On the Channel Statistics in Hybrid ARQ Systems for Correlated Channels

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***Abstract*—This paper discusses a result for error control techniques based on retransmissions, most notably Hybrid Automatic Repeat reQuest schemes. We assume that the underlying coding technique is described by the so-called *reliable region*. Under this assumption, we derive the channel distribution after a frame is either acknowledged or discarded. This is derived within an entirely analytical framework, where we show that such a distribution can be found as the result of an iterative process. Remarkably, this distribution is *not the same* as the steady-state channel distribution, and can be significantly different from it. Thus, using one instead of the other can lead to evaluation errors, which instead are avoided by our model.**

***Index Terms*—Channel coding, error correction, automatic repeat request, forward error correction, communication channels.**

I. Introduction

In this paper, we present a result concerning retransmission-based error control techniques, with particular reference to Hybrid Automatic Repeat reQuest (Hybrid ARQ), and especially Incremental Redundancy Hybrid ARQ (IR-HARQ) [1]. Such techniques counteract channel errors by using data coding and transmitting fragments of a longer codeword, which encodes an information frame. If the first transmission attempt fails, i.e., the receiver is unable to retrieve the information frame from the received fragment, another fragment is transmitted over a different channel realization. This time, the fragments are combined at the receiver's side in a longer codeword. If the receiver is still unable to decode the frame, another fragment is requested. This process continues until either a successful decoding happens, or a maximum number of transmission attempts is reached, in which case the frame is discarded.

Hybrid ARQ techniques are deemed to be effective especially for the wireless channel, and in this context they have been widely studied in the literature. Most of the investigations are performed from the physical layer standpoint, and usually by means of simulation [2], [3]. Nevertheless, there is a flourishing interest for analytical models [4], [5], which, however, generally use fading models with independent channel coefficients at every transmission.

To apply Hybrid ARQ techniques in higher layer contexts, analytical representations must track the evolution of the communication process over multiple transmissions. This can be done by means of Markov chains, as proposed in [4], [6]; however, this approach can be complicated, as the memory of the system may become very large [7]. Moreover, it is important, as we will argue in the following, that *channel correlation* between multiple retransmissions is properly kept into account [8].

Finally, we need a model to determine the outcome of each combined codeword, where it is considered that different codeword fragments are sent over different channel realizations. To this end, we exploit recent findings on practical near-capacity-achieving codes, which have drawn a significant research effort to derive information theoretic coding performance bounds. In particular, we consider the so-called *good code* ensembles [9], such as Turbo codes [10] and Low–Density Parity–Check (LDPC) codes [11], whose performance can be characterized by a threshold behavior. A code is said to be good when its block error probability asymptotically approaches zero as the codeword length increases, provided that the channel parameters fall within the so-called *reliable region* [12]. As the specific structure of good codes is usually not given, bounds are often derived for ensembles of codes, i.e., classes of codes characterized by basic features of structure and construction, such as the distance spectrum and the input-output weight enumeration function [13], [14].

The analytical study of an IR-HARQ scheme implies checking whether multiple, say n, codeword fragments transmitted over different channel conditions can be correctly decoded at the receiver side. This translates into checking whether the n-tuple of corresponding channel parameters falls within the reliable region. Even when the channel distribution and its evolution in a correlated fashion are known, this setup presents a further problem for the analysis, namely, how to determine the channel distribution of the first transmitted fragment. In a naïve setup, one can think of choosing this as the steady-state channel distribution, which is, however, incorrect.

Intuitively, the first fragment of a codeword is transmitted when a previous frame has been correctly decoded in the previous transmission, thus, if the channel distribution is correlated, it is more likely that the channel was, and is, in a "good" state, i.e., one that enables decoding with higher probability. Actually, this holds also for the case where the frame is discarded after a given number of failed incremental decoding attempts. However, in this case the channel is more likely to be in a bad state. Both these cases are taken into account by our analysis, where we derive the channel distribution after the acknowledgement or discard of a Hybrid ARQ frame, which is also directly correlated to the channel distribution encountered by the first fragment of the next Hybrid ARQ codeword. We show, with a formal proof making use of Banach fixed point theorem, that this distribution exists and can be derived through an iterative procedure. Numerical results are presented to show that the distribution sought is considerably different from the steady-state channel statistics.

The rest of this paper is organized as follows. In Section II we describe the IR-HARQ mechanism assumed in the analysis and we introduce some notation and terminology. Section III presents the main contribution of this paper, i.e., the analytical derivation of the channel distribution sought. In Section IV we present numerical evaluations where we compare the steady-state distribution with the one found by our analysis, showing that they are indeed considerably different. Finally, we draw the conclusions in Section V.

II. System Model

Assume that we have a flow of data, subdivided in what we call *information frames*, which are exchanged between a transmitter and a receiver. The realization of an IR-HARQ scheme implies that the frame transmission is translated into that of multiple *HARQ packets* [15]. This means that every information frame is coded into a single long *codeword*, which is in turn split into multiple *fragments*. Each fragment is transmitted individually as a single HARQ packet. For this reason, we will use the terms "codeword" and "fragment" as synonyms for "information frame" (or simply "frame") and "HARQ packet," respectively. In the following, we assume that F fragments (and therefore F HARQ packets) are generated from each codeword. Moreover, we assume that each single fragment contains the entire information amount of a codeword, so that its fully correct reception is sufficient to decode the frame. When this assumption holds, the IR-HARQ scheme is also said to be of Type III ARQ [1].

Also, the Incremental Redundancy (IR) property means that the receiver can decode the codeword combining symbols contained in different fragments; therefore, upon reception of a HARQ packet, the receiver sends a *feedback packet* to the transmitter. This indicates either positive or negative acknowledgement; the feedback is referred to as ACK in the former case, and NACK in the latter. Due to the IR property, the acknowledgement message refers to the whole information frame. Thus, an ACK means that the receiver was able to decode the frame based on all received HARQ packets associated with this frame (in this case, we speak of *frame resolution*), whereas a NACK means that the frame could not be decoded since the channel impairments exceeded the correction capability of the code formed by the set of currently received fragments. Due to the IR property, unlike in other retransmission-based techniques, a NACK does not trigger the retransmission of the same data; instead, a physically different HARQ packet (though associated with the same information frame) is transmitted. In such a case, we adopt a slight abuse of terminology and speak of *frame retransmission*.

In the present paper, we focus on Stop-and-Wait (SW) HARQ, i.e., the transmitter and receiver alternate in exchanging HARQ and feedback packets. The extensions to other ARQ schemes, such as Go-Back-N and Selective Repeat, are straightforward along the same lines.

Thus, we assume that, as long as the receiver sends back NACKs, HARQ packets associated with the same information frame are sequentially transmitted. However, two different events can cause the transmitter to move to the next information frame. One happens when the information frame is acknowledged, so that the receiver sends back an ACK. The other occurs when a maximum number of transmission is reached without the receiver being able to decode the information frame. We set this number to F, i.e., all generated fragments are transmitted exactly once without success. The information frame is then discarded.

For each frame, we define the *resolution instant*, denoted as τ_R, as follows. If the frame is resolved at the kth transmission, the resolution instant is equal to k. If the frame is *discarded*, τ_R is conventionally set to $F+1$. Note that this notation distinguishes the resolution instant from the *number of transmissions* experienced by a given frame, which is called $\tau_T = \min(\tau_R, F)$. In other words, if the frame is resolved, τ_R is equal to τ_T. However, since the frame cannot be transmitted more than F times, we need to distinguish the case where the frame is resolved at the Fth transmission (i.e., the last possible one) from the case where it is discarded. In the former case, $\tau_R = F$, whereas in the latter $\tau_R = F+1$ (while $\tau_T = F$ in both cases). For any $k \in \{1, 2, \ldots, \tau_T\}$, we will use the symbol w_k to denote the codeword fragment sent at the kth transmission. For completeness, we will denote with w_0 the codeword fragment (related to the previous frame) which has been transmitted previous to the first one of the current frame.

One important application of Hybrid ARQ techniques is to control errors in data transmissions over wireless fading channels. Thus, the scenario of reference in the following will consider that the transmission outcome, according to which the receiver sends either ACK or NACK, depends on the Signal-to-Noise Ratio (SNR) at the receiver. However, the same rationale is directly applicable to other kinds of noisy channels.

In the rest of the paper, we assume a block flat fading channel, i.e., each HARQ packet experiences a single SNR coefficient, denoted with $s_k \in \mathbb{R}_+$ for the kth fragment, $k = 0, 1, \ldots, \tau_T$. Similar block fading models have been investigated, for example, in [16] for coded modulation. However, since we consider a block duration equal to the transmission time of a HARQ packet, our model is more similar to that of [17]. For the sake of simplicity, feedback packets are instead considered to be error-free. Indeed, they are shorter packets and just contain a binary information (ACK/NACK) so that they can be protected with strong encoding. The additional effect of erroneous feedback, which is reasonably quite limited, can be evaluated as shown in [18].

Therefore, a description of the channel statistics, i.e., of the SNR values, is required. We consider the cumulative distribution function (cdf) and the probability density function (pdf) of the SNR, and denote them with F_Γ and f_Γ, respectively. This means that, if Γ is the random variable describing the channel SNR,

$$F_\Gamma(s) = \text{Prob}\{\Gamma \le s\}, \quad f_\Gamma(s) = \frac{\mathrm{d}F_\Gamma(s)}{\mathrm{d}s}\,. \tag{1}$$

Note that $F_\Gamma(s)$ takes values in $[0,1]$ and is zero for $s \le 0$.

Moreover, $f_\Gamma(s)$ is integrable since

$$\lim_{s\to+\infty} F_\Gamma(s) = \int_0^{+\infty} f_\Gamma(s)\mathrm{d}s \ = 1 \ . \qquad (2)$$

A key aspect of our analysis is that we want to take correlation of fading effects into account. This means that we also consider conditional SNR statistics and we write $f_\Gamma(s_k|s_{k-1})$ to denote the conditional pdf of s_k given that the SNR was equal to s_{k-1} during the previous transmission. We assume that the evolution of the SNR has the Markov property [19], which means that the knowledge of s_{k-1} makes any other condition on previous SNR values irrelevant. This includes the particular case where the channel state is selected independently from block to block, according to a known prior distribution [15], [17], if $f_\Gamma(s_k|s_{k-1}) = f_\Gamma(s_k)$.

Finally, we need a model to describe the outcome of the decoding process, i.e., to determine if the receiver is able to correctly reconstruct the codeword from the reception of k fragments, i.e., w_1 through w_k, whose SNR coefficients are $s_1, s_2, \ldots, s_k$. To this end, we utilize the *reliable region model* [12], [20]

The reliable region at the kth transmission, denoted with $\mathcal{R}(k)$, is defined as a subset of $\mathbb{R}_+^k$ which contains the k-tuples of SNR coefficients where the failure probability becomes negligible if the packets sent are sufficiently large. Hence, the receiver is able to decode the codeword after the reception of $w_1, w_2, \ldots, w_k$ if $(s_1, s_2, \ldots, s_k) \in \mathcal{R}(k)$. The reliable region $\mathcal{R}(k)$ is specifically determined by the code used, the decoding algorithm and the codeword fragments construction [12].

However, some general properties hold. For example, it is easy to show that if $(s_1, \ldots, s_{k-1}, s_k) \in \mathcal{R}(k)$ and $s'_k > s_k$, then also $(s_1, \ldots, s_{k-1}, s'_k) \in \mathcal{R}(k)$. In fact, if s_k enables the receiver to decode the codeword, any SNR value better than s_k does so. Therefore, we represent $\mathcal{R}(k)$ through a threshold function $\vartheta_k : \mathbb{R}_+^{k-1} \to \mathbb{R}_+$, defined as follows:

$$\vartheta_k(\mathbf{s}^{(k-1)}) = \inf\{s_k : (s_1, \ldots, s_{k-1}, s_k) \in \mathcal{R}(k)\}, \qquad (3)$$

where $\mathbf{s}^{(k-1)} = (s_1, s_2, \ldots, s_{k-1})$. That is, the edge of $\mathcal{R}(k)$ is the graph of $s_k = \vartheta_k(\mathbf{s}^{(k-1)})$ in $\mathbb{R}_+^k$. When one transmission is considered, this curve degenerates to a single point $\vartheta_1 \in \mathbb{R}_+$, which is the value of s_1 associated with the (constant) SNR threshold to obtain correct decoding with a single fragment, and the reliable channel region $\mathcal{R}(1)$ corresponds to the interval $[\vartheta_1, +\infty[$.

Similarly to the property mentioned above, it is also true that if $(s_1, \ldots, s_k) \in \mathcal{R}(k)$, then $(s_1, \ldots, s_k, s_{k+1}) \in \mathcal{R}(k+1)$ for all $s_{k+1} \in \mathbb{R}_+$. In other words, if the fragments $w_1, w_2, \ldots, w_k$ are sufficient to decode the codeword, adding another fragment cannot worsen the decoding process. Actually, when the frame resolution is acknowledged, no further fragment is sent, but the transmitter moves to the next information frame.

III. The Channel Distribution after the Final Fragment Transmission

We denote with $\psi_{\tau_R,S|S_0}(j, s|s_0)$ the joint pdf of the resolution instant τ_R and the SNR S being equal to j and s, respectively, when a frame is either resolved or discarded, conditioned on the SNR of the final transmission of the previous frame S_0 being equal to s_0. Similarly, we denote with $\psi_S(s)$ the pdf of the SNR S being equal to s in the global case of frame resolution or discarding. For brevity, we refer to this pdf as the one for the *final fragment*. What is meant with "final" is that the current transmission attempt is the last one for this specific information frame, because either the frame becomes resolved or the fragment is the Fth one (and henceforth no further retransmission is allowed). Thus, the next transmission will involve a new frame.

If $\mathcal{X} \subseteq \mathbb{R}_+^k$, we define $\mathbb{1}(\mathbf{s}^{(k)}, \mathcal{X})$ as equal to 1 if $\mathbf{s}^{(k)} \in \mathcal{X}$, and 0 otherwise. We have [1]:

$$\psi_{\tau_R,S|S_0}(j, s|s_0) =$$

$$\begin{cases} f_\Gamma(s|s_0)\, \mathbb{1}\big((s), \mathcal{R}(1)\big) & \text{if } j = 1 \\ \displaystyle\int_{\mathbb{R}_+^{j-1}\setminus\mathcal{R}(j-1)} f_\Gamma(s|s_{j-1})\, \mathbb{1}\big((\mathbf{s}^{(j-1)}, s), \mathcal{R}(j)\big) \\ \displaystyle\prod_{\ell=1}^{j-1} f_\Gamma(s_\ell|s_{\ell-1})\, \mathrm{d}s_\ell & \text{if } 2 \le j \le F \\ \displaystyle\int_{\mathbb{R}_+^{F-1}\setminus\mathcal{R}(F-1)} f_\Gamma(s|s_{F-1})\, \mathbb{1}\big((\mathbf{s}^{(F-1)}, s), \mathbb{R}_+^F \setminus \mathcal{R}(F)\big) \\ \displaystyle\prod_{\ell=1}^{F-1} f_\Gamma(s_\ell|s_{\ell-1})\, \mathrm{d}s_\ell & \text{if } j = F+1 \end{cases}$$

$$\Rightarrow \psi_S(s) = \int_0^{+\infty} \psi_S(s_0) \sum_{j=1}^{F+1} \psi_{\tau_R,S|S_0}(j, s|s_0)\, \mathrm{d}s_0 \, . \qquad (4)$$

If we take $\beta(s, s_0) = \sum_{j=1}^{F+1} \psi_{\tau_R,S|S_0}(j, s|s_0)$, we can write (4) in the form

$$\psi_S(s) = \int_0^{+\infty} \psi_S(s_0)\beta(s, s_0)\mathrm{d}s_0 \, , \qquad (5)$$

that is, $\psi_S(s)$ must be an eigenfunction of the transform operation, which maps a generic function $\zeta : \mathbb{R} \to \mathbb{R}$ into its transform $\widehat{\zeta}$, given by

$$\widehat{\zeta}(s) = \int_0^{+\infty} \zeta(s_0)\beta(s, s_0)\mathrm{d}s_0 \, . \qquad (6)$$

In the following, we prove that such an eigenfunction exists and can be determined by a recursive strategy, so that we also provide an operational method to determine it. The reasoning is a direct consequence of Banach fixed point theorem [21]. Consider $\mathcal{K}$ as the set of all pdfs over the real non-negative semi-axis and denote with d the Lebesgue measure, i.e., the distance related to the norm-1. This means that, for any $x(s)$, $y(s) \in \mathcal{K}$

$$\int_0^{+\infty} |x(s)|\mathrm{d}s = \int_0^{+\infty} |y(s)|\mathrm{d}s = 1 \qquad (7)$$

$$d(x, y) = \int_0^{+\infty} |x(s) - y(s)|\mathrm{d}s \, . \qquad (8)$$

The function $\psi_S(s)$ such that $\psi_S(s) = \widehat{\psi_S}(s)$ is found as a suitable fixed point in the metric space $(\mathcal{K}, d)$, whose existence is guaranteed by the application of Banach fixed point theorem. The only missing part is to prove the following theorem.

Theorem 1: The transform $\widehat{\ }: \zeta \to \widehat{\zeta}$ is a contraction in the metric space $(\mathcal{K}, d)$.

Proof: We need to prove that either $d(x, y) = 0$ or $d(\widehat{x}, \widehat{y}) < d(x, y)$, where the inequality is strict. We observe that $d(x, y) = 0$ only if x and y are the same pdf; otherwise, since they both are density functions, both sets $\mathcal{U}_x = \{s \in \mathbb{R}_+ : x(s) \geq y(s)\}$ and $\mathcal{U}_y = \{s \in \mathbb{R}_+ : x(s) < y(s)\}$ have non-zero measure. At this point we evaluate $d(\widehat{x}, \widehat{y})$ as

$$d(\widehat{x}, \widehat{y}) = \int_0^{+\infty} \Big| \int_0^{+\infty} \big(x(s_0) - y(s_0)\big)\beta(s, s_0)\mathrm{d}s_0 \Big| \mathrm{d}s$$

Since the innermost integral is equal to

$$\int_{\mathcal{U}_x} \big(x(s_0) - y(s_0)\big)\beta(s, s_0)\mathrm{d}s_0 + \int_{\mathcal{U}_y} \big(x(s_0) - y(s_0)\big)\beta(s, s_0)\mathrm{d}s_0$$

with both terms being greater than zero, we can write a strict inequality

$$d(\widehat{x}, \widehat{y}) < \int_0^{+\infty} \int_0^{+\infty} \big|x(s_0) - y(s_0)\big|\beta(s, s_0)\mathrm{d}s_0\mathrm{d}s \,. \quad (9)$$

Note that $\int_0^{+\infty} x(s_0)\beta(s, s_0)\mathrm{d}s_0$ has a finite integral as a function of s. Hence, Fubini's theorem can be applied, i.e., we can reverse the integration order. Thus,

$$\begin{aligned} d(\widehat{x}, \widehat{y}) &< \int_0^{+\infty} \big|x(s_0) - y(s_0)\big| \Big(\int_0^{+\infty} \beta(s, s_0)\mathrm{d}s \Big) \mathrm{d}s_0 \\ &= \int_0^{+\infty} \big|x(s_0) - y(s_0)\big|\mathrm{d}s_0 = d(x, y) \end{aligned} \quad (10)$$

■

Therefore, the metric space $(\mathcal{K}, d)$ and the contraction $\widehat{\ }$ satisfy the hypothesis of Banach fixed point theorem, which guarantees the existence of a fixed point $\psi_S(s)$ for the contraction, i.e., $\psi_S(s) = \widehat{\psi_S}(s)$. Moreover, the theorem enables a recursive strategy to determine $\psi_S(s)$ as the limit of a sequence $\big\{\zeta_n(s)\big\}_{n\in\mathbb{N}}$ of functions where $\zeta_0(s) = f_\Gamma(s)$ and $\zeta_{\ell+1}(s) = \widehat{\zeta_\ell}(s)$, as per (6). Finally, the pdf for an SNR value s_1 for a newly transmitted fragment can be determined as

$$\int_0^{+\infty} \psi_S(s) f_\Gamma(s_1|s)\mathrm{d}s \,. \quad (11)$$

Such a result can also be used to extend related analytical studies, for example those reported in [1], to the case of correlated channels.

IV. Numerical Evaluation

In this section we show, by means of some numerical examples, that the channel probability density function in steady-state, i.e., $f_\Gamma(s)$, can be significantly different from $\psi_S(s)$ derived in the previous section even for moderate values of the channel correlation.

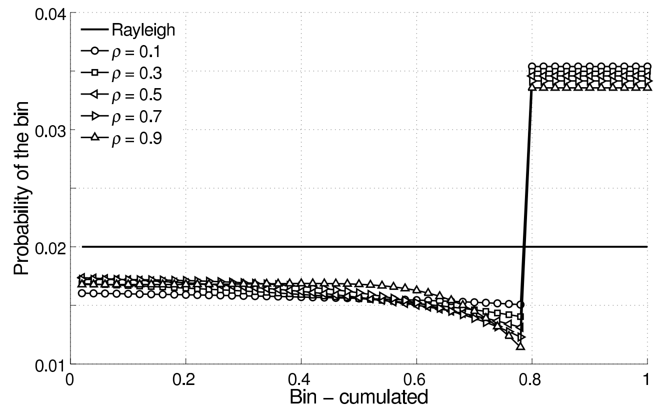

Fig. 1. The channel distribution after the last fragment for $F = 2$.

We assume that the source uses a capacity achieving code. The reliable channel region is then described by the following SNR thresholds:

$$\begin{aligned} \vartheta_1 &= 2^R - 1 \\ \vartheta_2(s_1) &= \frac{2^R - 1 - s_1}{1 + s_1} \\ \vartheta_3(s_1, s_2) &= \frac{2^R - 1 - s_1 - s_2 - s_1 s_2}{(1 + s_1)(1 + s_2)} \,, \end{aligned} \quad (12)$$

where R is the transmission rate in bit/s/Hz. The channel evolution has been computed according to a bivariate Rayleigh joint pdf $f_\Gamma(s_1, s_2)$ [22], equal to

$$f_\Gamma(s_1, s_2) = \frac{4 s_1 s_2}{(1-\rho)\Omega^2} e^{-\frac{\Omega s_1^2 + \Omega s_2^2}{\Omega^2(1-\rho)}} I_0 \left(2 \frac{\sqrt{\rho} s_1 s_2}{\Omega(1-\rho)} \right) \quad (13)$$

where Ω is the average SNR of the link, $\rho = J_0(2\pi f_d T_p)$ is the correlation of two samples of the underlying Gaussian process, spaced by T_p seconds, with T_p being the transmission time of a HARQ packet, f_d is the Doppler frequency, and $J_0(\cdot)$ and $I_0(\cdot)$ are the Bessel function and the modified Bessel function of the first kind and order zero, respectively.

To represent the probability density functions in a simple manner, we subdivided the positive real semi-axis $\mathbb{R}_+$ into 50 bins, determined with an equal probability criterion with respect to the Rayleigh distribution. This means that if the channel is Rayleigh, each of the bins has $1/50$ of probability of containing the SNR value in steady-state. Thus, the behavior of a Rayleigh pdf is flat to 0.02, which is exactly what we expect from the steady-state function $f_\Gamma(s)$. Conversely, the pdf of the SNR at the final fragment transmission will be different, and in this way it is possible to appreciate the differences. In the following plots, we set the transmission rate and the average SNR to R=2 bit/s/Hz and Ω=3 dB, respectively. Thus, the "good channel states", i.e., those where the frame is solved even with a single transmission, are in the bins above $1 - e^{-3/2} \approx 0.78$.

Figs. 1 and 2 report such evaluations for multiple values of ρ, i.e., for different correlation between two subsequent SNR values. Fig. 1 considers F=2, i.e., at most two transmission attempts are performed for each frame, whereas Fig. 2 considers a maximum of F=3 attempts. As is visible, the curves are significantly different already for low correlation values.

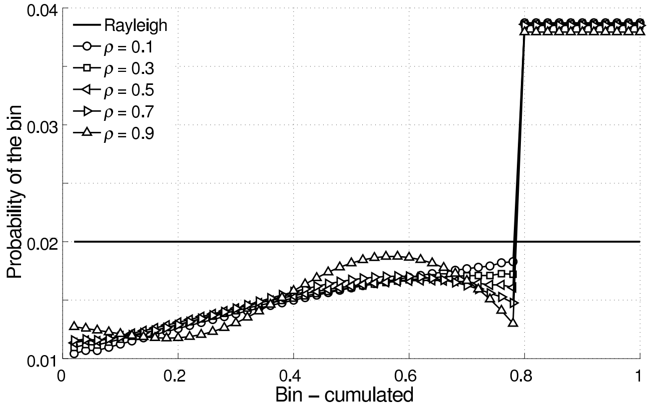

Fig. 2. The channel distribution after the last fragment for $F = 3$.

As argued in the introduction, we see an increase of the probability that the channel state is good at the final transmission of a packet, meaning that a bad channel is more likely to imply a retransmission, and therefore not be the final state. For $F = 2$ note also that very bad channel states are also slightly more likely than other bad channel states, which is due to the possibility of packet discarding, which is also included in the analysis. In fact, it is more likely that the frame is discarded after a burst of very bad channel states. The gap which makes good states more likely, which was already present for $F = 2$, is even more acute for $F = 3$. Also, it is no longer true that very bad channels are more likely, instead we see an interesting behavior with a local maximum in some intermediate channel value. All these effects are worth of further analysis, which will be an interesting subject for future work.

V. Conclusions

In this paper, we discussed the channel statistics at the final fragment transmission attempt of an Incremental Redundancy Hybrid ARQ codeword. Our proposal is motivated by the strong research interest around retransmission-based error control schemes and the renewed attention gained by practical coding techniques, such as LDPC and Turbo codes, that show a threshold behavior, i.e., they have error probability asymptotically going to zero if the channel parameters fall within the so-called reliable region.

To properly evaluate such schemes, a correct characterization of the channel statistics is required. We presented an analytical investigation, where the channel pdf at the final transmission attempt is found through a recursive methodology based on Banach fixed point theorem, and is shown to be different from the steady-state channel pdf. Finally, we reported numerical evaluations to highlight this difference.

One immediate practical implication is that over correlated channels IR-HARQ frames experience, at their first transmission, an average SNR which is in generally slightly above the steady-state value of the channel, due to the interdependence of IR-HARQ transmissions; other relationships are also present, depending on the maximum number of transmissions. These aspects must be carefully taken into account when deriving statistical properties of HARQ to avoid incorrect estimations of the error correction capabilities; therefore, they are important for the design of both coding and error control schemes.

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