

## Asymptotics for geometric spectral densities and a stochastic approach of the lattice-point problem

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**Abstract** How many points of a lattice hit a given planar domain? This question is known as the lattice-point problem. In stereology the problem is to assess the precision of the area estimator based on number of hits. A stochastic approach of the lattice-point problem has been first developed by D. Kendall. In this paper we extend this approach. The key point is a convergence result for the spectral density of random sets. As a statistical application we consider the stereological estimation of planar area. Approximation formulae for mean square errors are provided.

**Keywords** area, estimation, lattice, mean square error, random set, systematic sampling, spectral density, stereology.

## 1 Introduction

The lattice-point problem dates back (at least) from Gauss who stated that the number of points  $P$  of the integral lattice  $\mathbf{Z}^2$  hitting a disc with radius  $r$  (and centered at the origin) is such that

$$P - \pi r^2 = O(r), \quad r \rightarrow \infty. \quad (1)$$

Note that the asymptotic can be reversed by replacing the fixed lattice  $\mathbf{Z}^2$  by a lattice  $\sqrt{a}\mathbf{Z}^2$  (the area  $a$  of a fundamental region of the sampling lattice tends to 0) and by replacing the variable disc by the (fixed) unit disc. Then Formula (1) can be rewritten as

$$aP - \pi = O(\sqrt{a}).$$

The stereological interpretation of the formula above is as follows: the error of the disc area estimator  $aP$  is of the order  $\sqrt{a}$  (i.e. lattice spacing) for sufficiently dense sampling lattices. Since Gauss the exponent of  $a$  has been further and further refined. Moreover more general domains than discs (e.g. convex sets) have been considered.

A stochastic approach has been first proposed by D. Kendall [9, 10] where the location of the lattice with respect to the (convex) domain is randomized. Kendall derived an asymptotic approximation of the mean square error

$$E \left[ (aP - A)^2 \right], \quad a \rightarrow 0,$$

where  $A$  is the area of the domain. Kendall's result was further considered with a more practical point of view by Matheron [11, 12]. Matheron decomposed Kendall's approximation into 2 terms: the extension term and the fluctuating term. Although both terms are of the same asymptotic order, Matheron suggested to ignore the fluctuating term. Empirical studies e.g. [6] show that the fluctuating term is small compared to the extension term.

In this paper we further randomize the problem: both the sampling lattice and the domain under investigation are considered as random. Only the location of the lattice is supposed to be random. The domain under investigation is assumed to be random in size and orientation. In such a framework, it can be shown that the fluctuating term of the mean square error is of higher order compared to the extension term. Our result may be considered as a theoretical justification of approximations used in practical stereology, see e.g. [17, 18, 6, 2, 3, 4].

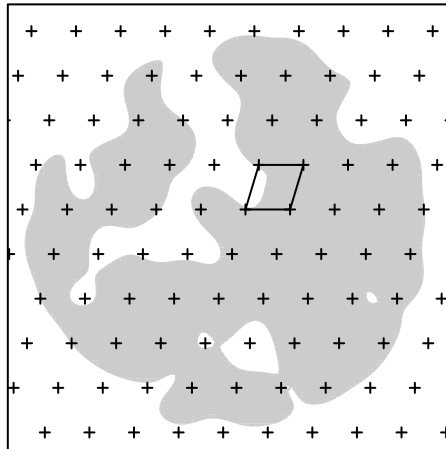


Figure 1: A compact planar domain (in gray) sampled by a lattice of points. The domain area is estimated by the number of lattice points hitting the domain multiplied by the area of a fundamental region of the lattice (parallelogram).

Section 2 introduces the estimation problem. In Section 3 the mean square error is expressed in terms of the so-called geometric spectral density of a (deterministic) domain. Previous asymptotic approximations of spectral densities are reviewed in Section 4.

In Section 5, asymptotics of geometric spectral densities are established in a stochastic framework. Finally applications to the lattice-point problem are discussed in Section 6.

## 2 Unbiased area estimation

Let  $X$  be a (fixed) compact subset of the Euclidean space  $\mathbf{R}^2$ . The parameter to be estimated is the area  $A$  of  $X$ . The planar domain  $X$  is not observed *in extenso*: only the intersection  $X \cap \mathbf{Y}$  of  $X$  with a stationary random lattice  $\mathbf{Y}$  is available, see Figure 1. Only the location of the lattice is randomized:  $\mathbf{Y}$  can be written as

$$\mathbf{Y} = \Lambda + \mathbf{U},$$

where  $\Lambda$  is a fixed lattice and  $\mathbf{U}$  is a random translation vector uniformly distributed in a fundamental region of the lattice  $\Lambda$ . It is easy to check that  $\mathbf{Y}$  is a stationary random set. The intensity  $\nu$  of  $\mathbf{Y}$  is defined by

$$\mathbf{E} [\lambda^0 (\mathbf{Y} \cap C)] = \nu \lambda^2 (C),$$

where  $\lambda^i$  is the  $i$ -dimensional Hausdorff measure for  $i = 0, 1, 2$  and  $C$  is any compact subset of  $\mathbb{R}^2$ . Simple calculations yield

$$\nu = a^{-1},$$

where  $a$  is the area of a fundamental region of  $\Lambda$ . It follows that the area estimator

$$\hat{\mathbf{A}} = a\lambda^0(\mathbf{Y} \cap X) \quad (2)$$

is unbiased.

Hence the mean estimation error is equal to 0. What about the precision of the estimator (2)? In statistics it is common to assess the precision of estimators from their mean square error:

$$\text{MSE}[\hat{\mathbf{A}}] = \mathbb{E} \left[ \left( \hat{\mathbf{A}} - A \right)^2 \right].$$

For an unbiased estimator, the variance and the mean square error coincide. In his book [15] on integral geometry and geometric probability, Luis A. Santaló wrote

The problem of determining the variance of the number of lattice points covered by a random region is in general difficult.

First let us mention some results on the error range which can be derived from deterministic geometric inequalities. A quick review of such inequalities involving lattices is provided by Santaló [15].

In particular, Blichfeldt's theorem yields the following lower bound for the worst positive error:

$$\sup \hat{\mathbf{A}} - A \geq a \left\lfloor \frac{A}{a} \right\rfloor - A + a,$$

where  $\lfloor \cdot \rfloor$  means "integral part". In case the domain  $X$  is convex, the estimator  $\hat{\mathbf{A}}$  is lower bounded:

$$\inf \hat{\mathbf{A}} - A \geq -\frac{B}{2}\sqrt{a},$$

where  $B$  is the boundary length of the convex  $X$ , see Bokowski et al. [1].

Further results are provided by asymptotic deterministic approaches where the lattice density is supposed to tend to infinity. Gauss result for a disc has already been mentioned in Section 1. A recent progress is due to Huxley [8]:

$$\hat{\mathbf{A}} - A = O \left( a^{\frac{50}{73}} \left( \log \frac{1}{a} \right)^{\frac{315}{146}} \right). \quad (3)$$

The latter result holds for a convex domain  $X$  with a  $\mathcal{C}^2$  boundary. Furthermore it is assumed that the curvature is non-zero on the boundary.

### 3 Mean square error and spectral density

Kendall [9] has been the first author to consider a stochastic approach where the location of the sampling lattice with respect to the domain  $X$  is randomized. Using Parseval equality, Kendall expressed the mean square error as

$$\text{MSE}[\widehat{\mathbf{A}}] = \sum'_{y \in \Lambda^*} \text{PSD}_X(y), \quad (4)$$

where  $\Lambda^*$  is the lattice dual of  $\Lambda$ , the prime exponent means that the origin is excluded from the summation and  $\text{PSD}_X$  is the (power) spectral density of the area measure restricted to  $X$ :

$$\text{PSD}_X(y) = \left| \int_X \exp(-2\pi i x \cdot y) dx \right|^2.$$

The dual  $\Lambda^*$  of  $\Lambda$  is the lattice such that for any pair  $(x, y) \in \Lambda \times \Lambda^*$  the scalar product  $x \cdot y$  is an integer, see Figure 2. The determinants of dual lattices are inverse:  $|\Lambda^*| = 1/|\Lambda|$ . So are their densities. In particular if we consider a very dense sampling lattice, the lattice sum in Equation (4) depends only on the behavior of the spectral density  $\text{PSD}_X$  for points  $y$  far from the origin.

The spectral density  $\text{PSD}_X$  may also be defined as the Fourier transform of the (geometric) covariogram  $k$  of  $I_X$ :

$$k(h) = \int_{\mathbf{R}^2} I_X(x+h) I_X(x) dx = \lambda^2(X \cap (X-h)), \quad h \in \mathbf{R}^2.$$

Note that the covariogram  $k$  may be written as the convolution product  $I_X * I_{-X}$ . The covariogram is continuous, admits a maximum at the origin and is a positive definite function.

The end of Section 4 involves spectral densities and covariograms of more general functions. The spectral density of a function  $f$  is defined as the squared modulus of its Fourier transform. It can be defined also as the Fourier transform of its covariogram  $g = f * \check{f}$  where  $\check{f}(x) = f(-x)$ .

### 4 Previous asymptotics for spectral densities

In general Equation (4) is not of direct use because the spectral density of the planar domain  $X$  is not known. Previous authors [9, 12] have derived

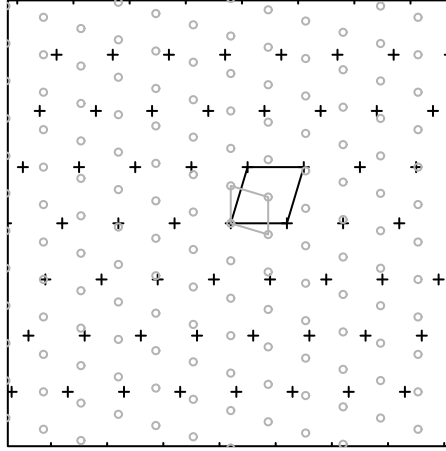


Figure 2: Dual lattices and their fundamental regions.

“simple” asymptotics for the spectral density of  $X$  which can be used in Equation (4).

In this section we do not restrict ourselves to the plane  $\mathbf{R}^2$ . The domain is a compact subset of the general Euclidean space  $\mathbf{R}^d$ .

Let us assume that the boundary  $\partial X$  of  $X$  is a continuously differentiable manifold. For any point  $x \in \partial X$ , let  $n(x)$  be the outer normal unit vector to  $X$  at  $x$ . The coordinates of  $n(x)$  are denoted by  $n_j(x)$ . The Fourier transform of the indicator function  $I_X$  fulfills the following equation

$$2\pi i y_j (\mathcal{F}I_X)(y) = (\mathcal{F}U_j)(y), \quad j = 1, \dots, d$$

where  $U_j$  is the measure on  $\mathbf{R}^d$  defined by

$$U_j(\phi) = \int_{\partial X} \phi(x) n_j(x) dx.$$

The relationship between the Fourier transform of  $I_X$  and  $U_j$  may be established using the Gauss-Green theorem and standard Fourier calculus. Summing the squared modulus for  $j = 1, \dots, d$  on both sides, one gets

$$4\pi^2 \|y\|^2 \text{PSD}_X(y) = \sum_{j=1}^d \text{PSD}_{U_j}(y). \quad (5)$$

Asymptotics for Fourier type integrals may be obtained using the method of the stationary phase, see e.g. [7]. Using such an approach, Kendall [9]

obtained the following convergence result:

$$4\pi^2 \|y\|^{d+1} \text{PSD}_X(y) \sim \sum_{\substack{x_1 \in \partial X: n(x_1) \parallel y \\ x_2 \in \partial X: n(x_2) \parallel y}} Z(x_1, x_2, y), \quad \|y\| \rightarrow +\infty \quad (6)$$

where

- $n(x) \parallel y$  stands for " $n(x)$  is parallel to  $y$ ".
- If  $K(x)$  denotes the Gauss-Kronecker curvature of  $\partial X$  at  $x$ , the function  $Z$  is defined by

$$Z(x_1, x_2, y) = \frac{\cos(2\pi(x_2 - x_1) \cdot y)}{\sqrt{|K(x_1)K(x_2)|}}$$

when both normals are equal. When the normals are opposite, the cosine function must be replaced by a  $-\sin$ .

Note that Kendall [9] only considered convex sets in  $\mathbf{R}^2$ . Therefore his result is somewhat simpler. Also the convergence result (6) holds under some regularity conditions. For instance in the paper [9], Kendall assumed that the boundary of the convex set  $X$  is  $\mathcal{C}^4$  and that the Gauss curvature is positive for all boundary points.

Note that computing the approximation (6) would require precise measurements of distance, normals and curvature along the boundary  $\partial X$ . The double sum in (6) can be decomposed into two partial sums. The partial sum obtained by summing over pairs of type  $(x, x)$  may be expressed in terms of the surface area  $S$  of  $\partial X$  and of the rose  $r$  of (non-oriented) normal directions. The rose of normal directions is characterized by the identity

$$\int_{\partial X} \phi(n(x)) dx = S \int_{\Omega_+} \phi(\omega) r(\omega) d\omega,$$

for any symmetric function  $\phi$  on the unit sphere ( $\Omega_+$  is the unit hemisphere). When the normals to the boundary  $\partial X$  are isotropically distributed, the rose  $r$  is constant:

$$r(\omega) \equiv \frac{2}{\sigma_d},$$

where  $\sigma_d$  is the  $(d - 1)$ -dimensional Hausdorff measure of the unit hypersphere in  $\mathbf{R}^d$ . The partial sum for  $x_1 = x_2$  can be written as

$$\sum_{x \in \partial X: n(x) \parallel y} Z(x, x, y) = Sr(\omega), \quad \omega \in \Omega_+, \omega \parallel y.$$

For pairs  $(x_1, x_2)$  where  $x_1 \neq x_2$ , the function  $y \rightarrow Z(x_1, x_2, y)$  is bounded and oscillates around 0. When the boundary  $\partial X$  has a complex shape, the sum of the oscillating functions may turn out to be rather small compared to  $Sr(\omega)$ . This is why in practice the oscillating part of the spectral density is just ignored although in view of Formula (6) the two partial sums are of the same order. Hence we get the following approximation of the spectral density

$$4\pi^2 \|y\|^{d+1} \text{PSD}_X(y) \simeq Sr(\omega), \quad \|y\| \rightarrow \infty. \quad (7)$$

Note that Matheron used a different approach in order to derive this asymptotic approximation. This is due to the fact that he did not consider only indicator functions. For a general function  $f$  there is no general convergence result for its spectral density  $\text{PSD}_f$ . We have seen in Section 3 that  $\text{PSD}_f$  is the Fourier transform of the covariogram  $g$ . Thus the asymptotic behavior of  $\text{PSD}_f$  depends on local differentiability properties of the covariogram. Special attention is paid to the behavior of the covariogram at the origin for if the covariogram is e.g. continuously differentiable at the origin then it is continuously differentiable everywhere. Let us consider the case where in a neighbourhood of the origin the covariogram behaves as

$$g(h) = a + b \|h\| + O\left(\|h\|^2\right). \quad (8)$$

Such a covariogram is not differentiable at the origin and the contribution of this singularity to its Fourier transform (i.e. to the spectral density of  $f$ ) is

$$-\frac{d-1}{2\pi^2 \sigma_{d-1}} b \|y\|^{-(d+1)}.$$

In the particular case where  $f = I_X$  and the normals to  $\partial X$  are isotropically distributed, Matheron [12, 13] showed that

$$g(h) = V - \frac{\sigma_{d-1}}{(d-1)\sigma_d} S \|h\| + O\left(\|h\|^2\right). \quad (9)$$

Thus the contribution of the singularity of the covariogram at the origin to its Fourier transform is

$$\frac{S}{2\pi^2 \sigma_d} \|y\|^{-(d+1)}.$$

This result is consistent with the approximation (7) when the normals are isotropically distributed. Singularities of the covariogram not occurring at



the origin contribute to the spectral density as oscillating functions. This was shown by Matheron [11] on a simple example.

The local model (9) holds under regularity conditions. Following Serra [16], (9) holds when  $X$  belongs to the convex ring and also when  $X$  is invariant under a combination of an opening and a closing by a ball (regular model).

Matheron considered other local models than (8) for the covariogram near the origin. Let us write these models as follows

$$g(h) \simeq a + \psi(h),$$

where  $\psi$  is an isotropic function on  $\mathbf{R}^d$ . Computing the Fourier transform of  $\psi$  and using a straightforward generalization of Equation (4), one can compute asymptotic approximations of mean square errors. The so-called correspondance principle given by Matheron provides a correspondance between a function  $\psi$  and its contribution to the mean square error.

Note that Matheron's methods are based on isotropy assumptions. An extension to a very simple case of anisotropy is provided in [12].

## 5 Asymptotics for the geometric spectral densities of random sets

In this section the results from the previous section are derived using another approach. We assume that the domain of interest is *random*. It is denoted by  $\mathbf{X}$  instead of  $X$ . We will see that an asymptotic approximation of the (mean) spectral density can be derived using tools much simpler than the method of the stationary phase. The obtained approximation does not involve any oscillating term: this provides a theoretical argument for ignoring the oscillating terms involved in the previous section. Also no isotropy assumption is required and regularity conditions turn out to be rather simple.

### 5.1 A weak convergence result

For a start let us stay in a deterministic framework: the domain  $X$  is fixed. It is assumed that the boundary  $\partial X$  is a compact continuously differentiable manifold. It follows from Equation (5) that asymptotics of  $\text{PSD}_X$  can be derived from asymptotics of the spectral densities of the surface measures  $U_j$ .

The key point is the following result provided by Hörmander [7].

**Theorem 1** *Let  $M$  be a  $C^1$  compact  $q$ -dimensional manifold in  $\mathbf{R}^d$ . Let  $U$  be a measure on  $M$  with density  $u$  with respect the Hausdorff measure  $\lambda^q$ . The density  $u$  is assumed to be squared integrable. Then we have*

$$\|\rho y\|^q \text{PSD}_U(\rho y) \xrightarrow{\text{weakly}} W, \tag{10}$$

where  $W$  is the measure on  $\mathbf{R}^d$  defined by

$$\phi \rightarrow \int_M \int_{N(x)} |u(x)|^2 \phi(n) \|n\|^q dndx.$$

Above,  $N(x)$  denotes the normal subspace to  $M$  at  $x$ .

A straightforward application of this theorem combined with Equation (5) yields the limit (in a weak sense) of  $4\pi^2 \|\rho y\|^{d+1} \text{PSD}_X(y)$ . There is a further simplification due to the fact that the sum of squared densities of the measures  $U_j$  is equal to 1. Hence we get

$$4\pi^2 \|\rho y\|^{d+1} \text{PSD}_X(\rho y) \xrightarrow{\text{weakly}} SR, \tag{11}$$

where  $R$  is the measure on  $\mathbf{R}^d$  defined by

$$\phi \rightarrow \int_{\Omega_+} \int_{\mathbf{R}} \phi(t\omega) t^{d-1} r(\omega) dt d\omega.$$

Note that above it is implicitly assumed that the rose of normal directions exists as a function (i.e. the boundary of  $\partial X$  has no flat parts). However it is easy to see that such an assumption is not necessary. A more general result is obtained by defining the rose of normal directions as a measure.

The convergence result (11) and the approximation (7) are somewhat equivalent. However in order to approximate the lattice sum (4) we need a pointwise convergence instead of a weak convergence as in (11).

## 5.2 Stochastic interpretation

We will see below that weak convergence can be reinterpreted as pointwise convergence through randomization. Let  $\mathbf{h}$  be a random positive real variable and  $\mathbf{P}$  be a random orthogonal transformation on  $\mathbf{R}^d$ . Consider the random compact set  $\mathbf{X} = \mathbf{hP}X$ . This random set has a fixed shape, but its size and orientation are random. Define the spectral density of  $\mathbf{X}$  as the mean squared modulus of the Fourier transform of the random function  $I_{\mathbf{X}}$ . It is easy to check that

$$\text{PSD}_{\mathbf{X}}(y) = \mathbf{E} \left[ \mathbf{h}^{2d} \text{PSD}_X(\mathbf{hP}'y) \right].$$

Now consider the random point  $\mathbf{y} = \mathbf{h}\mathbf{P}'\omega$  where  $\omega \in \Omega_+$  and  $\omega \parallel y$ . The spectral density of  $\mathbf{X}$  can be expressed as

$$\text{PSD}_{\mathbf{X}}(y) = \mathbb{E} \left[ \|y\|^{2d} \text{PSD}_X(\|y\| y) \right].$$

Let us assume for the moment that the joint distribution of the random variables  $\mathbf{h}$  and  $\mathbf{P}$  is such that the distribution of  $y$  has a continuous density  $\psi_\omega$  with compact support with respect to the Lebesgue measure on  $\mathbf{R}^d$ . The spectral density of  $\mathbf{X}$  can be rewritten as

$$\|y\|^{d+1} \text{PSD}_{\mathbf{X}}(y) = \int_{\mathbf{R}^d} \psi_\omega(z) \|z\|^{d-1} (\|y\| \|z\|)^{d+1} \text{PSD}_X(\|y\| z) dz.$$

In view of the convergence result (11), we have

$$4\pi^2 \|y\|^{d+1} \text{PSD}_{\mathbf{X}}(y) \xrightarrow{\|y\| \rightarrow \infty} S \int_{\Omega_+} \int_{\mathbf{R}} \psi_\omega(t\eta) t^{2d-2} r(\eta) dt d\eta.$$

Note that above the surface area  $S$  and the rose of normal directions are defined for the deterministic set  $X$ . The limit can be rewritten using the mean surface area  $S_{\mathbf{X}}$  and the rose  $r_{\mathbf{X}}$  of normal directions of the random set  $\mathbf{X}$ .

The rose  $r_{\mathbf{X}}$  of normal directions to  $\partial\mathbf{X}$  is defined by the identity

$$\mathbb{E} \int_{\partial\mathbf{X}} \phi(n(x)) dx = S_{\mathbf{X}} \int_{\Omega_+} \phi(\eta) r_{\mathbf{X}}(\eta) d\eta.$$

Both characteristics  $S_{\mathbf{X}}$  and  $r_{\mathbf{X}}$  can be derived from  $S$ ,  $r$  and some features of the random variables  $\mathbf{h}$  and  $\mathbf{P}$ . It turns out that only the density  $\psi_\omega$ , is required:

$$S_{\mathbf{X}} r_{\mathbf{X}}(\omega) = S \int_{\Omega_+} \int_{\mathbf{R}} \psi_\omega(t\eta) t^{2d-2} r(\eta) dt d\eta.$$

Hence the convergence result may be simplified into

$$4\pi^2 \|y\|^{d+1} \text{PSD}_{\mathbf{X}}(y) \xrightarrow{\|y\| \rightarrow \infty} S_{\mathbf{X}} r_{\mathbf{X}}(\omega), \tag{12}$$

where the convergence is *pointwise*.

Obviously the type of random set we have considered so far is quite particular (fixed shape, random size and orientation). Such random set does not seem to be a very realistic model for e.g. biological structures. Let us consider two arbitrary compact sets  $X_1, X_2 \subset \mathbf{R}^d$ . The shapes

of  $X_1$  and  $X_2$  coincide if there exist a translation vector  $v$ , an orthogonal transformation  $P$  and a dilatation by a factor  $h$  such that

$$X_2 = hPX_1 + v.$$

Shapes may be defined as orbits under the transformations above. Furthermore using orbit representative, any compact set  $X$  may be uniquely decomposed as

$$X = hPF + v, \quad F \subset \mathbf{R}^d.$$

The set  $F$  represents the shape of  $X$ . Thus any random compact set  $\mathbf{X}$  defines a random shape say  $\mathbf{F}$ . The decomposition

$$\mathbf{X} = \mathbf{hPF} + \mathbf{v}$$

defines also a random translation by a vector  $\mathbf{v}$ , a random orthogonal transformation  $\mathbf{P}$  and a random dilatation by a factor  $\mathbf{h}$ .

Obviously the random translation does not matter here: the random probe  $\mathbf{Y}$  is already stationary. Therefore without loss of generality, we may assume that  $\mathbf{v} \equiv 0$ :

$$\mathbf{X} = \mathbf{hPF}.$$

Now consider the “spectral density of  $\mathbf{X}$  conditionally to  $\mathbf{F}$ ”. Under some regularity conditions to be discussed below, the convergence of the “conditional” spectral density is given by Formula (12) where the means involved in the limit should be replaced by conditional means:

$$4\pi^2 \|y\|^{d+1} \text{PSD}_{\mathbf{X}/\mathbf{F}}(y) \xrightarrow{\|y\| \rightarrow \infty} S_{\mathbf{X}/\mathbf{F}} r_{\mathbf{X}/\mathbf{F}}(\omega). \quad (13)$$

Taking the mean with respect to the shape  $\mathbf{F}$  on both sides in the previous formula, we extend the convergence result (11) to “arbitrary” random compact sets. Of course deriving the unconditional convergence (12) from the conditional convergence (13) requires some extra argument. Using an auxiliary result provided together with Theorem 1 by Hörmander [7], one obtains an upper bound for the left-hand side of Formula (13). This upper bound is proportional to  $\sup \psi_\omega$ . Now the density  $\psi_\omega$  (density of the conditional distribution of  $\mathbf{y} = \mathbf{hP}'\omega$  given  $\mathbf{F}$ ) depends on the shape  $\mathbf{F}$ . If the density  $\psi_\omega$  is uniformly bounded, a deterministic upper bound for the left-hand side of Formula (13) may be derived:

$$4\pi^2 \|y\|^{d+1} \text{PSD}_{\mathbf{X}/\mathbf{F}}(y) \leq C < +\infty. \quad (14)$$

Thus using Lebesgue dominated convergence, one may derive (12) from (13). Further details are omitted here because they do not involve other technics than those used until now.

Some comments about regularity conditions are needed. In order to apply Theorem 1, the boundary of  $\mathbf{X}$  must be (almost surely) a compact  $\mathcal{C}^1$  manifold with finite mean area. Next the densities  $\psi_\omega$  need to be continuous with compact support. This condition is fulfilled if  $\mathbf{X}$  is a.s. uniformly bounded and if the conditional joint distribution of  $(\mathbf{h}, \mathbf{P})$  given  $\mathbf{F}$  has a continuous density with respect to the product of the Haar probability measure on the group of orthogonal transformations and the Lebesgue measure on the  $\mathbf{R}_+$ . Finally in order to derive the unconditional convergence (12) from the conditional convergence (13), we need a uniform upper bound for the densities  $\psi_\omega$ . The existence of such an upper bound can be proved if the density of the conditional distribution of  $(\mathbf{h}, \mathbf{P})$  given  $\mathbf{F}$  is uniformly bounded.

Compare results (12) and (7). In Formula (12) we have a strict asymptotic equivalence: omitted terms are of higher-order. In Formula (7) omitted terms are just assumed (based on empirical evidence) to be small compared to the approximating term.

## 6 Approximations of the mean square error of the area estimator

Let  $\mathbf{X}$  be a random compact set in the plane  $\mathbf{R}^2$ . This section provides asymptotic approximations of the mean square error of the estimator (2) based on the convergence results from Section 5. We assume that the regularity conditions discussed at the end of Section 5.2 are fulfilled. In order to stick to standard stereological notations, the mean perimeter length is denoted by  $B$  instead of  $S_{\mathbf{X}}$ .

In view of the convergence result (12), the inequality (14) and Equation (4), we can compute the following limit for the MSE of the area estimator

$$\text{MSE}[\widehat{\mathbf{A}}] \sim \frac{B}{4\pi^2} \sum_{y \in \Lambda^*} \frac{r(\omega(y))}{\|y\|^3}, \quad (15)$$

where  $r$  is the rose of normal directions to the boundary of  $\mathbf{X}$ . The rose  $r$  is considered as a function on a unit half circle  $\Omega_+$  and  $\omega(y)$  is the unit vector from  $\Omega_+$  which is parallel to  $y$ .

Note that the mean square error is now defined by

$$\text{MSE}[\widehat{\mathbf{A}}] = \mathbb{E} \left[ \left( \widehat{\mathbf{A}} - \mathbf{A} \right)^2 \right],$$

where  $\mathbf{A}$  is the area of the random set  $\mathbf{X}$ . Hence it is the mean square difference between two random variables instead of the mean square difference between a fixed parameter and its estimation. In some way  $\text{MSE}[\widehat{\mathbf{A}}]$  may be considered more like a prediction error than a mean square error as usually defined in statistics.

Let us consider the special case where the normals to the boundary  $\partial X$  are isotropically distributed:  $r \equiv \pi^{-1}$ . The mean square error simplifies into

$$\text{MSE}[\widehat{\mathbf{A}}] \sim \frac{B}{4\pi^3} \sum'_{y \in \Lambda^*} \frac{1}{\|y\|^3}.$$

Similar formulae based on the approaches described in Section 4 have been derived by Kendall [9] and Matheron [12].

The lattice sum is an instance of the zeta function defined by Epstein (see [5]) as

$$Z_{\Lambda}(s) = \sum'_{x \in \Lambda} \|x\|^{-s}, \quad s > 2. \quad (16)$$

In order to distinguish the density of the sampling lattice from its shape, let us write it as  $\Lambda = \sqrt{a}\Lambda_0$ . The determinant of the lattice  $\Lambda_0$  is equal to 1. All lattices associated with the same  $\Lambda_0$  only differ by scale. Using this decomposition, the mean square error can be written as

$$\text{MSE}[\widehat{\mathbf{A}}] \sim \frac{B}{4\pi^3} a^{\frac{3}{2}} Z_{\Lambda_0^*}(3). \quad (17)$$

Hence the MSE is of the same order as  $a^{\frac{3}{2}}$ : doubling the sampling density decreases the MSE by a factor 2.8. Also compare the rate of convergence in (17) with Huxley's result (3). If we consider the standard error (i.e. the square root of the MSE) of  $\widehat{\mathbf{A}}$  we get

$$\text{SE}[\widehat{\mathbf{A}}] = O\left(a^{\frac{3}{4}}\right).$$

Note that this result holds even if the normals to  $\partial \mathbf{X}$  are not isotropically distributed.

Furthermore the effect of the lattice shape can be assessed by comparing  $Z_{\Lambda_0^*}(3)$  for various "unit" lattices  $\Lambda_0$ . It has been shown by Rankin [14]

that  $Z_{\Lambda_0^*}(3)$  is least for the hexagonal lattice. For an hexagonal lattice  $\Lambda_0$  (note that the unit hexagonal lattice is self-dual), we have

$$Z_{\Lambda_0^*}(3) = 8 \left( \frac{\sqrt{3}}{2} \right)^{\frac{5}{2}} \zeta(3) \operatorname{Im} \operatorname{Li}_3(\exp(2\pi i/3)) = 8.89,$$

where  $\zeta$  is the Riemann zeta function and the  $\operatorname{Li}_s$ 's are polylogarithmic functions.

For a (self-dual) square lattice

$$Z_{\Lambda_0^*}(3) = 4\zeta(3) \operatorname{Im} \operatorname{Li}_3(\exp(2\pi i/2)) = 9.03.$$

Thus the difference between square and hexagonal lattices is not very important.

For arbitrary lattice shapes, the Epstein zeta function may be numerically computed from Formula (16). However the convergence of the lattice sum involved in Formula (16) is very slow. An alternative is to use the Chowla-Selberg expansion of the Epstein zeta function. Let  $e_1$  and  $e_2$  be two basis vectors of the unit lattice  $\Lambda_0$ . Let  $\alpha$  be the norm of the projection of  $e_2$  onto the axis orthogonal to  $e_1$ . And define  $\beta$  such that  $\beta e_1$  is the projection of  $e_2$  onto the axis spanned by  $e_1$ . Then the Chowla-Selberg expansion may be written as follows:

$$\begin{aligned} Z_{\Lambda_0}(s) &= 2\alpha^s \zeta(s) + 2\sqrt{\pi} \frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})} \alpha^{1-s} \zeta(s-1) + \frac{2^{\frac{3-s}{2}} \sqrt{\pi}}{\Gamma(\frac{s}{2})} \alpha^{2-s} \\ &\times \sum_{k_1, k_2 \neq 0} \exp(2\pi i \beta k_1 k_2) |k_2|^{1-s} (2\pi \alpha^2 |k_1 k_2|)^{\frac{s-1}{2}} K_{\frac{1-s}{2}}(2\pi \alpha^2 |k_1 k_2|). \end{aligned} \tag{18}$$

Matheron's correspondance rules mentionned at the end of Section 4 are based on the first two terms of the Chowla-Selberg expansion. Note that better convergence of the double sum is obtained by ordering the lattice basis so that  $e_1$  is shorter than  $e_2$ . Using this expansion, the computation of Epstein zeta function is much faster.

For a unit rectangular lattice ( $\beta = 0$ ) such that  $\|e_2\| = 3\|e_1\|$ , we get

$$Z_{\Lambda_0}(3) = 16.3.$$

Hence the performance of highly anisotropic sampling lattices is poor for isotropic random domains.

Note that all results for isotropic random domains hold for anisotropic random domains if the orientation of the sampling lattice is isotropically randomized.

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