ELSEVIER

ESS Patterns: Adding Pairs to an ESS

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ABSTRACT

The notion of a pattern of evolutionarily stable strategies was introduced by Cannings and Vickers in 1988 (*J. Theor. Biol.* 132:387–420). In this paper a specific class of patterns is considered. Suppose that there is an evolutionarily stable strategy (ESS) on some set of n strategies $\{1,2,\ldots,n\}$ and that new strategies $\{n+1,n+2,\ldots,n+k\}$ are added. Supposing that for this new enlarged conflict there is still an ESS on $\{1,2,\ldots,n\}$ and also that there are ESSs on $\{n+i,j\}$ for $1\leqslant i\leqslant k$ and $j\in \mathbf{S}_i\subset\{1,2,\ldots,n\}$, the authors investigate the restrictions on the $\mathbf{S}_i$. These restrictions are related to certain properties of strong tournaments introduced by Reid and Beineke. We also specify, given the $\mathbf{S}_i$, what ESSs of the form $\{n+i,n+j\}$ can be added.

1. INTRODUCTION

The notion of an evolutionary stable strategy (ESS) has become of major importance in studying the strategies adopted by organisms. An ESS corresponds to a strategy which if adopted by a population in some conflict cannot be invaded by any alternative introduced at low frequency. Such a strategy is therefore stable; it will persist if current payoffs and costs remain the same and if no new pure strategies become available.

We now formally define an ESS, restricting ourselves to a finite set of strategies for ease, as that is the focus of this paper. Suppose that pairwise contests are taking place within a species, each player choosing from a set $\mathbf{U}$ of pure strategies. The concept of an ESS corresponding to a noninvadable population strategy was introduced by Maynard Smith

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and Price [1]. Let $\mathbf{A}$ be an $n \times n$ matrix with entries a_{ij}. The expected payoff to an i player when it meets a j player is $E[i,j] = a_{ij}$. Let the proportion of the individuals in the population that play i be p_i (some of these may be zero). The payoff to each member of a group of individuals of which a proportion p_i play i against a group of which a proportion q_i play i is

$$E[\mathbf{p},\mathbf{q}] = \Sigma_{ij} p_i q_j E[i,j].$$

$\mathbf{p}$ is said to be ES (evolutionarily stable) against $\mathbf{q}$ with respect to $\mathbf{A}$ if

(i) $E[\mathbf{p},\mathbf{p}] > E[\mathbf{q},\mathbf{p}]$ or
(ii) $E[\mathbf{p},\mathbf{p}] = E[\mathbf{q},\mathbf{p}]$ and $E[\mathbf{p},\mathbf{q}] > E[\mathbf{q},\mathbf{q}]$.

$\mathbf{p}$ is said to be an ESS of $\mathbf{A}$ if, for all $\mathbf{q} \neq \mathbf{p}$, $\mathbf{p}$ is ES against $\mathbf{q}$ with respect to $\mathbf{A}$.

The *support* of an ESS $\mathbf{p}$ is $\mathbf{S}(\mathbf{p}) = \{i; i \in \mathbf{U}, p_i > 0\}$, that is, it is the set of strategies that have a nonzero probability of being played by an individual who plays the ESS $\mathbf{p}$.

An interesting feature of conflicts with a finite number of available pure strategies, where the payoffs can most naturally be specified by a (payoff) matrix, is the possible existence of multiple ESSs. The ESSs that exist will have supports that are restricted in various ways, and the discussion of these restrictions is the subject of a series of papers by the second and third of the current authors and various collaborators (see, e.g., [2,3]). Biological relevance lies in the possibility that for a particular conflict with a specific payoff matrix one may observe, in separated niches, different strategy combinations depending on which ESSs have evolved in those niches. Conversely, observation of a particular pattern of supports may imply that the niches do not all have the same payoff matrix. Additionally it is of interest to know how many different ESSs might exist for a given number of pure strategies and how large these ESSs might be.

The current paper discusses one particular type of pattern, the one with one ESS with support of size n, and all other ESSs with supports of size 2, although Lemma 3 is more general. We shall specify the necessary and sufficient conditions for that pattern of ESSs. We consider patterns specified by $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_k \rangle$, where $\mathbf{U} = \{1,2,3,\dots,n\}$ and $\mathbf{S}_i \subset \mathbf{U}$, $i = 1,\dots,k$, and the set of ESSs has supports $\mathbf{U}$ and $(n+i, j)$, $1 \leqslant i \leqslant k$ and $j \in \mathbf{S}_i$, so one has a support of size n to which has been added pairs.

Our approach will be the following:

(1) We discuss the possible sign patterns for the entries in a payoff matrix $\mathbf{A} = (a_{ij})$, $i,j = 1,\dots,n$, that supports an internal ESS.

(2) We consider the requirements if we are to add a set of pairs $(n+i,j)$, $j \in \mathbf{S}_i$ for some $i \leqslant k$, and these will be shown to be simple restrictions on the signs of certain entries in **A**.

(3) The considerations of the restrictions demonstrated in item 2 jointly for the set of $\mathbf{S}_i$'s together with the requirements of item 1 lead to a fairly simple specification of the possible sets of $\mathbf{S}_i$ (as described in Theorem 4), part of which is in terms of cycles in tournaments.

(4) We also specify, given the $\mathbf{S}_i$, what ESSs of the form $\{n+i, n+j\}$ can be added. This then solves completely the problem of patterns where all ESS supports, except one, have two elements (Theorem 5).

2. RESTRICTIONS ON A

We shall use three restrictions on the signs of the entries of a payoff matrix **A**. Note that from now on matrix **A** will be assumed, without further comment, to be $n \times n$ and to have $a_{ii} = 0$, $1 \leqslant i \leqslant n$. This involves no loss of generality, as the ESSs of any payoff matrix **B** are precisely those of the matrix $\mathbf{B}^*$, where $b_{ij}^* = b_{ij} - b_{jj}$, which, of course, has zeros on the diagonal [4].

CONDITION 1

A matrix **A** *is said to satisfy Condition* 1 *if* $a_{ij} + a_{ji} > 0$ *all* i, j *with* $1 \leqslant i < j \leqslant n$.

Note. For any i and j, at least one of a_{ij} and a_{ji} must be positive. The usefulness of this condition is well known and has been exploited by Haigh [5], among others.

CONDITION 2

A matrix **A** *is said to satisfy Condition* 2 *if* $\forall$ *nonempty* $\mathbf{V} \subset \mathbf{U}$ $\exists$ (i, j) *with* $j \in \mathbf{V}$ *and* $i \in \mathbf{U} - \mathbf{V}$ *with* $a_{ij} > 0$. *Equivalently (and in the form needed later), it is not possible for* **A** *to have* $a_{ij} < 0$ *all* $j \in \mathbf{V}$ *and all* $i \in \mathbf{U} - \mathbf{V}$.

This condition is related to reducibility, for if we define $\mathbf{B} = (b_{ij})$ *by* $b_{ij} = a_{ij}$ *if* $a_{ij} > 0$ *and* $b_{ij} = 0$ *otherwise, then* **A** *satisfies Condition* 2 *if and only if* **B** *is irreducible.*

A matrix **A** *satisfies Condition* 2 *if there exists a permutation* $(i_1, i_2, \ldots, i_n)$ *of* $(1, 2, \ldots, n)$ *such that* $a_{i_j i_{j+1}} > 0$, $1 \leqslant j \leqslant n-1$, *and* $a_{i_n i_1} > 0$ [6].

CONDITION 3

A matrix **A** *satisfies Condition* 3 *if there exists a permutation* $(i_1, \ldots, i_n)$ *of* $(1, \ldots, n)$ *such that* $a_{i_{j+1} i_j} < 0$, $1 \leqslant j \leqslant n-1$ *and* $a_{i_1 i_n} < 0$.

If a matrix satisfies Conditions 1 *and* 3, *then clearly it also satisfies Condition* 2.

2.1. *SIGN-PATTERN MATRICES AND SETS* **W** *AND* **Z**

We define the set $\mathbf{W} = \{\mathbf{A};\ \mathbf{A}$ satisfies Conditions 1 and 2$\}$.

Corresponding to any matrix $\mathbf{A}$ is a matrix $\mathbf{S(A)}$ whose entries $s(\mathbf{A})_{ij} = \mathrm{sgn}(a_{ij})$. $\mathbf{S(A)}$ is said to be derived from $\mathbf{A}$. Such an $\mathbf{S(A)}$ is referred to as the sign-pattern matrix of $\mathbf{A}$.

We define the set $\mathbf{Z}$ as the set of sign-pattern matrices derived from elements of $\mathbf{W}$. Similarly, we define the sets $\mathbf{W}^* = \{\mathbf{A};\ \mathbf{A}$ satisfies Conditions 1 and 3$\}$ and $\mathbf{Z}^*$ as the set of sign-pattern matrices derived from elements of $\mathbf{W}^*$. Clearly, $\mathbf{Z}^* \subseteq \mathbf{Z}$.

The next lemma and theorem draw on the specification of the conditions under which a specific strategy $\mathbf{p}$ is an ESS of a given payoff matrix A, Haigh's theorem [7].

LEMMA 1

If $\mathbf{A}$ *is a payoff matrix, then an ESS* $\mathbf{p}$ *(other than a pure ESS) of* $\mathbf{A}$ *(if one exists) has* $\mathbf{p}^T\mathbf{A}\mathbf{p} > 0$.

Proof. By Haigh's theorem, if a matrix $\mathbf{A}$ has an ESS, then $a_{ij} + a_{ji} > 0$ for all i, j in the support. Thus,

$$\mathbf{p}^T\mathbf{A}\mathbf{p} = \sum_{i,j} a_{ij} p_i p_j = \sum_{i<j} (a_{ij} + a_{ji}) p_i p_j > 0. \qquad \blacksquare$$

Stronger results hold (1) for a payoff matrix $\mathbf{D}$ (with no restriction on its diagonal entries) if a nonpure ESS $\mathbf{p}$ of $\mathbf{D}$ (if one exists) has $\mathbf{p}^T\mathbf{D}\mathbf{p} > \Sigma_{i\in \mathbf{S(p)}} d_{ii} p_i$ and (2) for an ESS $(\mathbf{p}^T\mathbf{A})_j > 0$ all j in the support of $\mathbf{p}$. This follows because for all $\mathbf{q} \neq \mathbf{p}$ we have $\mathbf{p}^T\mathbf{A}\mathbf{q} > \mathbf{q}^T\mathbf{A}\mathbf{q}$, and so choosing $\mathbf{q}$ to correspond to the pure strategy $\mathbf{j}$ we have $(\mathbf{p}^T\mathbf{A})_j > a_{jj} = 0$.

THEOREM 1

If a matrix $\mathbf{A}$ *has an internal ESS (i.e., with support* $\mathbf{U}$*), then* $\mathbf{A}$ *satisfies both Conditions* 1 *and* 2 *(i.e., if* $\mathbf{A}$ *has an internal ESS, then* $\mathbf{A} \in \mathbf{W}$*).*

Proof. (1) Haigh's theorem requires that if there is an internal ESS, then Condition 1 is satisfied.

(2) The matrix $\mathbf{A}$ has C matrix ($c_{ij} = a_{ij} - a_{il} - a_{lj} + a_{ll}$ for some l) negative definite and hence by [8] possesses a unique ESS. Every $\mathbf{V} \subseteq \mathbf{U}$ specifies a submatrix formed from the appropriate rows and columns of $\mathbf{A}$ whose C matrix is negative definite. Therefore there must be an ESS in $\mathbf{V}$, but if all the a_{ij} were negative for $j \in \mathbf{V}$ and $i \in \mathbf{U}-\mathbf{V}$, then this ESS, suitably augmented by zeros, would also be an ESS in $\mathbf{U}$, because the payoff for any ESS for a matrix with diagonal elements 0 will be

nonnegative (by Lemma 1) and hence could not be invaded by any other strategy. However, there is only one such ESS. ■

LEMMA 2

A payoff matrix **A** *that has constant row sums* T *and* $a_{ij} + a_{ji} = c > 0$ *all* i *and* j *with* $i \neq j$ *has an internal ESS.*

Proof. Since the row sums are constant, there is an internal equilibrium at the unit vector (suitably scaled), that is, $\mathbf{A}1 = T1$.

We need to demonstrate that $\mathbf{z}^T\mathbf{A}\mathbf{z} \leqslant 0 \ \forall \ \mathbf{z}$, with $\mathbf{z}^T 1 = 0$ and equality only if $\mathbf{z} = 0$ [7]. We have

$$\mathbf{z}^T\mathbf{A}\mathbf{z} = \sum_{1 \leqslant i < j \leqslant n} (a_{ij} + a_{ji}) z_i z_j = \sum_{1 \leqslant i < j \leqslant n} c z_i z_j = -c \sum_{1 \leqslant i \leqslant n} z_i^2 \leqslant 0 \quad (1)$$

with equality only if $\mathbf{z} = 0$. ■

THEOREM 2

For any $\mathbf{Y} \in \mathbf{Z}^*$ *there exists an* $\mathbf{A} \in \mathbf{W}^*$ *that has an internal ESS and such that* $\mathbf{S}(\mathbf{A}) = \mathbf{Y}$.

Proof. We are given **Y** and construct an appropriate **A** with $\mathbf{S}(\mathbf{A}) = \mathbf{Y}$. Define **A** as follows, in terms of $\alpha > 0$ and b_i, $1 \leqslant i \leqslant n$.

$$\text{If } j = (i+1) \bmod(n), \text{ then set } a_{ij} = b_i \text{ and } a_{ji} = 2\alpha - b_i;$$

otherwise

$$\text{If } y_{ij} = y_{ji} = +, \text{ then set } a_{ij} = a_{ji} = \alpha,$$

and

$$\text{if } y_{ij} = + \text{ while } y_{ji} = -, \text{ then set } a_{ij} = 1 + 2\alpha \text{ and } a_{ji} = -1.$$

We now demonstrate that one can choose the b_i and α such that each row of **A** adds to the same value, $T = (n-1)\alpha$.

$$\sum_j a_{ij} = b_i + 2\alpha - b_{(i-1)\bmod(n)} + r_i,$$

where r_i is the sum of the elements of the row other than those from the permutation, that is, $n-2$ entries.

Each of these sums is equal to T, so we can easily express each b_i, for $2 \leqslant i \leqslant n$, in terms of b_1 as

$$b_i = b_1 + \sum_{j=2}^{j=i} (T - r_j - 2\alpha),$$

where $T = (n-1)\alpha$.

Hence $\mathbf{A}$ satisfies the requirements of Lemma 2 and so has an internal ESS.

We now choose $b_1 = M (M$ very large) so that $b_1 \gg 0$ and thus all $b_i \gg 0$, and so $2\alpha - b_i$ is negative for all i. This means that $\mathbf{A}$ satisfies Condition 3 with permutation the identity and that $(a_{ij} + a_{ji}) = 2\alpha$ for all $i \neq j$, so $\mathbf{A} \in \mathbf{W}^*$ and $\mathbf{S}(\mathbf{A}) = \mathbf{Y}$. ■

A stronger result holds. For any element of $\mathbf{Y} \in \mathbf{Z}^*$ and probability vector $\mathbf{p} > 0$, there exists an $\mathbf{A} \in \mathbf{W}^*$ that has an internal ESS $\mathbf{p}$ and such that $\mathbf{S}(\mathbf{A}) = \mathbf{Y}$. This requires that $\sum_j a_{ij} p_j = T$ all i, and the b_i's can easily be suitably defined to achieve this.

3. RESTRICTIONS ON THE $\mathbf{S}_i$

We now turn our attention to adding pairs, that is, defining the sets of $\mathbf{S}_i$ that can coexist.

THEOREM 3

Suppose $\mathbf{A}$ *is a payoff matrix defined on* $\mathbf{U}$, $\mathbf{A}^+$ *is a payoff matrix defined on* $\mathbf{U} \cup \{n+1\}$ *with* $a^+_{ij} = a_{ij}$ $\forall$ $i, j \leqslant n$, $a^+_{n+1\,n+1} = 0$, *and* $0 < a^+_{1\,n+1} < a^+_{2\,n+1} < \cdots < a^+_{k\,n+1}$ *for some fixed* $k \leqslant n$, *and there are ESSs on* $(i, n+1)$ *for* $1 \leqslant i \leqslant k$. *Then* $a_{ij} < 0$ *whenever* $1 \leqslant j < i \leqslant k$.

Proof. We require there to be an ESS on $(j, n+1)$ for $j \leqslant k$ that requires $a^+_{j\,n+1} > 0$, which we have already specified, and $a^+_{n+1\,j} > 0$. We thus have $a^+_{jj} = 0$, $a^+_{j\,n+1} > 0$, so if $i \in \mathbf{U}$ is not to invade when $a^+_{i\,n+1} > a^+_{j\,n+1}$, we require $a_{ij} < 0$. Thus since $a^+_{1\,n+1} < a^+_{2\,n+1} < \cdots < a^+_{k\,n+1}$, we require $a_{ij} < 0$ for all $1 \leqslant j < i \leqslant k$. ■

If, as is our objective, we wish to add additional strategies $n + i$, each of which is associated with a set $\mathbf{S}_i$, then each of these addition will require corresponding negative a_{ij} in specific positions in $\mathbf{A}$. It is thus necessary that these conditions be consistent with any restrictions imposed on $\mathbf{A}$ due to it having an internal ESS.

Suppose that $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$, where $\mathbf{U} = \{1,2,3,\ldots,n\}$ and $\mathbf{S}_i \subset \mathbf{U}$, $1 \leqslant i \leqslant k$, specifies the set of supports $\mathbf{U}$ and $(n+i, j)$, $1 \leqslant i \leqslant k$ and $\forall$ $j \in \mathbf{S}_i$ (so one has a support of size n to which have been added pairs).

We refer to $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ as attainable if the corresponding pattern is attainable.

The following theorems, which specify the conditions imposed on $\mathbf{A}$ for $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ to be attainable, are best expressed in terms of tournaments, so we digress briefly to define some important terms [9].

A *tournament* $\mathbf{T}(\mathbf{V}, \mathbf{E})$ is a directed simple graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$, where $\mathbf{V}$ is a set of vertices and $\mathbf{E}$ a set of edges (i.e., ordered pairs of vertices that are said to join those vertices), and every pair of vertices is joined precisely once. It therefore can be used to represent the results in a round robin competition.

A vertex u *dominates* a vertex v if their common edge is directed from u to v.

A tournament (or subtournament) is *transitive* if the binary relation *dominates* is transitive (i.e., u *dominates* v, v *dominates* $w \Rightarrow u$ *dominates* w).

A tournament is *reducible* if its set of vertices can be partitioned into nonempty sets V_0 and V_1 in such a way that every $u \in V_0$ dominates every $v \in V_1$.

A tournament that is not reducible is called irreducible or *strong*.

A fundamental result we shall need is that a tournament is strong if and only if there exists a spanning circuit, that is, there is a permutation $(i_1, i_2, \ldots, i_n)$ of $(1,2,3,\ldots,n)$ such that i_j dominates i_{j+1} for all j, and i_n dominates i_1 [9].

LEMMA 3

If an $n \times n$ matrix $\mathbf{A}$ has negative entries in positions a_{ij}, $1 \leqslant j < i \leqslant k < n$, and no pure ESSs, then one can find an $(n+1) \times (n+1)$ matrix $\mathbf{A}^+$ whose ESSs are precisely those of $\mathbf{A}$ together with ESSs that have the supports $(i, n+1)$, $i = 1, \ldots, k$.

Proof. Define $\mathbf{A}^+$ by

$$a_{ij}^+ = \begin{cases} a_{ij} & \text{if } 1 \leqslant i;\ j \leqslant n, \\ (1-c^i)/(1-c) & \text{if } j = n+1;\ 1 \leqslant i \leqslant k, \\ (wr)/c^j & \text{if } i = n+1;\ 1 \leqslant j \leqslant k, \\ -M & \text{otherwise,} \end{cases}$$

where c and r are defined to satisfy $0 < 2c < r < m/2w < 1$, where w is the largest element of $\mathbf{A}$, $m = \min_{i \neq j} |a_{ij}|$, and M is a large positive constant.

The ESSs of $\mathbf{A}$ (Note no ESS of $\mathbf{A}$ can have support within $\{1,2,\ldots,k\}$ because $a_{ij} < 0$, $1 \leqslant j < i \leqslant k$) are also ESSs of $\mathbf{A}^+$, since for each ESS of **A** there is an i in its support with $a^+_{n+1\,i} = -M$, so the payoff to $n+1$ against the ESS must be less than for the ESS against itself.

For each i, $1 \leqslant i \leqslant k$, the pair $(i, n+1)$ has $a^+_{i\,n+1} > 0$, $a^+_{n+1\,i} > 0$, $a^+_{ii} = 0$, and $a^+_{n+1\,n+1} = 0$, so it is an ESS in its own space. We need to prove that it cannot be invaded by any j.

For $j > k$, the entry $a^+_{jn+1} = -M$, so j cannot invade.

For j where $1 \leqslant j < i \leqslant k$,

$$\begin{aligned}
wa^+_{in+1}/\left(a^+_{in+1} - a^+_{jn+1}\right) &\leqslant w(1-c^i)/(1-c)c^{i-1} \leqslant 2w/c^{i-1}, \\
&< rw/c^i = a^+_{n+1\,i} \\
&\Rightarrow wa^+_{in+1} + a^+_{jn+1}a^+_{n+1\,i} < a^+_{in+1}a^+_{n+1\,i} \\
&\Rightarrow a^+_{ji}a^+_{in+1} + a^+_{jn+1}a^+_{n+1\,i} < a^+_{in+1}a^+_{n+1\,i} \qquad (2)
\end{aligned}$$

and the final inequality is precisely the noninvadability (by j) condition for the putative ESS on $(i, n+1)$.

For j where $1 \leqslant i < j \leqslant k$,

$$\begin{aligned}
a^+_{in+1}(1 + m/a^+_{n+1\,i}) &= ((1-c^i)/(1-c))\left[1 + (mc^i/wr)\right] \\
&\geqslant \left[(1-c^i)/(1-c)\right](1+2c^i) \\
&= (1 + c^i - 2c^{2i})/(1-c) \\
&> 1/(1-c) \\
&> a^+_{jn+1} \\
&\Rightarrow a^+_{in+1}a^+_{n+1\,i} \\
&> a^+_{jn+1}a^+_{n+1\,i} - ma^+_{in+1} \\
&\geqslant a^+_{jn+1}a^+_{n+1\,i} + a^+_{ji}a^+_{in+1} \qquad (3)
\end{aligned}$$

which is again the condition for noninvasion.

We have thus proved that the original ESSs of **A** and the extra pairs of the statement are indeed ESSs of $\mathbf{A}^+$. It remains to prove that there are no other ESSs. There are clearly no ESSs with supports in $(1, 2, \ldots, n)$, as those would have been ESSs of **A**. There are no ESSs involving $n+1$ and elements of $(1, 2, \ldots, n)$ other than those specified. For $k < j \leqslant n$, both $a^+_{n+1\,j}$ and a^+_{jn+1} are negative, so no ESS can contain both $n+1$ and such a j by Haigh's theorem. For $1 \leqslant i \leqslant k$, $(i, n+1)$ is an ESS, so no other support of an ESS can contain both $(n+1)$ and i.

THEOREM 4

$\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ *is attainable if and only if there exists an* **F** *such that* **T(U,F)** *is a strong tournament and each subtournament* $\mathbf{T}(\mathbf{S}_i, \mathbf{F})$, $1 \leqslant i \leqslant k$, *is transitive.*

Proof. (1) Suppose there exists a strong tournament $\mathbf{T}(\mathbf{U},\mathbf{F})$ such that each subtournament $\mathbf{T}(\mathbf{S}_i,\mathbf{F})$ is transitive. Then define $\mathbf{Y}$ to have $y_{ij} = -\text{if}(i,j) \in \mathbf{F}$ and positive values elsewhere. Thus the sign pattern of $\mathbf{Y}$ satisfies Theorem 2, so there exists a payoff matrix $\mathbf{A}$ with $\mathbf{S}(\mathbf{A}) = \mathbf{Y}$ with an internal ESS. Further, this $\mathbf{A}$ satisfies the conditions of Lemma 3 with respect to $\mathbf{S}_1$ [we can relabel the elements of $\mathbf{S}_1$ as $1,2,\ldots,|\mathbf{S}_1| < n$; $|\mathbf{S}_1| < n$ since $\mathbf{T}(\mathbf{U},\mathbf{F})$ is strong and $\mathbf{T}(\mathbf{S}_1,\mathbf{F})$ is transitive], and hence the matrix $\mathbf{A}^+$ of Lemma 3 has as ESSs the pairs $(i, n+1)$, $i \in \mathbf{S}_1$ and $\mathbf{U}$. This new matrix $\mathbf{A}^+$ itself satisfies the condition of Lemma 3 with respect to $\mathbf{S}_2$, and we can apply an inductive argument to add the pairs required.

(2) Suppose that $\langle \mathbf{U},\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_k \rangle$ is attainable. Then there exists a payoff matrix $\mathbf{A}^+$ with this pattern of ESSs. $\mathbf{A}^+$ has an ESS on $\mathbf{U}$, so this submatrix, $\mathbf{A}$ say, satisfies Condition 2. Consider the set of negative entries of $\mathbf{A}$. If this set is of size $n(n-1)/2$, then the tournament $\mathbf{T}(\mathbf{U},\mathbf{F})$ where $\mathbf{F} = (i,j)$; $a_{ij} < 0$ is strong, since it is the complement of $\mathbf{T}(\mathbf{U},\mathbf{G})$, where $\mathbf{G} = (i,j)$; $a_{ij} > 0$, which is strong by Theorem 1. The existence of the pairwise ESSs defined in terms of the $\mathbf{S}_i$ requires that the subtournaments $\mathbf{T}(\mathbf{S}_1,\mathbf{F})$ are transitive by Theorem 3.

If the number of negative entries is of size less than $n(n-1)/2$, then we may identify a spanning cycle within the positive elements of $\mathbf{A}$ and add elements to $\mathbf{F}$ so that if (i,j) belongs to the spanning cycle then $(j,i) \in \mathbf{F}$. $\mathbf{T}(\mathbf{U},\mathbf{F})$ is now strong and, as above, all $\mathbf{T}(\mathbf{S}_i,\mathbf{F})$, $1 \leqslant i \leqslant k$, are transitive.

In fact, stronger results than those of Theorem 4 are possible. When adding new pure strategies and pairs to an existing setup we can also add pairwise ESSs with supports $(n+i, n+j)$. We demonstrate this possibility only in the case corresponding to Theorem 4. We can add any pairs from the $\{n+1, n+2, \ldots, n+k\}$ provided that by so doing we do not create any "triangles," that is, all pairs from a triple $\{u,v,w\}$. It was shown in [2] that one could not have a set $\{u,v,w\}$ with ESSs on (u,v), (v,w), and (u,w); this is the "triangle exclusion rule." Thus Theorem 5 proves that this is the only restriction in this context.

THEOREM 5

Suppose that $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ *is attainable. Then* $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m \rangle$ *[where this notation indicates that the original set of ESSs have been augmented by a new set of pairs with supports* $\mathbf{T}_i, i = 1, \ldots, m$*, each* $\mathbf{T}_i$ *being of the form* $(n+u, n+v), 1 \leqslant u < v \leqslant m$*] is attainable if and only if*

(1) $\mathbf{T}_i = (n+u, n+v) \Rightarrow \mathbf{S}_u \cap \mathbf{S}_v = \varnothing$, *and*

(2) $\mathbf{T}_i = (n+u, n+v)$ *and* $\mathbf{T}_j = (n+v, n+w)$ *implies that there is no* l *with* $\mathbf{T}_l = (n+u, n+w)$.

Proof. (a) Suppose that $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m \rangle$ is attainable. Then if $\mathbf{S}_u \cap \mathbf{S}_v \neq \emptyset$, there is some element $j \in \mathbf{S}_u \cap \mathbf{S}_v$ such that $(n + u, j)$ and $(n + v, j)$ both support ESSs. It follows, from the triangle exclusion rule, that $(n + u, n + v)$ cannot support an ESS. Thus there is no $\mathbf{T}_i = (n + u, n + v)$. Thus Condition 1 holds, and Condition 2 is simply the triangle exclusion rule.

(b) Suppose that the $\mathbf{T}_i$ satisfy Conditions 1 and 2. Suppose that $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ is attainable on matrix $\mathbf{A}^+$ defined sequentially as in Theorem 4, using Lemma 3. Define $\mathbf{A}^*$ by

$$a_{ij}^* = \begin{cases} +1 & \text{if } \exists\, x\, \mathbf{T}_x = (n+i, n+j), \\ +1 & \text{if } \exists\, x\, \mathbf{T}_x = (n+j, n+i), \\ a_{ij}^+ & \text{otherwise.} \end{cases}$$

Now the change from $\mathbf{A}^+$ to $\mathbf{A}^*$ does not affect the existence of the ESS of $\mathbf{U}$, since only elements with both indices exceeding n have been changed. The ESSs of the form $(n + i, j)$ for $j \in \mathbf{S}_i$ remain, because for any l either $\mathbf{S}_i \cap \mathbf{S}_l = \emptyset$, in which case $a_{n+lj}^* = a_{n+lj}^+ = -M$, or $j \in \mathbf{S}_i \cap \mathbf{S}_l$, in which case $a_{n+ln+i}^* = a_{n+ln+i}^+ = -M$. Thus $(n + i, j)$ will not be invaded.

Finally, we see that for any $\mathbf{T}_x = (n + i, n + j)$ there is an ESS. Within the set of strategies $\{n+1, n+2, \ldots, n+k\}$, $(n + i, n + j)$ is clearly an ESS [3]. We have that $\mathbf{S}_i \cap \mathbf{S}_j = \emptyset$, so that for any $1 \leqslant u \leqslant n$, either $a_{un+i}^* = -M$, $a_{un+j}^* = -M$, or both, so that $(n + i, n + j)$ is not invadable. ■

4. COROLLARIES

COROLLARY 1

If $\{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k\}$ *is such that* $\exists\,(i, j)$ *such that* $\{i, j\} \not\subseteq \mathbf{S}_u$ *for an* $\mathbf{u}$, *then* $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ *is attainable.*

Proof. The tournament $\mathbf{T}(\mathbf{U}, \mathbf{E})$, where $\mathbf{E} = \{\bigcup_{1 \leqslant i < j \leqslant n} (i, j)\} \cup (n, 1) \setminus (1, n)$, is strong, because $(1, 2, 3, \ldots, n)$ defines a spanning circuit, and each subtournament that does not contain both 1 and n is transitive. ■

COROLLARY 2

If $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ *is attainable, then* $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k, \mathbf{S}_{k+1}, \ldots, \mathbf{S}_{k+m} \rangle$ *where* $|\mathbf{S}_{k+i}| \leqslant 2$ *for* $1 \leqslant i \leqslant m$ *is attainable.*

Proof. $\langle \mathbf{U}, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k \rangle$ attainable implies that there exists a strong tournament with transitive subtournaments $\mathbf{T}(\mathbf{S}_i, \mathbf{F})$. Clearly any $\mathbf{S}_i$ of size 1 or 2 can be added because $\mathbf{T}(\mathbf{S}_i, \mathbf{F})$ will be automatically transitive. ■

COROLLARY 3

If $\mathbf{T}(\mathbf{U},\mathbf{E})$ *is strong, then* $\langle \mathbf{U},\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_k\rangle$ *is attainable if none of the* $\mathbf{S}_i$ *contains a triple* (r,s,t) *that is a cyclic triangle in* $\mathbf{T}(\mathbf{U},\mathbf{E})$.

Proof. $\mathbf{T}(\mathbf{U},\mathbf{E})$ is strong implies that $\langle \mathbf{U},\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_k\rangle$ is attainable if $\forall\ i$ $\mathbf{T}(\mathbf{S}_i,\mathbf{E})$ is transitive, and this occurs if and only if $\mathbf{T}(\mathbf{S}_i,\mathbf{E})$ contains no cycles. Since any cycle implies the existence of a three-cycle, the result follows. ■

COROLLARY 4

If $\langle \mathbf{U},\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_k\rangle$ *is attainable, then* $\langle \mathbf{U},\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_k,\mathbf{S}_{k+1},\ldots,\mathbf{S}_{k+m}\rangle$ *where* $\mathbf{S}_{k+j}\subseteq\mathbf{S}_i$ *for some i and all* $1\leqslant j\leqslant m$ *is attainable.*

Proof. Since there exists a strong tournament with transitive subtournaments on each $\mathbf{S}_i$, all $1\leqslant i\leqslant k$, all subtournaments of these are transitive. ■

The above corollaries allow one to specify the set of possible attainable patterns of interest in terms of certain maximal sets of sets. Suppose that for given n, $\Phi=\{\mathbf{T}_1,\mathbf{T}_2,\ldots,\mathbf{T}_l\}$ is such that there exists a strong tournament in which each $\mathbf{T}_i$ is transitive and such that every transitive subtournament is a subset of some $\mathbf{T}_i$. Then $\langle \mathbf{U},\Delta\rangle$ where $|\Delta|=m$ and $\Delta\in\Phi^m$ is attainable (for any m). Thus each such Φ specifies a family of attainable patterns and the set of possible Φ specifies the set of attainable patterns. We shall write a possible Φ as $[\mathbf{T}_1,\mathbf{T}_2,\ldots,\mathbf{T}_l]$.

An alternative specification is via cycle triangles. We say that a set of triples $\Gamma=\{\mathbf{T}_1,\mathbf{T}_2,\ldots,\mathbf{T}_l\}$ is a minimal cyclic set for given n if there exists a strong tournament that has cyclic triangles corresponding to the $\mathbf{T}_i$ and there is no strong tournament with given n whose cyclic triangles are a subset of Γ. We shall display the sets Γ as $((\gamma_1,(\gamma_2),\ldots,(\gamma_l))$. Φ and Γ are related in a simple manner; for any Γ there is a matching Φ whose elements are those that are minimal and do not contain any of the members of Γ.

5. CASES $n=4$ AND $n=5$

THE CASE $n=4$

There is essentially only one case to consider; we ignore $\mathbf{S}_i$ of size 1 or 2 because they may be added, by Corollary 2.

(1) Suppose that $(1,4)$ is not in any $\mathbf{S}_i$. Then we can take $\mathbf{S}_1=(1,2,3)$ and $\mathbf{S}_2=(2,3,4)$.

(2) If there is no missing pair in the $\mathbf{S}_i$, then we obtain the same result because the unique strong tournament on four vertices (suitably numbered) is precisely the one constructed in Corollary 1.

Accordingly for $n = 4$ we have that the $\mathbf{S}_i$ are a subset of

$$(1),(2),(3),(4),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4),(1,2,3),(2,3,4),$$

so that the unique attainable pattern is specified by the cycles $[(1,2,3),(2,3,4)]$ up to a permutation

THE CASE $n = 5$

For $n = 5$ there are four permutationally distinct strong tournaments [9] but only two minimal sets of cyclic triangles,

$$((1,2,3),(1,2,4),(1,2,5)) \qquad \text{and} \qquad ((1,2,3),(2,3,4),(3,4,5)).$$

Thus there are two permutationally distinct patterns of $\mathbf{S}_i$,

$$[(1,3,4,5),(2,3,4,5)]$$

and

$$[(1,3,4),(1,3,5),(2,3,5),(1,2,4,5)].$$

To each of these may be added the seven noncyclic triangles and all the pairs and singletons.

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