recursively until a grid is reached at which direct summation can be done in at most $O(N)$ operations.

2.4 Pre-calculated tables for local corrections

Since $y_m - x_j$ assumes only uniform grid values, there are only $O(s)$ values of $G_+ - G_S$ different from each other, for *all* the sums ($j = 1, \ldots N$) in (17). These values may be stored in a pre-calculated table of size $O(s)$. Coarser levels require similar tables (e.g. storing the values of $G_S - G_{2S}$ for $l = 1$, and so on). The local correction procedure may be further improved by designing the table of $l = 0$ to annihilate the interpolation errors for $|y-x| \leq S+(ph/2)$. This is done by storing the values $G_+ - \tilde{G}_S$ instead of $G_+ - G_S$, where $\tilde{G}_S$ is the *interpolated* value of G_S from the next coarser level, i.e. the right-hand side of (18) (the coarse level tables are modified similarly). As a result, the local corrections require $O(pN)$ additional operations, but the main interpolation errors are eliminated, since the derivatives of G_S inside the softening region are typically much larger than their values outside (e.g., for $\mathrm{J}(y) := \delta(y - \frac{1}{2})$ and $s = p = 6$, the maximal ε_I is reduced from $5 \cdot 10^{-3}$ to $2 \cdot 10^{-7}$ at the cost of 6 operations per node). In this approach, G_S need not be smooth across $r = S$. For example, $G_S(r)$ could be set to 0 for $r \in [0, S]$. However, a smooth G_S may be needed to form a more accurate discretization of (1).

2.5 Computational cost and evaluation error

Steps (i) and (iii) of the anterpolation and interpolation cost $O(pN)$ each; the local corrections (when implemented with the tables described in §2.4) require $O(sN)$ operations. Thus, the complexity of the fine level work in the

algorithm of §2.3 is $W = O((p+s)N)$. Generally, if the order of anterpolation/interpolation from/to level l to/from $l-1$ is denoted by p_l and the corresponding softening distance is denoted by $S_l := 2^l s_l h$, the total work W per fine grid point in evaluating (5) (neglecting the direct evaluation at the coarsest level) is given by

$$\frac{W}{N} = A\sum_{l=0}^{t-1} 2^{-l}(p_l + Bs_l), \tag{25}$$

where $t = O(\log N)$ is the number of levels and the constants $A, B > 0$ are of order 1 (in our implementation, $A \approx 10$ and $B \approx 0.5$). The error ε_E in evaluating E satisfies (as implied by (14))

$$\varepsilon_E \lesssim 2\sum_{l=0}^{t-1}\left(\frac{p_l}{2es_l}\right)^{p_l}.$$

2.6 Parameter optimization

The values of s and p at each of the levels $l = 0, \ldots, t-1$ should be determined to minimize the computational work (25) under the constraint of a controlled evaluation error, $\varepsilon_E = \varepsilon$.

Let us first consider the case $t = 2$. Discarding the coarse level summation portion of the work, the constrained minimization problem for $p := p_0, s := s_0$ is

$$\begin{cases} W/N & \propto p(1 + B\kappa/(2e)) \quad \longrightarrow \text{min.}, \qquad \kappa := 2es/p, \\ \varepsilon_E & \propto \kappa^{-p} \qquad\qquad\qquad\quad = \varepsilon. \end{cases}$$

Replacing p by $\log(1/\varepsilon)/\log(\kappa)$, the optimum is attained if and only if

$$\left[\frac{d}{d\kappa}\left(\frac{1 + B\kappa/(2e)}{\log(\kappa)}\right)\right]_{\kappa=\kappa_{\text{opt}}} = 0, \quad p_{\text{opt}} = \frac{\log(1/\varepsilon)}{\log(\kappa_{\text{opt}})}.$$

This implies

$$\kappa_{\text{opt}}(\log(\kappa_{\text{opt}}) - 1) = (2e)/B \quad \Longrightarrow \quad \text{(e.g.)} \quad \kappa_{\text{opt}} \approx 9.045 \text{ for } B = 0.5.$$

Thus,

$$p_{\text{opt}} = p_{\text{opt}}(\varepsilon) = K_1 \log\left(\frac{1}{\varepsilon}\right), \quad s_{\text{opt}} = s_{\text{opt}}(\varepsilon) = K_2 \log\left(\frac{1}{\varepsilon}\right). \tag{26}$$

In our implementation, $K_1 \approx 0.45$ and $K_2 \approx 0.75$. Consequently, the computational complexity of evaluating (5) to accuracy ε is

$$W = A(K_1 + BK_2)N \log(1/\varepsilon) =: KN \log(1/\varepsilon). \tag{27}$$

Next, we optimize $\{s_l, p_l\}_l$ for the multilevel summation algorithm. Clearly, if we use $p_l = p_{\text{opt}}(\varepsilon)$, $s_l := s_{\text{opt}}(\varepsilon)$ at all levels $l = 0, \ldots, t-1$, the error ε_E would be $(t-1)\varepsilon$. Instead, the coarse levels should add little error (compared with the fine grid) at a reasonable cost. Thus, we use

$$p_l = p_{\text{opt}}(2^{-l-1}\varepsilon), \quad s_l = s_{\text{opt}}(2^{-l-1}\varepsilon), \qquad l = 0, \ldots, t-1, \tag{28}$$

so that $\varepsilon_E = \sum_{l=0}^{t-1} 2^{-l-1}\varepsilon \le \varepsilon$ and

$$W \le KN \sum_{l=0}^{t-1} 2^{-l} \log\left(\frac{2^{l+1}}{\varepsilon}\right) \le 2KN \left(\log\left(\frac{1}{\varepsilon}\right) + 4\log(2)\right), \tag{29}$$

using (25). Therefore, the total complexity in computing the original sum (5) is also $W = O(N \log(1/\varepsilon))$. In our implementation, $W \approx 8.5N \log(1/\varepsilon)$.

2.7 Numerical results

We constructed a "practical" test case by prescribing in (4) a *left*-hand side function

$$\mathrm{E}(x_j) := e^{-ik\sin(\theta)(j-\frac{1}{2})h}, \qquad j = 1, \ldots, N, \tag{30}$$

which describes a plane wave field hitting the strip at angle θ (see Fig. (1)). The number of gridpoints was proportional to k: $N = [(10k)/(2\pi)]$. The corresponding current values $\{\mathrm{J}(x_j)\}_j$ were then reconstructed by solving (directly) the system of equations (4), and used in the integral evaluation tests described below. First, we performed two-level ($t = 2$) evaluation experiments of (5) for different values of N, p, s to show that the evaluation error satisfies $\varepsilon_E = O(s^{-p})$ independently of N. Figure 3 shows that this indeed happens in practice. Second, we performed the multilevel evaluation of (5) for various N and ε values ($t = O(\log N)$ being the maximum possible, so that at level $l = t-1$ the grids contained $O(p_{t-1})$ points), using $\{p_l, s_l\}_{l=0}^{t-1}$, which were computed using the table generated at the two-level stage and (28). Figure 4 reveals that the relative l_∞ evaluation error ε_E in evaluating E was always less than the desired ε. Moreover, the cost per node of a direct evaluation of E increases linearly with N, whereas it remains constant for our proposed method, as desired. The cross-over was detected at $N \approx 40$ for $\varepsilon = 10^{-2}$, at $N \approx 95$ for $\varepsilon = 10^{-4}$, and at $N \approx 210$ for $\varepsilon = 10^{-8}$.

3 Concluding remarks

In the previous section we described the basic fast evaluation of E for the single strip scattering problem. The algorithm can be extended in various directions such as the following.

The multiple strip problem can be addressed by the same algorithm of §2, where J is defined to be zero outside the strips. Since the coarse grids usually extend beyond the edges of the physical strips, at some coarsening stage they cover the gaps between the strips, thereby reducing the computational task to a single-strip-type task. The algorithm can also be extended for problems with large *quasi-planar* surfaces. Complicated geometries can be efficiently addressed by local refinements (see for example [9]).

The presented approach can be adapted to the fast *multilevel solution* of (2), basically at the cost of *one* evaluation of its right-hand side (see [5]). The softening technique (§2.2) can be used to design discretization schemes whose resolution (to a given accuracy) *does not depend on* k.

The extension to d-dimensional scattering problems can be effected by following the approach of [2, §5]. The main idea is to separate out more directions $y - x$, and to "increase the number of these directions by the factor 2^{d-1} at each coarsening level" [2]. In this way, a coarser grid of J and E is defined for *each* of these directions. The grid should be aligned with the propagation direction of the corresponding eikonal, its meshsize growing faster in that direction than in the perpendicular direction (as in [4]; see a relevant comment

in [3, §7] or [1, §7]). Since the number of points in each grid decreases by the factor 2^d at each level of coarsening, the overall work-per-point will still be independent of the size of the system [2].

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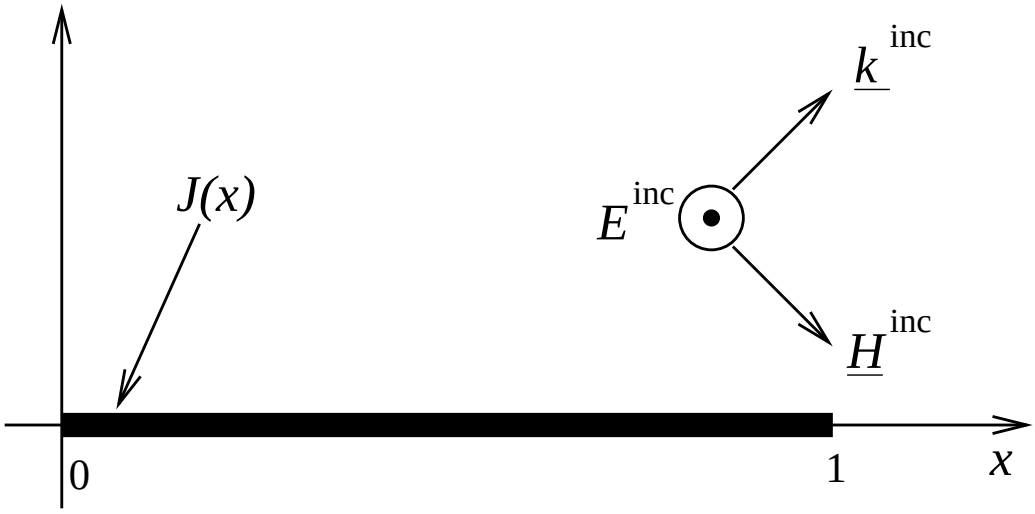


Figure 1: The single strip scattering problem.

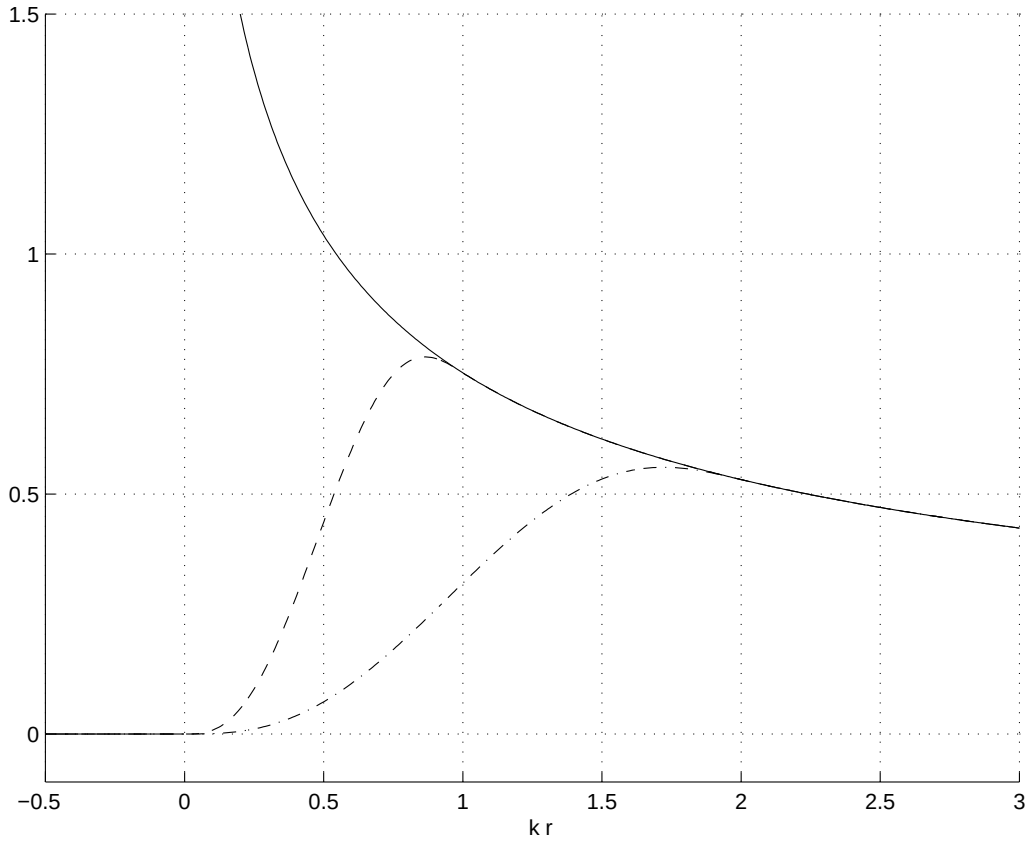


Figure 2: The kernel $G_+(r)$ (solid line) and its softenings $G_S(r)$ for $p = 2, S = 1$ (dashed line) and $p = 2, S = 2$ (dashed-dotted line).

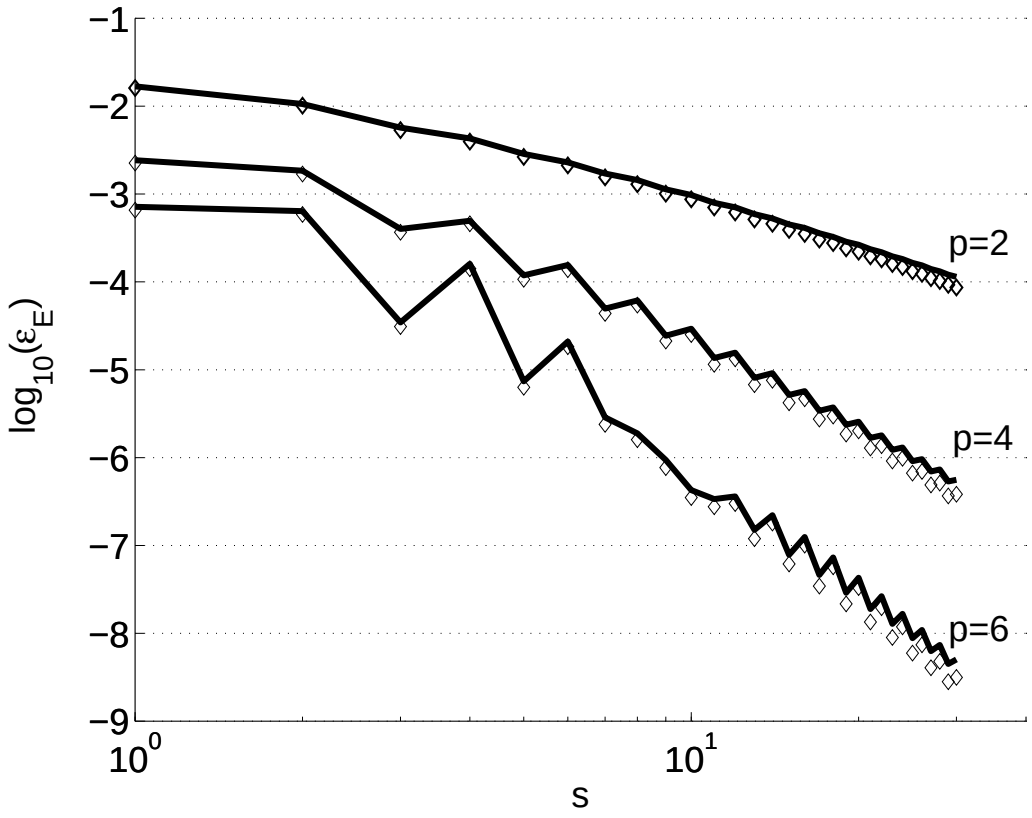
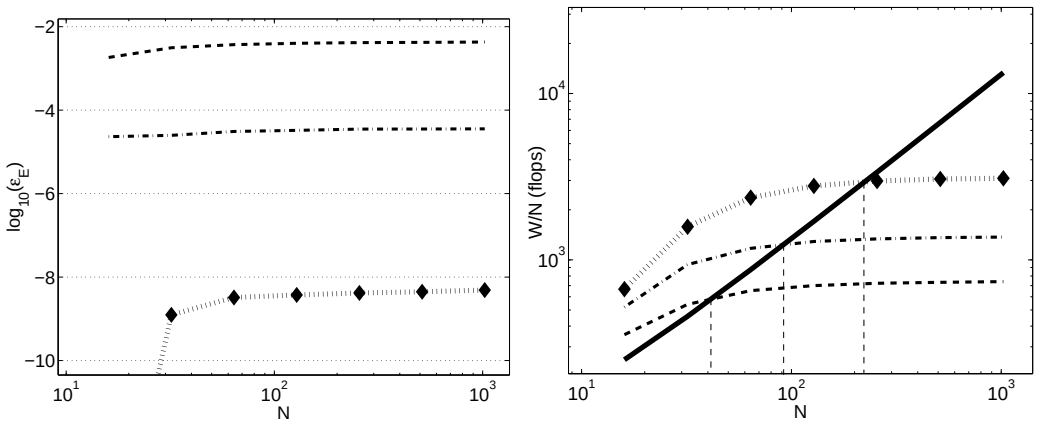


Figure 3: The two-level relative l_∞ evaluation error ε_E in evaluating E as a function of s and p, for $N = 64$ (diamonds) and $N = 1024$ (solid line).



(a) (b)

Figure 4: (a) The multi-level relative l_∞ evaluation error ε_E in evaluating E, and (b) the multilevel computational cost-per-node W/N, both versus N for a prescribed $\varepsilon = 10^{-2}$ (dashed line), 10^{-4} (dashed-dotted line) and 10^{-8} (dotted line with diamonds). The computational cost-per-node of direct evaluation is also shown in (b) (solid line).